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# New Hybrid Algorithms for Global Minimization of Common Best Proximity Points of Some Generalized Nonexpansive Mappings

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**Abstract.** In this paper, we introduce two hybrid algorithms for finding a common best proximity point of two best proximally nonexpansive mappings. We establish strong convergence theorems of the proposed algorithms under some control conditions in a real Hilbert space. Moreover, some numerical examples are given for supporting our main theorems.

## 1. Introduction

Best proximity point problems can be applied to study the existence of various nonlinear equations in science, applied science and including equilibrium point problems in economics. Many interesting results, by several authors, concerning best proximity point problems can be found in the following works [2–13], for examples.

Let $(X, \|\cdot\|)$ be a normed linear space and $U$ a nonempty subset of $X$. An operator $T : U \to X$ is said to have a fixed point in $U$ if the fixed point equation $Tx = x$ has at least one solution, that is, there exists a point $x \in U$ such that $\|x - Tx\| = 0$. Throughout this article, we consider in the case that the fixed point equation does not have a solution, i.e., $\|x - Tx\| > 0$ for all $x \in U$. Our aim is to find an element $x \in U$ such that the error $\|x - Tx\|$ is minimum. The point $x$ is said to be a *best approximation* of $T$. This is the idea behind the best approximation theory.

One of the well-known best approximation theorems was proved by Fan [1] in 1961 as the following theorem:

**Theorem 1.1.** *([1]) Let $U$ be a nonempty, compact and convex set in a normed linear space $X$. If $T$ is a continuous mapping from $U$ into $X$, then there exists a point $x$ in $U$ such that* $\|x - Tx\| = d(Tx, U) := \inf\{\|Tx - u\| : u \in U\}$.

An element $x$ in the previous theorem is called a *best approximation point* of $T$ in $U$. Now, we consider the optimization problem,

$$\text{minimize } f(x) \text{ subject to } x \in U,$$

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where a real valued function $f : U \to \mathbb{R}$ is an objective function. A point $\overline{x} \in U$ such that $f(x) \geq f(\overline{x})$ for all $x \in U$ is called a *global minimizer* of $f$ over $U$ and we write

$$f(\overline{x}) = \min_{x \in U} f(x).$$

In this regard, the optimization problem is solved only when its global minimizer exists. Best proximity point theorems have been explored to find sufficient conditions so that, the minimization problem $\min_{x \in U} \|x - Tx\|$ has at least one solution.

To have a concrete lower bound, let $U$ and $V$ be two nonempty subsets of a normed linear space $X$ and $T : U \to V$ a mapping. The natural problem is whether we can find an element $x_0 \in U$ such that $\|x_0 - Tx_0\| = \min_{x \in U} \|x - Tx\|$. We denote that $d(U, V) := \inf\{\|x - y\| : x \in U, y \in V\}$. Since $\|x - Tx\| \geq d(U, V)$ for any $x \in U$, the interesting problem is to find a point $x \in U$ such that

$$\|x - Tx\| = d(U, V).$$

It is called a *best proximity point of* $T$. In particular case, if $d(U, V) = 0$, then the best proximity points of $T$ are exactly fixed points of $T$.

A interesting way to solve the problems of fixed point theory is to introduce and employ some iterative methods which now have received vast investigations. In 2003, the hybrid algorithm for nonexpansive mappings was firstly introduced by Nakajo and Takahashi [14]. They proved that the iterative sequence, generated by the CQ method, converges strongly to fixed points of such kind of mapping. Martinez-Yanes and Xu [22] used the ideas of Nakajo and Takahashi to prove some strong convergence theorems for nonexpansive mappings in Hilbert spaces. At a later time, Takahashi *et al.* [15] proved a strong convergence theorem by their hybrid method for a family of nonexpansive mappings which generalizes Nakajo and Takahashi theorems [14]. Jacob *et al.* [16], in 2017, introduced the hybrid algorithms for nonself nonexpansive mappings on real Hibert spaces and proved that the iterative sequence of these algorithms converges strongly to the proximity point. Recently, Suparatulatorn and Suantai [17] introduced a best proximally nonexpansive mapping which is more general than nonexpansive mappings. They presented a new hybrid algorithm for finding a global minimization of best proximity points for this type of mapping.

It is our purpose in this paper to introduce two hybrid algorithms and prove some results which assure the proposed algorithms converge strongly to common best proximity points of two nonself best proximity nonexpansive mappings under some certain conditions in real Hilbert spaces. Moreover, we compare the convergence behavior between our proposed algorithms with the previous work.

