Filomat 33:8 (2019), 2367–2380
https://doi.org/10.2298/FIL1908367S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hermite-Hadamard and Hermite-Hadamard-Fejer Type Inequalities Involving Fractional Integral Operators

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Abstract. Since the so-called Hermite-Hadamard type inequalities for convex functions were presented, their generalizations, refinements, and variants involving various integral operators have been extensively investigated. Here we aim to establish several Hermite-Hadamard inequalities and Hermite-Hadamard-Fejer type inequalities for symmetrized convex functions and Wright-quasi-convex functions with a weighted function symmetric with respect to the midpoint axis on the interval involving the fractional conformable integral operators initiated by Jarad et al. [9]. We also point out that certain known inequalities are particular cases of the results presented here.

1. Introduction and preliminaries

We begin by recalling the following classical Hermite-Hadamard type inequalities for convex functions (see, e.g., [2]): Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a convex function. Then, for $a, b \in I$ with $a < b$,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

holds. Here and in the following, let $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ be the sets of real numbers, positive real numbers, and positive integers, respectively, and let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The inequality (1) has attracted a remarkable number of researchers' attention. For new proofs, refinements, generalizations, and numerous applications of this inequality (1), we refer, for example, to [4, 10] and the references cited therein.

Here, Definitions 1.1, 1.2, and 1.3 are recalled (see [5]).

Definition 1.1. *Let I be a nonempty interval on $\mathbb{R}$. Then a function $f : I \to \mathbb{R}$ is called quasi-convex on I (denoted by $f \in QC(I)$) if*

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\} \quad (0 \le t \le 1;\ x,\ y \in I). \tag{2}$$

2010 *Mathematics Subject Classification*. 26A33, 26D10, 26D15, 33B20
Keywords. Convex function, Quasi-convex function, Symmetrized convex function, Wright-quasi-convex functions, Hermite-Hadamard type inequalities, Generalized fractional integral operators, Hermite-Hadamard-Fejér type inequalities
Received: 13 August 2018; Accepted: 27 December 2018
Communicated by Miodrag Spalević
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Clearly, any convex function is quasi-convex. Furthermore, there exists a quasi-convex function which is not convex.

Definition 1.2. *Let I be a nonempty interval on $\mathbb{R}$. Then a function $f : I \to \mathbb{R}$ is called Wright-quasi-convex on I (denoted by $f \in WQC(I)$) if*

$$\frac{1}{2}\Big[f(tx + (1-t)y) + f((1-t)x + ty)\Big] \le \max\{f(x), f(y)\}. \tag{3}$$

$(0 \le t \le 1;\ x,\ y \in I)$.

Definition 1.3. *Let I be a nonempty interval on $\mathbb{R}$. Then a function $f : I \to \mathbb{R}$ is called Jensen-quasi-convex on I (denoted by $f \in JQC(I)$) if*

$$f\left(\frac{x+y}{2}\right) \le \max\{f(x), f(y)\} \quad (x,\ y \in I). \tag{4}$$

The following strict inclusions holds (see [5])

$$QC(I) \subsetneq WQC(I) \subsetneq JQC(I). \tag{5}$$

We recall the following theorem (see [5]).

Theorem A. *Let I be a nonempty interval on $\mathbb{R}$ and $a,\ b \in I$ with $a < b$. Also let $f \in WQC(I)$ be integrable on $[a, b]$. Then the following Hermite-Hadamard type inequality holds*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \max\{f(a), f(b)\}. \tag{6}$$

We also recall Definition 1.4 (see [12]).

Definition 1.4. *Let I and J be intervals on $\mathbb{R}$ with $(0, 1) \subseteq J$. Also let $f : I \to \mathbb{R}_0^+$ be a function and $h : J \to \mathbb{R}_0^+$ a function with $h \not\equiv 0$. Then f is called h-convex if*

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) \quad (0 < t < 1;\ x,\ y \in I). \tag{7}$$

Definition 1.5. *Let $[a, b]$ $(a < b)$ be an interval on $\mathbb{R}$ and $f : [a, b] \to \mathbb{C}$ a function. Then the symmetrical transform of f, denoted by $\check{f}$, is defined by*

$$\check{f}(t) := \frac{1}{2}\,[f(t) + f(a+b-t)] \quad (t \in [a, b]). \tag{8}$$

The anti-symmetrical transform of f on $[a, b]$, denoted by denoted by $\tilde{f}$, is defined by

$$\tilde{f}(t) := \frac{1}{2}\,[f(t) - f(a+b-t)] \quad (t \in [a, b]). \tag{9}$$

Obviously, for any function f, $\check{f} + \tilde{f} = f$.

Definition 1.6. *[2, 6] We say that the function $f : [a, b] \to \mathbb{R}$ is symmetrized convex (concave) on the interval $[a, b]$ if the symmetrical transform $\check{f}$ is convex (concave) on $[a, b]$.*

Now, if we denote by Con$[a, b]$ the closed convex cone of convex functions defined on $[a, b]$ and by SCon $[a, b]$ the class of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a, b] \subsetneq SCon[a, b] \tag{10}$$

Also, if $[c, d] \subset [a, b]$ and $f \in SCon[a, b]$, then this does not imply in general that $f \in SCon[c, d]$.

Example 1.7. *Let the function f be defined as follows:*

$$f:[a,b]\to\mathbb{R}\quad f(x)=\frac{e^{x}-e^{-x}}{2}\quad a<0<b\ \text{ and }\ a+b>0.$$

Then f is a symmetrized convex function.

Definition 1.8. *Let h be the function in Definition 1.4. A function $f:[a,b]\to\mathbb{R}_0^+$ is called h-symmetrized convex (concave) on the interval $[a,b]$ if the symmetrical transform $\check{f}$ is h-convex (concave) on $[a,b]$.*

Example 1.9. *Let h be a function defined as $h(t)=t$, $t>0$ and let the function f be defined as follows:*

$$f:[-2,2]\to\mathbb{R}^{+}\quad f(x)=x^{3}+1\ \text{ and }\ \check{f}(t)=\frac{1}{2}[f(t)+f(a+b-t)]=1\ \ h(t)=t\ \ t>0.$$

Then f is a h-symmetrized convex function.