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $U$ be a nonempty, closed and convex subset of $H$. For each $x \in H$, there exists a unique point in $U$, say $P_U x$, such that

$$\|x - P_U x\| \leq \|x - y\| \text{ for all } y \in U.$$

The map $P_U$ is called the *metric projection* of $H$ onto $U$. We also know that $P_U$ is a nonexpansive mapping of $H$ onto $U$.

Let $U$ and $V$ be two nonempty, closed and convex subsets of a real Hilbert space $H$. We denote by $Fix(T)$ the set of fixed points of $T$ and $Best_U(T)$ the set of best proximity points of $T$, that is,

$$Best_U(T) := \{x \in U : \|x - Tx\| = d(U, V)\},$$

where $d(U, V) := \inf\{\|x - y\| : x \in U, y \in V\}$. Next, we will recall some notations for convenience. Let

$$U_0 := \{x \in U : \|x - y\| = d(U, V), \text{ for some } y \in V\},$$

$$V_0 := \{y \in V : \|x - y\| = d(U, V), \text{ for some } x \in U\}.$$

Actually, we can see that every element of $Best_U(T)$ is in the set $U_0$. We shall use the notations: $x_n \to x$ means that a sequence $\{x_n\}$ converges strongly to $x$ and $x_n \rightharpoonup x$ means that a sequence $\{x_n\}$ converges weakly to $x$. The following some important definitions and useful lemmas will be used in the sequel.

**Definition 2.1.** *([17]) Let $U$ and $V$ be two nonempty subsets of a real Hilbert space $H$ and $C$ a subset of $U$. A mapping $T : U \to V$ is said to be $C$-nonexpansive if*

$$\|Tx - Tz\| \leq \|x - z\| \text{ for all } x \in U, z \in C.$$

*If $C = Best_U(T) \neq \emptyset$, we say that $T$ is best proximally nonexpansive.*

It is obvious that if $T$ is nonself nonexpansive, then it is $C$-nonexpansive for every subset $C$ of $U$. Moreover, a $C$-nonexpansive mapping is a quasi-nonexpansive mapping where $C = Fix(T) \neq \emptyset$.

**Definition 2.2.** *([4, 18]) A pair $(U, V)$ of nonempty subsets of a normed linear space $X$ with $U_0 \neq \emptyset$ is said to have the $P$-property if and only if*

$$\left.\begin{array}{l}\|x_1 - y_1\| = d(U, V)\\ \|x_2 - y_2\| = d(U, V)\end{array}\right\} \Longrightarrow \|x_1 - x_2\| = \|y_1 - y_2\|,$$

*whenever $x_1, x_2 \in U_0$ and $y_1, y_2 \in V_0$.*

We know that every pair $(U, V)$ of nonempty, closed and convex subsets of a real Hilbert space $H$ has the $P$-property.

**Definition 2.3.** *([3]) Let $U$ and $V$ be nonempty subsets of a metric space $(X, d)$. Then $(U, V)$ is said to satisfy the property $UC$ if the following holds:*

*If $\{x_n\}$ and $\{x'_n\}$ are two sequences in $U$ and $\{y_n\}$ is a sequence in $V$ such that $\lim_{n\to\infty} d(x_n, y_n) = d(U, V)$ and $\lim_{n\to\infty} d(x'_n, y_n) = d(U, V)$, then $\lim_{n\to\infty} d(x_n, x'_n) = 0$ holds.*

For $U$ and $V$ are nonempty subsets of a uniformly convex Banach space, if $U$ is convex, then the pair $(U, V)$ has the property $UC$.

**Lemma 2.4.** *([17]) Let $U$ and $V$ be two nonempty, closed and convex subsets of a metric space $(X, d)$. Then $d(x, P_V x) = d(U, V)$ for all $x \in U_0$ and $d(y, P_U y) = d(U, V)$ for all $y \in V_0$.*

**Lemma 2.5.** *([17]) Let $U$ and $V$ be two nonempty subsets of a uniformly convex Banach space $X$ such that $U$ is closed and convex. Suppose that $T : U \to V$ is a mapping such that $T(U_0) \subset V_0$. Then*

$$Fix(P_U T|_{U_0}) = Fix(P_U T) \cap U_0 = Best_U(T).$$

**Lemma 2.6.** *([17]) Let $U$ and $V$ be two nonempty subsets of a uniformly convex Banach space $X$ such that $U$ is closed and convex. Suppose that $T : U \to V$ is a best proximally nonexpansive mapping such that $T(U_0) \subset V_0$. Then $P_U T|_{U_0}$ is a quasi-nonexpansive mapping.*