Theorem B. *Let h be the function in Definition 1.4 and a function $f:[a,b]\to\mathbb{R}_0^+$ be h-symmetrized convex on the interval $[a,b]$. Then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\le\frac{f(x)+f(a+b-x)}{2}\le\left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2}.\tag{11}$$

For Definitions 1.5 and 1.8, and Theorem B, we refer to [2, 6].

We recall Hermite-Hadamard inequalities for symmetrized convex functions (see [2]) in the following theorem and corollary.

Theorem C. *Let $f:[a,b]\to\mathbb{R}$ $(a<b)$ be a symmetrized convex function. Then, for any $x\in[a,b]$, we have*

$$f\left(\frac{a+b}{2}\right)\le\check{f}(x)\le\frac{f(a)+f(b)}{2}.\tag{12}$$

Corollary D. *Let $f:[a,b]\to\mathbb{R}$ $(a<b)$ be a symmetrized convex and integrable function. Then we have the following Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{f(a)+f(b)}{2}.\tag{13}$$

Fejér [7] established a so-called Hermite-Hadamard-Fejér inequality related to the integral mean of a convex function f which is a weighted generalization of Hermite-Hadamard inequality (1), which is recalled in the following theorem.

We recall the following theorem (see [7]).

Theorem E. *Let $f:[a,b]\to\mathbb{R}$ $(a<b)$ be a convex function and $f\in L_1(a,b)$. Also let $g:[a,b]\to\mathbb{R}$ be nonnegative, integrable and symmetric to $(a+b)/2$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\le\frac{1}{b-a}\int_a^b f(x)\,g(x)dx\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.\tag{14}$$

We recall the following Theorem F and Theorem G (see [8]).

Theorem F. *Let $\alpha\in\mathbb{R}^{+}$, $f:[a,b]\to\mathbb{R}$ $(a<b)$ be a convex function and $f\in L_1[a,b]$. Also let $g:[a,b]\to\mathbb{R}$ be nonnegative, integrable and symmetric to $(a+b)/2$. Then*

$$f\left(\frac{a+b}{2}\right)\left[\left(J_{a+}^{\alpha}g\right)(b)+\left(J_{b-}^{\alpha}g\right)(a)\right]\le\left[\left(J_{a+}^{\alpha}fg\right)(b)+\left(J_{b-}^{\alpha}fg\right)(a)\right]\le\frac{f(a)+f(b)}{2}\left[\left(J_{a+}^{\alpha}g\right)(b)+\left(J_{b-}^{\alpha}g\right)(a)\right].\tag{15}$$

Theorem G. *Let $\rho, \lambda \in \mathbb{R}^+$. Also let $f : [a,b] \to \mathbb{R}$ $(a < b)$ be a convex function and $f \in L_1[a,b]$. Further let $g : [a,b] \to \mathbb{R}$ be nonnegative, integrable and symmetric to $\frac{a+b}{2}$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}g\right)(a)\right] &\le \left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}fg\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}fg\right)(a)\right] \\ &\le \frac{f(a)+f(b)}{2}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}g\right)(a)\right]. \end{aligned} \tag{16}$$

Definition 1.10. *[9] The left and right-fractional conformable integrals of order $\beta \in \mathbb{C}$, $Re(\beta) > 0$, are defined by*

$${}^{\beta}_{a}\mathfrak{J}^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)dt}{(t-a)^{1-\alpha}}, \tag{17}$$

$${}^{\beta}\mathfrak{J}^{\alpha}_{b}f(x)=\frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)dt}{(b-t)^{1-\alpha}}. \tag{18}$$

Notice that, if $(Qf)(t) = f(a+b-t)$, then we have $({}^{\beta}_{a}\mathfrak{J}^{\alpha}Qf)(x) = Q({}^{\beta}\mathfrak{J}^{\alpha}_{b}f)(x)$. The fractional integral in (17) coincides with the Riemann-Liouville fractional integral when $a = 0$ and $\alpha = 1$. Moreover (18) coincides with the Riemann-Liouville fractional integral when $b = 0$ and $\alpha = 1$.

In this paper, we aim to establish several Hermite-Hadamard inequalities for symmetrized convex functions and Wright-quasi-convex functions with a weighted function symmetric with respect to the midpoint axis on the interval involving the fractional integral operators (17) and (18). We also point out that certain known inequalities are particular cases of the results presented here.

2. Hermite-Hadamard type inequalities

In this section, we investigate certain Hermite-Hadamard inequalities involving the fractional integral operators (17) and (18). We begin by presenting some useful equalities associated (17) and (18), asserted in the following lemma.

Lemma 2.1. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$. Also let $[a,b]$ $(a < b)$ be an interval on $\mathbb{R}$, $f : [a,b] \to \mathbb{C}$ be an integrable function. Then*

$$\frac{1}{2}\left[{}^{\beta}_{a}\mathfrak{J}^{\alpha}f(x)+{}^{\beta}\mathfrak{J}^{\alpha}_{b}f(a+b-x)\right]=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\check{f}(t)dt}{(t-a)^{1-\alpha}} \tag{19}$$

and

$$\frac{1}{2}\left[{}^{\beta}_{a}\mathfrak{J}^{\alpha}f(a+b-x)+{}^{\beta}\mathfrak{J}^{\alpha}_{b}f(x)\right]=\frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\check{f}(t)dt}{(b-t)^{1-\alpha}}. \tag{20}$$

Proof. We prove (19). we find from (18) that, for $a < x \le b$,

$${}^{\beta}\mathfrak{J}^{\alpha}_{b}f(a+b-x)=\frac{1}{\Gamma(\beta)}\int_{a+b-x}^b\left(\frac{(x-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)dt}{(b-t)^{1-\alpha}}. \tag{21}$$

Setting $t = a+b-u$ to change the variable in the right side of (21), we obtain

$${}^{\beta}\mathfrak{J}^{\alpha}_{b}f(a+b-x)=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^{\alpha}-(t-u)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(a+b-u)dt}{(u-a)^{1-\alpha}}. \tag{22}$$

From (17), we also have

$$ {}^{\beta}_{a}\mathfrak{I}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)dt}{(t-a)^{1-\alpha}}. \tag{23} $$

Adding (22) and (23) sides by sides and using the definition (8), we obtain the desired equality (19).