**Lemma 2.7.** *([19]) Let $U$ be a closed and convex subset of a real Hilbert space $H$. Let $T : U \to U$ be a quasi-nonexpansive mapping. Then $Fix(T)$ is a closed and convex subset.*

**Definition 2.8.** *Let $U$ be a nonempty subset of a Banach space $X$ and $T : U \to X$ a mapping. Then $T$ is said to be demiclosed at $y \in X$ if for any sequence $\{x_n\}$ in $U$ such that $x_n \rightharpoonup x \in U$ and $Tx_n \to y$ imply $Tx = y$.*

**Definition 2.9.** *([20]) Let $U$ and $V$ be nonempty subsets of a normed space $X$ and $T : U \to V$ a mapping. Then $T$ is said to satisfy the proximal property if for any sequence $\{x_n\}$ in $U$ such that $x_n \rightharpoonup x \in U$ and $\|x_n - Tx_n\| \to d(U, V)$ imply $\|x - Tx\| = d(U, V)$.*

It is obvious from above definition that if $d(U, V) = 0$, the proximal property reduces to the usual demiclosedness property of the map $I - T$ at 0.

**Lemma 2.10.** *([21]) Let $U$ and $V$ be two nonempty subsets of a uniformly convex Banach space $X$ such that $U$ is closed and convex. Suppose that $T : U \to V$ is mapping such that $T(U_0) \subset V_0$. Then $T|_{U_0}$ satisfies the proximal property if and only if $I - P_A T|_{U_0}$ is demiclosed at zero.*

In the sequel, we shall need the following facts and tools in a real Hilbert space.

**Lemma 2.11.** *In a real Hilbert space H, the identity following holds:*

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \text{ for any } x, y \in H.$$

**Lemma 2.12.** *([22]) Let H be a real Hilbert space. Given a closed and convex subset $U \subset H$ and points $w, x, y \in H$ and a real number $a \in \mathbb{R}$. The set*

$$D := \{u \in U : \|y-u\|^2 \le \|x-u\|^2 + \langle w, u\rangle + a\}$$

*is a closed convex subset of U.*

**Lemma 2.13.** *Let U be a closed and convex subset of a real Hilbert space H and let $P_U$ be the metric projection mapping from H onto U. Given $x \in H$ and $z \in U$. Then $z = P_U x$ if and only if there holds the relation:*

$$\langle x-z, y-z\rangle \le 0 \text{ for all } y \in U.$$

**Lemma 2.14.** *([22]) Let U be a closed and convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $x \in H$. Let $q = P_U x$. Suppose that $\omega_w(x_n) := \{u :$ there is a sequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup u\}$, the weak $\omega$-limit set of $\{x_n\}$, is a subset of U and satisfies the condition*

$$\|x_n - x\| \le \|x-q\| \text{ for all } n.$$

*Then $x_n \to q$.*

## 3. Main Results

In this section, we introduce new algorithms for finding a global minimization of common best proximity points of two nonself best proximally nonexpansive mappings and prove that the iterate sequence generated by the proposed algorithms converges strongly to a common best proximity point of those mappings.

Now, we start with our first main result.

**Theorem 3.1.** *Let U and V be two nonempty, closed and convex subsets of a real Hilbert space H. Let $S, T : U \to V$ be best proximally nonexpansive mappings such that $S(U_0) \subset V_0$ and $T(U_0) \subset V_0$. Suppose that S and T satisfy the proximal property and $\Omega := Best_U(S) \cap Best_U(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ such that $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$. For an initial guess $x_0 \in U_0$, define the sequence $\{x_n\}$ by*

$$\begin{cases} z_n = \beta_n P_V x_n + (1-\beta_n) T x_n, \\ y_n = \alpha_n z_n + (1-\alpha_n) S x_n, \\ C_n = \{u \in U_0 : \|y_n - u\| \le \|x_n - u\| + d(U,V) \text{ and } \|z_n - u\| \le \|x_n - u\| + d(U,V)\}, \\ Q_n = \{u \in U_0 : \langle x_n - u, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, n \ge 0. \end{cases} \tag{1}$$

*Then the sequence $\{x_n\}$ defined by (1) converges strongly to a point $q^* \in \Omega$, where $q^* = P_\Omega x_0$.*

*Proof.* Choose $x_0 \in U_0$ arbitrarily: First observe that $C_n$ and $Q_n$ are closed convex subsets of U by Lemma 2.12 (see more details Lemma 3.1[16]). Next, we claim that $\Omega \subset C_n$ for all n. Indeed, we have, for all $q \in \Omega$,

$$\begin{aligned} \|z_n - q\| &= \|\beta_n P_V x_n + (1-\beta_n) T x_n - q\| \\ &= \|\beta_n (P_V x_n - q) + (1-\beta_n)(T x_n - q)\| \\ &\le \beta_n \|P_V x_n - q\| + (1-\beta_n)\|T x_n - q\| \\ &\le \beta_n \|P_V x_n - Tq\| + \beta_n \|Tq - q\| + (1-\beta_n)\|T x_n - Tq\| + (1-\beta_n)\|Tq - q\| \\ &\le \beta_n \|P_V x_n - Tq\| + \beta_n d(U,V) + (1-\beta_n)\|x_n - q\| + (1-\beta_n) d(U,V) \\ &= \beta_n \|P_V x_n - Tq\| + (1-\beta_n)\|x_n - q\| + d(U,V). \end{aligned}$$

*Since $\|x_n - P_V x_n\| = d(U,V)$ and $\|q - Tq\| = d(U,V)$, using the P-property we obtain that $\|P_V x_n - Tq\| = \|x_n - q\|$. Therefore, the above inequality becomes*

$$\begin{aligned}\|z_n - q\| &\le \beta_n \|P_V x_n - Tq\| + (1-\beta_n)\|x_n - q\| + d(U,V)\\ &= \beta_n\|x_n - q\| + (1-\beta_n)\|x_n - q\| + d(U,V)\\ &= \|x_n - q\| + d(U,V). \end{aligned} \tag{2}$$

*Moreover, by (2) and the best proximally nonexpansivity of $S$, we have*

$$\begin{aligned}\|y_n - q\| &= \|\alpha_n z_n + (1-\alpha_n)Sx_n - q\|\\ &= \|\alpha_n(z_n - q) + (1-\alpha_n)(Sx_n - q)\|\\ &\le \alpha_n\|z_n - q\| + (1-\alpha_n)\|Sx_n - q\|\\ &\le \alpha_n\|z_n - q\| + (1-\alpha_n)\|Sx_n - Sq\| + (1-\alpha_n)\|Sq - q\|\\ &\le \alpha_n(\|x_n - q\| + d(U,V)) + (1-\alpha_n)\|x_n - q\| + (1-\alpha_n)d(U,V)\\ &= \|x_n - q\| + d(U,V). \end{aligned} \tag{3}$$

*Hence $q \in C_n$ and $\Omega \subset C_n$ for all $n$. Next, we show that*

$$\Omega \subset Q_n \ \text{ for all } \ n \ge 0. \tag{4}$$

*We prove this by induction. For $n = 0$, we have $Q_0 = U_0$. Assume that $\Omega \subset Q_n$. Since $x_{n+1}$ is the projection of $x_0$ onto $C_n \cap Q_n$ (such an element exist since $C_n$ and $Q_n$ are closed and convex), by Lemma 2.13, we have*

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \ge 0 \ \text{ for all } \ z \in C_n \cap Q_n.$$

*As $\Omega \subset C_n \cap Q_n$, by the induction assumption, the last inequality holds, in particular, for all $z \in \Omega$. This together with the definition of $Q_{n+1}$ imply $\Omega \subset Q_{n+1}$. Hence (4) holds true for all $n \ge 0$.*

*By the definition of $Q_n$ and Lemma 2.13, we get $x_n = P_{Q_n} x_0$. Since $\Omega \subset Q_n$, we have $\|x_n - x_0\| \le \|q - x_0\|$ for all $q \in \Omega$. It follows that $\{x_n\}$ is bounded and*

$$\|x_n - x_0\| \le \|q^* - x_0\|, \ \text{where } q^* = P_\Omega x_0. \tag{5}$$

*From $\|P_V x_n - Tx_n\| \le \|P_V x_n - Tq\| + \|Tq - Tx_n\|$, we also get that $\{P_V x_n - Tx_n\}$ is bounded. By $x_{n+1} \in Q_n$, we have $\langle x_{n+1} - x_n, x_n - x_0\rangle \ge 0$. This together with Lemma 2.11 imply*

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2\\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle\\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned}$$

*Hence $\sum_{n=0}^{\infty} \|x_{n+1} - x_n\|^2 < \infty$, which implies $\|x_{n+1} - x_n\| \to 0$.*
*Since $x_{n+1} \in C_n$, we have*

$$\begin{aligned}\|z_n - x_n\| &\le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|\\ &\le \|x_n - x_{n+1}\| + d(U,V) + \|x_{n+1} - x_n\| \end{aligned}$$