A similar argument as in the proof of (19) will establish the equality (20). We omit the details. □

We present Hermite-Hadamard inequalities involving the fractional integral operators (17) and (18), asserted by the following theorem.

Theorem 2.2. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$. Also let $[a, b]$ $(a < b)$ be an interval on $\mathbb{R}$, $f : [a, b] \to \mathbb{C}$ is a symmetrized convex and integrable function. Then*

$$ f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta}{2(x-a)^{\alpha\beta}} \left[{}^{\beta}_{a}\mathfrak{I}^{\alpha} f(x) + {}^{\beta}\mathfrak{I}^{\alpha}_{b} f(a+b-x) \right] \le \frac{f(a)+f(b)}{2} \tag{24} $$

$(a < x \le b)$;

$$ f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta}{2(b-x)^{\alpha\beta}} \left[{}^{\beta}_{a}\mathfrak{I}^{\alpha} f(a+b-x) + {}^{\beta}\mathfrak{I}^{\alpha}_{b} f(x) \right] \le \frac{f(a)+f(b)}{2} \tag{25} $$

$(a \le x < b)$.

Proof. Since f is symmetrized convex on $[a, b]$, in view of Theorem C, we have

$$ f\left(\frac{a+b}{2}\right) \le \check{f}(t) \le \frac{f(a)+f(b)}{2} \quad (t \in [a, b]). \tag{26} $$

Multiplying both sides of (26) by $\frac{1}{\Gamma(\beta)} \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{1}{(t-a)^{1-\alpha}}$ and integrating each term of the resulting inequalities with respect to t from a to x $(a < x \le b)$, we get

$$ \begin{aligned} & f\left(\frac{a+b}{2}\right) \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{1}{(t-a)^{1-\alpha}} dt \\ & \le \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{\check{f}(t)dt}{(t-a)^{1-\alpha}} \\ & \le \frac{f(a)+f(b)}{2} \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{1}{(t-a)^{1-\alpha}} dt. \end{aligned} \tag{27} $$

Using (19) for the second integral in (27) and, for the first and third integrals in (27), considering the following easily-derivable integral

$$ \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{1}{(t-a)^{1-\alpha}} dt = \frac{(x-a)^{\alpha\beta}}{\beta\alpha^\beta}, \tag{28} $$

we obtain the desired inequality (24).

Similarly as in the proof of (24), we can prove the inequality (25). We omit the details. □

As in Lemma 2.1, we present some useful equalities associated (17) and (18), asserted in the following lemma.

Lemma 2.3. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$. Also let $[a,b]$ $(a < b)$ be an interval on $\mathbb{R}$, $f : [a,b] \to \mathbb{C}$ be an integrable function. Then*

$$\frac{1}{2}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{x}f(a) + {}^{\beta}_{a+b-x}\mathfrak{I}^{\alpha}f(b)\right] = \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(x-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\check{f}(t)dt}{(x-t)^{1-\alpha}} \tag{29}$$

and

$$\frac{1}{2}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{a+b-x}f(a) + {}^{\beta}_{x}\mathfrak{I}^{\alpha}f(b)\right] = \frac{1}{\Gamma(\beta)}\int_x^b \left(\frac{(b-x)^{\alpha}-(t-x)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\check{f}(t)dt}{(t-x)^{1-\alpha}}. \tag{30}$$

Proof. Using (17), we have

$${}^{\beta}_{a+b-x}\mathfrak{I}^{\alpha}f(b) = \frac{1}{\Gamma(\beta)}\int_{a+b-x}^b \left(\frac{(x-a)^{\alpha}-(t-(a+b-x))^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)dt}{(t-(a+b-x))^{1-\alpha}}. \tag{31}$$

Setting $u = a+b-t$ in (31), we obtain

$${}^{\beta}_{a+b-x}\mathfrak{I}^{\alpha}f(b) = \int_a^x \left(\frac{(x-a)^{\alpha}-(x-u)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(a+b-u)du}{(x-u)^{1-\alpha}}.$$

We use (18) to have

$${}^{\beta}\mathfrak{I}^{\alpha}_{x}f(a) = \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(x-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)dt}{(x-t)^{1-\alpha}}. \tag{32}$$

Finally, adding (31) and (32) sides by sides, in view of the definition (8), we obtain the desired equality (29).
The proof of the equality (30) would run parallel to that of (29). We omit the details. □

We present Hermite-Hadamard inequalities involving the fractional integral operators (17) and (18), asserted by the following theorem.

Theorem 2.4. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$. Also let $[a,b]$ $(a < b)$ be an interval on $\mathbb{R}$, $f : [a,b] \to \mathbb{C}$ is a symmetrized convex and integrable function. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(x-a)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{x}f(a) + {}^{\beta}_{a+b-x}\mathfrak{I}^{\alpha}f(b)\right] \le \frac{f(a)+f(b)}{2} \tag{33}$$

$(a < x \le b)$;

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-x)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{a+b-x}f(a) + {}^{\beta}_{x}\mathfrak{I}^{\alpha}f(b)\right] \le \frac{f(a)+f(b)}{2} \tag{34}$$

$(a \le x < b)$.

Proof. A similar argument as in the proof of Theorem 2.2, here, using the equalities in Lemma 2.3, will establish the results here. We omit the details. □

Theorem 2.5. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$, $[a,b]$ $(a < b)$ be an interval on $\mathbb{R}$, $f : [a,b] \to \mathbb{C}$ be an integrable function. Also let $f : [a,b] \to \mathbb{R}$ be Wright-quasi-convex and integrable on $[a,b]$. Then*

$$\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(x-a)^{\alpha\beta}}\left[{}^{\beta}_{a}\mathfrak{I}^{\alpha}f(x) + {}^{\beta}\mathfrak{I}^{\alpha}_{b}f(a+b-x)\right] \le \max\{f(a), f(b)\} \tag{35}$$

$(a < x \le b)$;

$$\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\mathfrak{I}^{\alpha}f(b)+{}^{\beta}\mathfrak{I}_{b}^{\alpha}f(a)\right]\leq\max\{f(a),f(b)\};\tag{36}$$

$$\frac{\Gamma(\beta+1)\alpha^{\beta}2^{\alpha\beta-1}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\mathfrak{I}^{\alpha}f(\frac{a+b}{2})+{}^{\beta}\mathfrak{I}_{b}^{\alpha}f(\frac{a+b}{2})\right]\leq\max\{f(a),f(b)\}.\tag{37}$$

Proof. Since $f:[a,b]\to\mathbb{R}$ is Wright-quasi-convex on $[a,b]$, setting $x=a$, $y=b$ and $t=\frac{s-a}{b-a}\in[0,1]$ for $s\in[a,b]$ in (3), we have

$$\check{f}(s)=\frac{1}{2}\Big[f(a+b-s)+f(s)\Big]\leq\max\{f(a),f(b)\}.\tag{38}$$

Multiplying both sides of (38) by $\frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^{\alpha}-(s-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(s-a)^{1-\alpha}}$ and integrating each term of the resulting inequalities with respect to s from a to x ($a<x\leq b$), we obtain

$$\frac{1}{\Gamma(\beta)}\int_{a}^{x}\left(\frac{(x-a)^{\alpha}-(s-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\check{f}(s)ds}{(s-a)^{1-\alpha}}\leq\max\{f(a),f(b)\}\frac{1}{\Gamma(\beta)}\int_{a}^{x}\left(\frac{(x-a)^{\alpha}-(s-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{ds}{(s-a)^{1-\alpha}}.\tag{39}$$

Applying (19) to the left side of (39), we get

$$\frac{1}{2}\left[{}_{a}^{\beta}\mathfrak{I}^{\alpha}f(x)+{}^{\beta}\mathfrak{I}_{b}^{\alpha}f(a+b-x)\right]\leq\max\{f(a),f(b)\}\frac{(x-a)^{\alpha\beta}}{\Gamma(\beta+1)\alpha^{\beta}},$$

leads to the desired inequality (35).

Setting $x=b$ and $x=\frac{a+b}{2}$ in (35) yields, respectively, the inequalities in (36) and (37). □

Theorem 2.6. *Let $\beta\in\mathbb{C}$, $Re(\beta)>0$, $[a,b]$ $(a<b)$ be an interval on $\mathbb{R}$, $f:[a,b]\to\mathbb{C}$ be an integrable function. Also let $f:[a,b]\to\mathbb{R}$ be Wright-quasi-convex and integrable on $[a,b]$. Then*

$$\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(x-a)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}_{x}^{\alpha}f(a)+{}_{a+b-x}^{\beta}\mathfrak{I}^{\alpha}f(b)\right]\leq\max\{f(a),f(b)\}\tag{40}$$

$(a<x\leq b)$;

$$\frac{\Gamma(\beta+1)\alpha^{\beta}2^{\alpha\beta-1}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}_{\frac{a+b}{2}}^{\alpha}f(a)+{}_{\frac{a+b}{2}}^{\beta}\mathfrak{I}^{\alpha}f(b)\right]\leq\max\{f(a),f(b)\}.\tag{41}$$

Proof. A similar argument as in the proof of Theorem 2.5 will establish the inequality (40). The inequality (41) is just a special case of the inequality (40) when $x=\frac{a+b}{2}$. We omit the details. □

Theorem 2.7. *Assume that the function $f:[a,b]\to[0,\infty)$ is h-symmetrized convex on the interval $[a,b]$ with h is integrable on $[0,1]$ and f is integrable on $[a,b]$. Then we have*

$$\begin{aligned}\frac{f\left(\frac{a+b}{2}\right)}{2h\left(\frac{1}{2}\right)\beta} &\leq \frac{\Gamma(\beta)\alpha^{\beta}\left[{}_{a}^{\beta}\mathfrak{I}^{\alpha}f(x)+{}^{\beta}\mathfrak{I}_{b}^{\alpha}f(a+b-x)\right]}{2(x-a)^{\alpha\beta}}\\ &\leq \alpha\frac{f(a)+f(b)}{2}\int_{0}^{1}(1-s^{\alpha})^{\beta-1}s^{1-\alpha}\left[h\left(1-\frac{x-a}{b-a}s\right)+h\left(\frac{x-a}{b-a}s\right)\right]ds.\end{aligned}\tag{42}$$

Proof. Since h-symmetrized convex function, we have;

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\leq\check{f}(t)\leq\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2}.$$

To prove the first inequality, multiplying each terms with $\frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}$ and integrating on $[a,x]$ with respect to t, we get

$$\frac{1}{2h(\frac{1}{2})\Gamma(\beta)}f\left(\frac{a+b}{2}\right)\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}dt\leq\frac{\left[{}_a^\beta\mathfrak{I}^\alpha f(x)+{}^\beta\mathfrak{I}_b^\alpha f(a+b-x)\right]}{2}.$$

After simple calculation, we get

$$\frac{f\left(\frac{a+b}{2}\right)}{2h\left(\frac{1}{2}\right)\beta}\leq\frac{\Gamma(\beta)\alpha^\beta\left[{}_a^\beta\mathfrak{I}^\alpha f(x)+{}^\beta\mathfrak{I}_b^\alpha f(a+b-x)\right]}{2(x-a)^{\alpha\beta}}\tag{43}$$

thus the first inequality is proved.