*and*

$$\begin{aligned}\|y_n - x_n\| &\le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|\\ &\le \|x_n - x_{n+1}\| + d(U,V) + \|x_{n+1} - x_n\|. \end{aligned}$$

*The last two inequalities imply that $\|z_n - x_n\| \to d(U,V)$ and $\|y_n - x_n\| \to d(U,V)$.*

*Now, we consider*

$$y_n - x_n = \alpha_n z_n + (1-\alpha_n)Sx_n - x_n = \alpha_n(z_n - x_n) + (1-\alpha_n)(Sx_n - x_n)$$

*and*

$$z_n - x_n = \beta_n P_V x_n + (1-\beta_n)Tx_n - x_n = \beta_n(P_V x_n - Tx_n) + (Tx_n - x_n).$$

*Then*

$$(1-\alpha_n)(Sx_n - x_n) = (y_n - x_n) - \alpha_n(z_n - x_n)$$

*and*

$$Tx_n - x_n = (z_n - x_n) - \beta_n(P_V x_n - Tx_n).$$

*From above equalities, we obtain*

$$\|Sx_n - x_n\| \le \frac{1}{1-\alpha_n}\left(\|y_n - x_n\| + \alpha_n\|z_n - x_n\|\right)$$

*and*

$$\|Tx_n - x_n\| \le \|z_n - x_n\| + \beta_n\|P_V x_n - Tx_n\|.$$

*Taking $n \to \infty$ in above inequalities and use the assumptions $\alpha_n \to 0$ and $\beta_n \to 0$ together with boundedness of $\{P_V x_n - Tx_n\}$ we obtain*

$$\|Sx_n - x_n\| \to d(U,V) \text{ and } \|Tx_n - x_n\| \to d(U,V).$$

*Since $\|P_U Sx_n - Sx_n\| = d(U,V)$ and $\|P_U Tx_n - Tx_n\| = d(U,V)$, by Property UC, we get*

$$\|P_U Sx_n - x_n\| \to 0 \text{ as } n \to \infty \tag{6}$$

*and*

$$\|P_U Tx_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{7}$$

*Now, we define two mappings $S^*, T^* : U_0 \to U_0$ by $S^*(x) = P_U Sx, x \in U_0$ and $T^*(x) = P_U Tx, x \in U_0$. By Lemma 2.6, we get $S^*$ and $T^*$ are quasi-nonexpansive. Also, by Lemma 2.7, we have $Fix(S^*), Fix(T^*)$ are closed and convex subsets of $U_0$. Since $Fix(S^*) = Best_U(S)$ and $Fix(T^*) = Best_U(T)$ (by Lemma 2.5), it follows that $Best_U(S)$ and $Best_U(T)$ are closed convex subsets of $U_0$. Since $S$ and $T$ satisfy the proximal property, by Lemma 2.10, we have $I - S^*$ and $I - T^*$ are demiclosed at zero. Hence, by (6) and (7), we obtain the inclusion $w_\omega(x_n) \subset Fix(S^*) \cap Fix(T^*)$. This together with (5) and Lemma 2.14 guarantees that $\{x_n\}$ converges strongly to $q^* \in \Omega$, where $q^* = P_\Omega x_0$ and this completes the proof.* □

Next, we prove our second main result.

**Theorem 3.2.** *Let $U$ and $V$ be two nonempty, closed and convex subsets of a real Hilbert space $H$. Let $S, T : U \to V$ be best proximally nonexpansive mappings such that $S(U_0) \subset V_0$ and $T(U_0) \subset V_0$. Suppose that $S$ and $T$ satisfy the proximal property and $\Omega := Best_U(S) \cap Best_U(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\limsup_{n\to\infty}\beta_n < 1$. Let $C_1 = U_0$ and an initial guess $x_0 \in H$, define the sequence $\{x_n\}$ by $x_1 = P_{C_1}x_0$ and*

$$\begin{cases} z_n = \beta_n x_n + (1-\beta_n)P_U Tx_n, \\ y_n = \alpha_n P_V z_n + (1-\alpha_n)Sx_n, \\ C_{n+1} = \{u \in C_n : \|y_n - u\| \le \|x_n - u\| + d(U,V) \text{ and } \|z_n - u\| \le \|x_n - u\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, n \ge 1. \end{cases} \tag{8}$$

*Then the sequence $\{x_n\}$ defined by (8) converges strongly to a point $q^* \in \Omega$, where $q^* = P_\Omega x_0$.*