To prove the second inequality, multiplying each terms with $\frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}$ and integrating on $[a,x]$ with respect to t, we get

$$\frac{\left[{}_a^\beta\mathfrak{I}^\alpha f(x)+{}^\beta\mathfrak{I}_b^\alpha f(a+b-x)\right]}{2}\leq\frac{f(a)+f(b)}{2\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]dt$$

for any $a<x\leq b$.

If we change the variable with $t=(1-s)a+sx$ for $s\in[0,1]$, *i.e.* $dt=(x-a)ds$, $\frac{b-t}{b-a}=1-\frac{x-a}{b-a}s$, $\frac{t-a}{b-a}=\frac{x-a}{b-a}s$ and $x-t=(1-s)(x-a)$, then we have

$$\begin{aligned}&\frac{\left[{}_a^\beta\mathfrak{I}^\alpha f(x)+{}^\beta\mathfrak{I}_b^\alpha f(a+b-x)\right]}{2}\\ \leq\ &\left(\frac{f(a)+f(b)}{2}\right)\int_0^1\left(\frac{(x-a)^\alpha-(s(x-a))^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{\left[s(x-a)\right]^{1-\alpha}}\left[h\left(1-\frac{x-a}{b-a}s\right)+h\left(\frac{x-a}{b-a}s\right)\right](x-a)ds.\end{aligned}$$

After simple calculation, we get

$$\frac{\Gamma(\beta)\alpha^\beta\left[{}_a^\beta\mathfrak{I}^\alpha f(x)+{}^\beta\mathfrak{I}_b^\alpha f(a+b-x)\right]}{2(x-a)^{\alpha\beta}}\leq\alpha\frac{f(a)+f(b)}{2}\int_0^1(1-s^\alpha)^{\beta-1}s^{1-\alpha}\left[h\left(1-\frac{x-a}{b-a}s\right)+h\left(\frac{x-a}{b-a}s\right)\right]ds.\tag{44}$$

The proof is completed. □

Remark 2.8. The case $a=0$ and $\alpha=1$ in Lemma 2.1, Theorem 2.2, Lemma 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.6 and Theorem 2.7 reduces to the known results, respectively, in [3, Lemma 1],[3, Theorem 2],[3, Lemma 2],[3, Theorem 3],[3, Theorem 4],[3, Theorem 5] and [3, (4.10) - (4.13)].

3. Hermite-Hadamard-Fejér type inequalities

In this section, we investigate certain Hermite-Hadamard-Fejér type inequalities involving the fractional integral operators (17) and (18). We begin by presenting some useful equalities associated (17) and (18), asserted in the following lemma.

Lemma 3.1. *Let $\beta\in\mathbb{C}$, $Re(\beta)>0$. Also let $[a,b]$ $(a<b)$ be an interval on $\mathbb{R}$, $f:[a,b]\to\mathbb{C}$ be an integrable function and $g:[a,b]\to\mathbb{R}$ be integrable and symmetric to $(a+b)/2$. Then*

$$\frac{1}{2}\left[({}_a^\beta\mathfrak{I}^\alpha fg)(x)+({}^\beta\mathfrak{I}_b^\alpha fg)(a+b-x)\right]=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{\check{f}(t)g(t)dt}{(t-a)^{1-\alpha}}\tag{45}$$

and

$$\frac{1}{2}\Big[({}^{\beta}_{a}\mathfrak{J}^{\alpha} fg)(a+b-x) + ({}^{\beta}\mathfrak{J}^{\alpha}_{b} fg)(x)\Big] = \frac{1}{\Gamma(\beta)}\int_{x}^{b}\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\check{f}(t)g(t)dt}{(b-t)^{1-\alpha}}. \tag{46}$$

Proof. Firstly, we will prove (45). We find from (18) that, for $a < x \le b$,

$$({}^{\beta}\mathfrak{J}^{\alpha}_{b} fg)(a+b-x) = \frac{1}{\Gamma(\beta)}\int_{a+b-x}^{b}\left(\frac{(x-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)g(t)dt}{(b-t)^{1-\alpha}}. \tag{47}$$

Setting $t = a + b - u$ to change the variable in the right side of (21), we obtain

$$({}^{\beta}\mathfrak{J}^{\alpha}_{b} fg)(a+b-x) = \frac{1}{\Gamma(\beta)}\int_{a}^{x}\left(\frac{(x-a)^{\alpha}-(t-u)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(a+b-u)g(a+b-u)du}{(u-a)^{1-\alpha}}. \tag{48}$$

Since $g(t)$ is symmetric with respect to the axis $t = (a + b)/2$ on $[a, b]$, from (22), we get

$$({}^{\beta}\mathfrak{J}^{\alpha}_{b} fg)(a+b-x) = \frac{1}{\Gamma(\beta)}\int_{a}^{x}\left(\frac{(x-a)^{\alpha}-(t-u)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(a+b-u)g(u)du}{(u-a)^{1-\alpha}}. \tag{49}$$

From (17), we also have

$$({}^{\beta}_{a}\mathfrak{J}^{\alpha} fg)(x) = \frac{1}{\Gamma(\beta)}\int_{a}^{x}\left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)g(t)dt}{(t-a)^{1-\alpha}}. \tag{50}$$

Adding (49) and (50) sides by sides and using the definition (8), we obtain the desired equality (45).

A similar argument as in the proof of (45) will establish the equality (46). We omit the details. □

We present Hermite-Hadamard inequalities involving the fractional integral operators (17) and (18), asserted by the following theorem.

Theorem 3.2. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$. Also let $[a, b]$ $(a < b)$ be an interval on $\mathbb{R}$, $f : [a, b] \to \mathbb{C}$ is a symmetrized convex and integrable function and $g : [a, b] \to \mathbb{R}$ be integrable and symmetric to $(a + b)/2$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^{\beta}\Big[{}^{\beta}_{a}\mathfrak{J}^{\alpha} f(x) + {}^{\beta}\mathfrak{J}^{\alpha}_{b} f(a+b-x)\Big]}{2(x-a)^{\alpha\beta}\|g\|_{\min}} \tag{51}$$

$(a < x \le b)$;

$$\frac{\Gamma(\beta+1)\alpha^{\beta}\Big[{}^{\beta}_{a}\mathfrak{J}^{\alpha} f(x) + {}^{\beta}\mathfrak{J}^{\alpha}_{b} f(a+b-x)\Big]}{2(x-a)^{\alpha\beta}\|g\|_{\infty}} \le \frac{f(a)+f(b)}{2} \tag{52}$$

$(a < x \le b)$.

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^{\beta}\Big[{}^{\beta}_{a}\mathfrak{J}^{\alpha} f(a+b-x) + {}^{\beta}\mathfrak{J}^{\alpha}_{b} f(x)\Big]}{2(b-x)^{\alpha\beta}\|g\|_{\min}} \tag{53}$$

$(a \le x < b)$;

$$\frac{\Gamma(\beta+1)\alpha^{\beta}\Big[{}^{\beta}_{a}\mathfrak{J}^{\alpha} f(a+b-x) + {}^{\beta}\mathfrak{J}^{\alpha}_{b} f(x)\Big]}{2(b-x)^{\alpha\beta}\|g\|_{\infty}} \le \frac{f(a)+f(b)}{2} \tag{54}$$

$(a \le x < b)$.

Proof. Since f is symmetrized convex on $[a,b]$, in view of Theorem C, we have

$$f\left(\frac{a+b}{2}\right) \le \check{f}(t) \le \frac{f(a)+f(b)}{2} \quad (t \in [a,b]). \tag{55}$$

Multiplying both sides of (26) by $\frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}g(t)$ and integrating each term of the resulting inequalities with respect to t from a to x ($a < x \le b$), we get

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}g(t)dt\\
&\le \frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{\check{f}g(t)(t)dt}{(t-a)^{1-\alpha}}\\
&\le \frac{f(a)+f(b)}{2}\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}g(t)dt.
\end{aligned} \tag{56}$$

Using (45) for the second integral in (56) and, for the first and third integrals in (56), considering the following easily-derivable integral

$$\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}dt = \frac{(x-a)^{\alpha\beta}}{\beta\alpha^\beta}, \tag{57}$$

we obtain the desired inequality (51) and (52)

Similarly as in the proof of (51) and (52), we can prove the inequality (53) and (54). We omit the details. □

As in Lemma 3.1, we present some useful equalities associated (17) and (18), asserted in the following lemma.

Lemma 3.3. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$. Also let $[a,b]$ ($a < b$) be an interval on $\mathbb{R}$, $f : [a,b] \to \mathbb{C}$ be an integrable function and $g : [a,b] \to \mathbb{R}$ be integrable and symmetric to $(a+b)/2$. Then*

$$\frac{1}{2}\Big[({}^\beta\mathfrak{J}_x^\alpha fg)(a) + ({}_{a+b-x}^{\beta}\mathfrak{J}^\alpha fg)(b)\Big] = \frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(x-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{\check{f}(t)g(t)dt}{(x-t)^{1-\alpha}} \tag{58}$$

and

$$\frac{1}{2}\Big[({}^\beta\mathfrak{J}_{a+b-x}^\alpha fg)(a) + ({}_x^\beta\mathfrak{J}^\alpha fg)(b)\Big] = \frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^\alpha-(t-x)^\alpha}{\alpha}\right)^{\beta-1}\frac{\check{f}(t)g(t)dt}{(t-x)^{1-\alpha}}. \tag{59}$$

Proof. Using (17), we have

$$({}_{a+b-x}^{\beta}\mathfrak{J}^\alpha fg)(b) = \frac{1}{\Gamma(\beta)}\int_{a+b-x}^b\left(\frac{(x-a)^\alpha-(t-(a+b-x))^\alpha}{\alpha}\right)^{\beta-1}\frac{f(t)g(t)dt}{(t-(a+b-x))^{1-\alpha}}. \tag{60}$$

Setting $u = a+b-t$ in (60), we obtain

$$({}_{a+b-x}^{\beta}\mathfrak{J}^\alpha fg)(b) = \int_a^x\left(\frac{(x-a)^\alpha-(x-u)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(a+b-u)g(a+b-u)du}{(x-u)^{1-\alpha}}.$$

Since $g(t)$ is symmetric with respect to the axis $t = (a+b)/2$ on $[a,b]$, we get

$$({}_{a+b-x}^{\beta}\mathfrak{J}^\alpha fg)(b) = \int_a^x\left(\frac{(x-a)^\alpha-(x-u)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(a+b-u)g(u)du}{(x-u)^{1-\alpha}}. \tag{61}$$

We use (18) to have

$$({}^{\beta}\mathfrak{I}_{x}^{\alpha} fg)(a) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (x-t)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)g(t)dt}{(x-t)^{1-\alpha}}. \tag{62}$$

Finally, adding (61) and (62) sides by sides, in view of the definition (8), we obtain the desired equality (58).
The proof of the equality (59) would run parallel to that of (58). We omit the details. □

We present Hermite-Hadamard inequalities involving the fractional integral operators (17) and (18), asserted by the following theorem.