*Proof.* *Choose $x_0 \in H$ arbitrarily and put $q^* = P_\Omega x_0$. By Lemma 2.12, we know that $C_n$ is convex and closed (see more details Lemma 3.1[16]). We now show that $\Omega \subset C_n$ for all $n \geq 1$. The proof is by induction. It is clear that $\Omega \subset U_0 = C_1$. Assume that $\Omega \subset C_n$ for some $n \in \mathbb{N}$. Then, for any $q \in \Omega$, we have $q \in C_n$ and*

$$\begin{aligned}\|z_n - q\| &= \|\beta_n x_n + (1-\beta_n)P_U T x_n - q\| \\ &= \|\beta_n(x_n - q) + (1-\beta_n)(P_U T x_n - q)\| \\ &\leq \beta_n\|x_n - q\| + (1-\beta_n)\|P_U T x_n - q\|.\end{aligned}$$

*From $\|P_U T x_n - T x_n\| = d(U,V)$ and $\|q - Tq\| = d(U,V)$, by the P-property, we obtain $\|P_U T x_n - q\| = \|T x_n - Tq\|$. This together with above inequality, we have*

$$\begin{aligned}\|z_n - q\| &\leq \beta_n\|x_n - q\| + (1-\beta_n)\|T x_n - Tq\| \\ &\leq \beta_n\|x_n - q\| + (1-\beta_n)\|x_n - q\| \\ &= \|x_n - q\|. \end{aligned} \tag{9}$$

*Using (8) and property of S, we get*

$$\begin{aligned}\|y_n - q\| &= \|\alpha_n P_V z_n + (1-\alpha_n)S x_n - q\| \\ &= \|\alpha_n(P_V z_n - q) + (1-\alpha_n)(S x_n - q)\| \\ &\leq \alpha_n\|(P_V z_n - q)\| + (1-\alpha_n)\|(S x_n - q)\| \\ &\leq \alpha_n\|P_V z_n - Sq\| + \alpha_n\|Sq - q\| + (1-\alpha_n)\|S x_n - Sq\| + (1-\alpha_n)\|Sq - q\| \\ &= \alpha_n\|P_V z_n - Sq\| + \alpha_n d(U,V) + (1-\alpha_n)\|x_n - q\| + (1-\alpha_n)d(U,V) \\ &= \alpha_n\|P_V z_n - Sq\| + (1-\alpha_n)\|x_n - q\| + d(U,V).\end{aligned}$$

*From $\|P_V z_n - z_n\| = d(U,V)$ and $\|Sq - q\| = d(U,V)$, using P-property again, we obtain $\|P_V z_n - Sq\| = \|z_n - q\|$. Hence, we have*

$$\begin{aligned}\|y_n - q\| &\leq \alpha_n\|z_n - q\| + (1-\alpha_n)\|x_n - q\| + d(U,V) \\ &\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|x_n - q\| + d(U,V) \\ &= \|x_n - q\| + d(U,V). \end{aligned} \tag{10}$$

*Using (9) and (10) together with the induction hypothesis, we have $q \in C_{n+1}$. Thus, by induction, $\Omega \subset C_n$ for all $n \in \mathbb{N}$. From $\Omega \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} x_0$ for all $n \geq 0$, it follows that*

$$\|x_{n+1} - x_0\| \leq \|q^* - x_0\|, \tag{11}$$

*which implies $\{\|x_n - x_0\|\}$ is a bounded sequence.*
*From the fact that $x_n = P_{C_n} x_0$ for each $n \in \mathbb{N}$, it implies by Lemma 2.13 that*

$$\langle x_0 - x_n, x_n - y\rangle \geq 0 \text{ for all } y \in C_n. \tag{12}$$

*Since $x_{n+1} \in C_n$, (12) implies*

$$\begin{aligned}0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_n - x_0, x_{n+1} - x_0\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_n - x_0\|\|x_{n+1} - x_0\|.\end{aligned}$$

*It follows that $\{\|x_n - x_0\|\}$ is nondecreasing. Because $\{\|x_n - x_0\|\}$ is bounded, we can conclude that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. By Lemma 2.11 and (12), we have*

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.\end{aligned}$$

*It follows that* $\|x_{n+1} - x_n\| \to 0$. *By* $x_{n+1} \in C_{n+1}$, *we have*

$$\begin{aligned}\|z_n - x_n\| &\le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\|\end{aligned}$$

*and*

$$\begin{aligned}\|y_n - x_n\| &\le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le \|x_n - x_{n+1}\| + d(U,V) + \|x_{n+1} - x_n\|,\end{aligned}$$