Theorem 3.4. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$. Also let $[a, b]$ $(a < b)$ be an interval on $\mathbb{R}$, $f : [a, b] \to \mathbb{C}$ is a symmetrized convex and integrable function and $g : [a, b] \to \mathbb{R}$ be integrable and symmetric to $(a + b)/2$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta \left[{}^{\beta}\mathfrak{I}_{x}^{\alpha} f(a) + {}^{\beta}_{a+b-x}\mathfrak{I}^{\alpha} f(b) \right]}{2(x-a)^{\alpha\beta} \|g\|_{\min}} \tag{63}$$

$(a < x \le b)$;

$$\frac{\Gamma(\beta+1)\alpha^\beta \left[{}^{\beta}\mathfrak{I}_{x}^{\alpha} f(a) + {}^{\beta}_{a+b-x}\mathfrak{I}^{\alpha} f(b) \right]}{2(x-a)^{\alpha\beta} \|g\|_{\infty}} \le \frac{f(a)+f(b)}{2} \tag{64}$$

$(a < x \le b)$.

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta \left[{}^{\beta}\mathfrak{I}_{a+b-x}^{\alpha} f(a) + {}^{\beta}_{x}\mathfrak{I}^{\alpha} f(b) \right]}{2(b-x)^{\alpha\beta} \|g\|_{\min}} \tag{65}$$

$(a \le x < b)$;

$$\frac{\Gamma(\beta+1)\alpha^\beta \left[{}^{\beta}\mathfrak{I}_{a+b-x}^{\alpha} f(a) + {}^{\beta}_{x}\mathfrak{I}^{\alpha} f(b) \right]}{2(b-x)^{\alpha\beta} \|g\|_{\infty}} \le \frac{f(a)+f(b)}{2} \tag{66}$$

$(a \le x < b)$.

Proof. A similar argument as in the proof of Theorem 3.2, here, using the equalities in Lemma 3.3, will establish the results here. We omit the details. □

Theorem 3.5. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$, $[a, b]$ $(a < b)$ be an interval on $\mathbb{R}$, $f : [a, b] \to \mathbb{C}$ be an integrable function. Also let $f : [a, b] \to \mathbb{R}$ be Wright-quasi-convex and integrable on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ be integrable and symmetric to $(a + b)/2$. Then*

$$\frac{\Gamma(\beta+1)\alpha^\beta \left[({}^{\beta}_{a}\mathfrak{I}^{\alpha} fg)(x) + ({}^{\beta}\mathfrak{I}_{b}^{\alpha} fg)(a+b-x) \right]}{2(x-a)^{\alpha\beta} \|g\|_{\infty}} \le \max\{f(a), f(b)\} \tag{67}$$

$(a < x \le b)$;

$$\frac{\Gamma(\beta+1)\alpha^\beta \left[({}^{\beta}_{a}\mathfrak{I}^{\alpha} fg)(b) + ({}^{\beta}\mathfrak{I}_{b}^{\alpha} fg)(a) \right]}{2(b-a)^{\alpha\beta} \|g\|_{\infty}} \le \max\{f(a), f(b)\}; \tag{68}$$

$$\frac{\Gamma(\beta+1)\alpha^\beta 2^{\alpha\beta-1} \left[({}^{\beta}_{a}\mathfrak{I}^{\alpha} fg)(\frac{a+b}{2}) + ({}^{\beta}\mathfrak{I}_{b}^{\alpha} fg)(\frac{a+b}{2}) \right]}{(b-a)^{\alpha\beta} \|g\|_{\infty}} \le \max\{f(a), f(b)\}. \tag{69}$$

Proof. Since $f : [a,b] \to \mathbb{R}$ is Wright-quasi-convex on $[a,b]$, setting $x = a$, $y = b$ and $t = \frac{s-a}{b-a} \in [0,1]$ for $s \in [a,b]$ in (3), we have

$$\check{f}(s) = \frac{1}{2}\Big[f(a+b-s) + f(s)\Big] \le \max\{f(a), f(b)\}. \tag{70}$$

Multiplying both sides of (38) by $\frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^\alpha-(s-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{g(s)}{(s-a)^{1-\alpha}}$ and integrating each term of the resulting inequalities with respect to s from a to x ($a < x \le b$), we obtain

$$\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(s-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{\check{f}(s)g(s)ds}{(s-a)^{1-\alpha}} \le \max\{f(a), f(b)\}\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(s-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{g(s)ds}{(s-a)^{1-\alpha}}. \tag{71}$$

Applying (45) to the left side of (71), we get

$$\frac{1}{2}\Big[({}^\beta_a\mathfrak{J}^\alpha fg)(x) + ({}^\beta\mathfrak{J}^\alpha_b fg)(a+b-x)\Big] \le \max\{f(a), f(b)\}\frac{(x-a)^{\alpha\beta}||g||_\infty}{\Gamma(\beta+1)\alpha^\beta},$$

leads to the desired inequality (67).

Setting $x = b$ and $x = \frac{a+b}{2}$ in (67) yields, respectively, the inequalities in (68) and (69). □

Theorem 3.6. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$, $[a,b]$ ($a < b$) be an interval on $\mathbb{R}$, $f : [a,b] \to \mathbb{C}$ be an integrable function. Also let $f : [a,b] \to \mathbb{R}$ be Wright-quasi-convex and integrable on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ be integrable and symmetric to $(a+b)/2$. Then*

$$\frac{\Gamma(\beta+1)\alpha^\beta\Big[({}^\beta\mathfrak{J}^\alpha_x fg)(a) + ({}^\beta_{a+b-x}\mathfrak{J}^\alpha fg)(b)\Big]}{2(x-a)^{\alpha\beta}||g||_\infty} \le \max\{f(a), f(b)\} \tag{72}$$