*which imply* $\|z_n - x_n\| \to 0$ *as* $n \to \infty$ *and* $\|y_n - x_n\| \to d(U,V)$ *as* $n \to \infty$.
*From* $\|P_V z_n - Sx_n\| \le \|P_V z_n - z_n\| + \|z_n - x_n\| + \|x_n - Sq\| + \|Sq - Sx_n\|$, *it follows that* $\{P_V z_n - Sx_n\}$ *is a bounded sequence. Now, we note that*

$$y_n - x_n = \alpha_n P_V z_n + (1-\alpha_n)Sx_n - x_n = \alpha_n(P_V z_n - Sx_n) + (Sx_n - x_n).$$

*Then* $\|Sx_n - x_n\| \le \|y_n - x_n\| + \alpha_n\|P_V z_n - Sx_n\|$ *which implies that* $\|Sx_n - x_n\| \to d(U,V)$. *Since* $\|P_U Sx_n - Sx_n\| = d(U,V)$ *and* $\|Sx_n - x_n\| \to d(U,V)$, *by Property UC, we have*

$$\|P_U Sx_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{13}$$

*From (8),*

$$z_n - x_n = \beta_n x_n + (1-\beta_n)P_U Tx_n - x_n = (1-\beta_n)(P_U Tx_n - x_n).$$

*Then* $(1-\beta_n)\|P_U Tx_n - x_n\| = \|z_n - x_n\|$.
*From* $\|z_n - x_n\| \to 0$ *and* $\limsup_{n\to\infty} \beta_n < 1$, *we get*

$$\|P_U Tx_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{14}$$

*Using the same proof as in Theorem 3.1, we can show that the sequence* $\{x_n\}$ *generated by (8) converges strongly to* $P_\Omega x_0$. □

## 4. Numerical Examples

We finish my paper by giving some numerical experiment results for supporting our main methods. Consider $H = \mathbb{R}^2$ with the Euclidean norm, that is, $\|(x,y)\|_2 = \sqrt{x^2 + y^2}$. For our numerical examples, we let

$$U := \{(x,y) \in \mathbb{R}^2 : x \le 2, -2 \le y \le 2\}$$

and

$$V := \{(x,y) \in \mathbb{R}^2 : x \ge 0, y \ge 0\}.$$

Then $U$ and $V$ are nonempty, closed and convex subsets of $\mathbb{R}^2$ with the value of $d(U,V) = 2$. We also see that $U_0 = \{(-2,y) : 0 \le y \le 2\}$, and $V_0 = \{(0,y) : 0 \le y \le 2\}$.

**Example 4.1.** *Define two mappings* $S : U \to V$ *and* $T : U \to V$ *by*

$$S(x,y) = \begin{cases} (\arctan(-x), -1-y) & \text{if } (x,y) \in U \text{ and } y < -1; \\ \left(-2-x, \frac{5+2y}{7}\right) & \text{if } (x,y) \in U \text{ and } y \ge -1. \end{cases}$$

*and*

$$T(x,y) = \left(-2-x, \frac{3-|y|}{2}\right) \text{ for all } (x,y) \in U.$$

Table 1: Numerical result of algorithm of Theorem 3.1

| $n$ | $x_n$ | $y_n$ | $z_n$ | $\|x_n - (-2, 1)\|$ |
|---|---|---|---|---|
| 0 | (-2.0000, 2.0000) | (0.0000, 1.1964) | (0.0000, 0.7500) | 1.0000 |
| 1 | (-2.0000, 1.7596) | (0.0000, 1.1550) | (0.0000, 0.7830) | 0.7596 |
| 2 | (-2.0000, 1.5993) | (0.0000, 1.1712) | (0.0000, 0.7004) | 0.5993 |
| 3 | (-2.0000, 1.4599) | (0.0000, 1.1314) | (0.0000, 0.7701) | 0.4599 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 113 | (-2.0000, 1.0136) | (0.0000, 1.0039) | (0.0000, 0.9932) | 0.0136 |
| 114 | (-2.0000, 1.0134) | (0.0000, 1.0038) | (0.0000, 0.9933) | 0.0134 |
| 115 | (-2.0000, 1.0133) | (0.0000, 1.0038) | (0.0000, 0.9934) | 0.0133 |