($a < x \le b$);

$$\frac{\Gamma(\beta+1)\alpha^\beta 2^{\alpha\beta-1}\Big[{}^\beta\mathfrak{J}^\alpha_{\frac{a+b}{2}} f(a) + {}^\beta_{\frac{a+b}{2}}\mathfrak{J}^\alpha f(b)\Big]}{(b-a)^{\alpha\beta}||g||_\infty} \le \max\{f(a), f(b)\}. \tag{73}$$

Proof. A similar argument as in the proof of Theorem 3.5 will establish the inequality (72). The inequality (73) is just a special case of the inequality (72) when $x = \frac{a+b}{2}$. We omit the details. □

Theorem 3.7. *Assume that the function $f : [a,b] \to [0,\infty)$ is h-symmetrized convex on the interval $[a,b]$ with h is integrable on $[0,1]$ and f is integrable on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ be integrable and symmetric to $(a+b)/2$. Then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{h\left(\frac{1}{2}\right)\Gamma(\beta+1)\alpha^\beta\Big[({}^\beta_a\mathfrak{J}^\alpha fg)(x) + ({}^\beta\mathfrak{J}^\alpha_b fg)(a+b-x)\Big]}{(x-a)^{\alpha\beta}||g||_{\min}}, \tag{74}$$

and

$$\begin{aligned}&\frac{\Gamma(\beta)\alpha^{\beta-1}\Big[{}^\beta_a\mathfrak{J}^\alpha f(x) + {}^\beta\mathfrak{J}^\alpha_b f(a+b-x)\Big]}{2(x-a)^{\alpha\beta}||g||_\infty}\\ \le\ &\frac{f(a)+f(b)}{2}\int_0^1(1-s^\alpha)^{\beta-1}s^{1-\alpha}\Big[h\Big(1-\frac{x-a}{b-a}s\Big) + h\Big(\frac{x-a}{b-a}s\Big)\Big]ds.\end{aligned} \tag{75}$$

Proof. Since h-symmetrized convex function, we have;

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \breve{f}(t) \leq \left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2}.$$

To prove (74), multiplying each terms with $\frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{g(t)}{(t-a)^{1-\alpha}}$ and integrating on $[a,x]$ with respect to t we get

$$\frac{1}{2h(\frac{1}{2})\Gamma(\beta)}f\left(\frac{a+b}{2}\right)\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{g(t)}{(t-a)^{1-\alpha}}dt \leq \frac{\left[{}_a^\beta\mathfrak{J}^\alpha f(x)+{}^\beta\mathfrak{J}_b^\alpha f(a+b-x)\right]}{2}.$$

After simple calculating we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{h\left(\frac{1}{2}\right)\Gamma(\beta+1)\alpha^\beta\left[({}_a^\beta\mathfrak{J}^\alpha fg)(x)+({}^\beta\mathfrak{J}_b^\alpha fg)(a+b-x)\right]}{(x-a)^{\alpha\beta}\|g\|_{\min}}. \tag{76}$$

Thus (74) is proved. To prove (75), multiplying each terms with $\frac{1}{\Gamma(\beta)}\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{g(t)}{(t-a)^{1-\alpha}}$ and integrating on $[a,x]$ with respect to t, we get

$$\frac{\left[{}_a^\beta\mathfrak{J}^\alpha f(x)+{}^\beta\mathfrak{J}_b^\alpha f(a+b-x)\right]}{2} \leq \frac{f(a)+f(b)}{2\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{g(t)}{(t-a)^{1-\alpha}}\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]dt.$$

for any $a < x \leq b$.

If we change the variable with $t = (1-s)a + sx$ for $s \in [0,1]$, *i.e.* $dt = (x-a)ds$, $\frac{b-t}{b-a} = 1 - \frac{x-a}{b-a}s$, $\frac{t-a}{b-a} = \frac{x-a}{b-a}s$ and $x - t = (1-s)(x-a)$, then we have

$$\begin{aligned}&\frac{\left[{}_a^\beta\mathfrak{J}^\alpha f(x)+{}^\beta\mathfrak{J}_b^\alpha f(a+b-x)\right]}{2}\\ \leq\ &\left(\frac{f(a)+f(b)}{2}\right)\int_0^1\left(\frac{(x-a)^\alpha-(s(x-a))^\alpha}{\alpha}\right)^{\beta-1}\frac{g\left((1-s)a+sx\right)}{\left[s(x-a)\right]^{1-\alpha}}\left[h\left(1-\frac{x-a}{b-a}s\right)+h\left(\frac{x-a}{b-a}s\right)\right](x-a)ds.\end{aligned}$$

After simple calculation, we get

$$\frac{\Gamma(\beta)\alpha^{\beta-1}\left[{}_a^\beta\mathfrak{J}^\alpha f(x)+{}^\beta\mathfrak{J}_b^\alpha f(a+b-x)\right]}{2(x-a)^{\alpha\beta}\|g\|_\infty} \leq \frac{f(a)+f(b)}{2}\int_0^1(1-s^\alpha)^{\beta-1}s^{1-\alpha}\left[h\left(1-\frac{x-a}{b-a}s\right)+h\left(\frac{x-a}{b-a}s\right)\right]ds. \tag{77}$$

The proof is completed. □

Remark 3.8. The case $g(t) = 1$ in Lemma 3.1, Theorem 3.2, Lemma 3.3, Theorem 3.4, Theorem 3.5, Theorem 3.6 and Theorem 3.7 reduces to the known results, respectively, Lemma 2.1, Theorem 2.2, Lemma 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.6 and Theorem 2.7.

Remark 3.9. The case $a = 0$ and $\alpha = 1$ in Lemma 3.1, Theorem 3.2, Lemma 3.3, Theorem 3.4, Theorem 3.5, Theorem 3.6 and Theorem 3.7 reduces to the known results, respectively, in [11, Corollary 1],[11, Corollary 2],[11, Corollary 3],[11, Corollary 4],[11, Corollary 5],[11, Corollary 6] and [11, Corollary 7].

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