Table 2: Numerical result of algorithm of Theorem 3.2

| $n$ | $x_n$ | $y_n$ | $z_n$ | $\|x_n - (-2, 1)\|$ |
|---|---|---|---|---|
| 1 | (-2.0000, 2.0000) | (0.0000, 1.2041) | (0.0000, 0.7143) | 1.0000 |
| 2 | (-2.0000, 1.3571) | (0.0000, 1.0729) | (0.0000, 0.8980) | 0.3571 |
| 3 | (-2.0000, 1.1275) | (0.0000, 1.0364) | (0.0000, 0.9363) | 0.1275 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 7 | (-2.0000, 1.0004) | (0.0000, 1.0001) | (0.0000, 0.9998) | 0.0004 |
| 8 | (-2.0000, 1.0001) | (0.0000, 1.0000) | (0.0000, 1.0000) | 0.0001 |

*Note that $S$ and $T$ are best proximally nonexpansive mappings such that $S(U_0) \subset V_0$ and $T(V_0) \subset U_0$. Moreover, we can see that the map $S$ is not nonexpansive because it is not a continuous mapping.*

*Define the real sequences $\alpha_n = \frac{1}{n^{100}+6}$ and $\beta_n = \frac{1}{n^{200}+6}$ for all $n \in \mathbb{N} \cup \{0\}$. We now choose the initial point $x_0 = (-2, 2)$. Then we obtain the following tables of numerical experiment for a common best proximity point in $U$ (see Table 1 and Table 2).*

*We observe from Table 1 and Table 2 that the sequence $\{x_n\}$, generated by our algorithms, converges to $(-2, 1)$ which is the common best proximity point of the maps $S$ and $T$. Moreover, we see that the convergence speed of algorithm of Theorem 3.2 is faster than that of algorithm of Theorem 3.1 under the same control conditions.*

**Example 4.2.** *In Jacob et al. (Algorithm 3.1) [16], we choose $x_0 = (-2, 2)$ and $\alpha_n = \frac{1}{n+6}$. Define the mapping $\widetilde{T} : U \to V$ by*

$$\widetilde{T}(x, y) = \begin{cases} (\arctan(-x), -1 - y) & \text{if } (x, y) \in U \text{ and } y < -1; \\ \left(-2 - x, \frac{5+2y}{7}\right) & \text{if } (x, y) \in U \text{ and } y \geq -1. \end{cases}$$

*For algorithm of Theorem 3.2, we choose $x_0 = (-2, 2)$, $\alpha_n = \frac{1}{n+6}$ and $\beta_n = \frac{1}{n^2+6}$. Define two mappings $S : U \to V$ and $T : U \to V$ by*

$$S = T = \widetilde{T}.$$

*Then we obtain the following tables of this numerical experiment for a common best proximity point in $U$ (see Table 3 and Table 4).*

*The stopping rule for both algorithms is $\|x_{n+1} - x_n\| < 10^{-4}$. So, from Table 3 and Table 4, we see that our algorithm of Theorem 3.2 requires less number of iterations than the corresponding algorithm of Jacob et al. [16]. Therefore the performance of approximation solution of our proposed algorithms is better than that. However, the performance of our studied algorithms depend on those control parameters.*

Table 3: Numerical example of algorithm of Jacob *et al.* [16]

| $n$ | $x_n$ | $\|x_n - (-2, 1)\|$ |
|---|---|---|
| 0 | (-2.0000, 2.0000) | 1.0000 |
| 1 | (-2.0000, 1.9317) | 0.9317 |
| 2 | (-2.0000, 1.8684) | 0.8684 |
| 3 | (-2.0000, 1.8104) | 0.8104 |
| ⋮ | ⋮ | ⋮ |
| 246 | (-2.0000, 1.0280) | 0.0280 |

Table 4: Numerical example of algorithm of Theorem 3.2

| $n$ | $x_n$ | $\|x_n - (-2, 1)\|$ |
|---|---|---|
| 1 | (-2.0000, 2.0000) | 1.0000 |
| 2 | (-2.0000, 1.6938) | 0.6938 |
| 3 | (-2.0000, 1.4814) | 0.4814 |
| ⋮ | ⋮ | ⋮ |
| 20 | (-2.0000, 1.0001) | 0.0001 |

## 5. Conclusion

The best proximity point problem plays an important role for studying the existence of various nonlinear equations in several fields. Existence problems of best proximity points for contractive type mappings were wild studied by many authors but there are a few papers paying attention on approximation methods for best proximity points. In this work, we purposed two new algorithms for finding a common best proximity point of some generalized nonexpansive mappings in a real Hilbert space. We analyzed convergence behavior of the proposed methods under some control conditions, see Theorem 3.1 and 3.2. Moreover, we also gave some numerical examples supporting our main results and comparisons of our two algorithms and the known existing algorithm in our literature.

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