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Subclasses of Meromorphic Functions Associated with a Convolution Operator

Mohamed K. Aouf[a], Teodor Bulboacă[b], Tamer M. Seoudy[c]

[a] *Department of Mathematics, Faculty of Science, Mansoura 35516, Egypt*
[b] *Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania*
[c] *Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt*

Abstract. The purpose of the present paper is to introduce a subclass of meromorphic functions by using the convolution operator, that generalizes some well-known classes previously defined by different authors. We discussed inclusion results, radius problems, and some connections with a certain integral operator.

1. Introduction

Let $H(\mathrm{U})$ be the class of functions analytic in the open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\Sigma(p, n)$ denote the class of all meromorphic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{j=n}^{\infty} a_j z^j,\ z \in \dot{\mathrm{U}} = \mathrm{U} \setminus \{0\} \quad (p, n \in \mathbb{N} = \{1, 2, 3, \dots\}). \tag{1}$$

Let $\mathcal{P}_k(\alpha)$ be the class of functions g, analytic in U, satisfying the condition $g(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} g(z) - \alpha}{1 - \alpha} \right| d\theta \le k\pi, \tag{2}$$

where $z = re^{i\theta}$, $0 < r < 1$, $k \ge 2$ and $0 \le \alpha < 1$. This class was introduced by Padmanabhan and Parvatham [15], and as a special case we note that the class $\mathcal{P}_k(0)$ was introduced by Pinchuk [16]. Moreover, $\mathcal{P}(\alpha) := \mathcal{P}_2(\alpha)$ is the class of analytic functions g in U, with $g(0) = 1$, and the real part greater than α.

Remark 1.1. *(i) Like in [13] and [14], from the definition* (2) *it can easily be seen that the function g, analytic in* U, *with $g(0) = 1$, belongs to $\mathcal{P}_k(\alpha)$ if and only if there exists the functions $g_1, g_2 \in \mathcal{P}(\alpha)$ such that*

$$g(z) = \left(\frac{k}{4} + \frac{1}{2}\right) g_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) g_2(z). \tag{3}$$

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Email addresses: mkaouf127@yahoo.com (Mohamed K. Aouf), bulboaca@math.ubbcluj.ro (Teodor Bulboacă), tms00@fayoum.edu.eg (Tamer M. Seoudy)

(ii) Notice that, if $g \in H(\mathrm{U})$ with $g(0) = 1$, then there exist functions $g_1, g_2 \in H(\mathrm{U})$ with $g_1(0) = g_2(0) = 1$, such that the function g can be written in the form (3). *For example, taking*

$$g_1(z) = \frac{g(z)-1}{k} + \frac{g(z)+1}{2} \quad \text{and} \quad g_2(z) = \frac{g(z)+1}{2} - \frac{g(z)-1}{k},$$

then $g_1, g_2 \in H(\mathrm{U})$, and $g_1(0) = g_2(0) = 1$.

(iii) Using the fact that $\mathcal{P}(\alpha)$ is the class of functions with real part greater than α, from the above representation formula it follows that

$$\mathcal{P}_k(\alpha_2) \subset \mathcal{P}_k(\alpha_1), \quad \text{if} \quad 0 \le \alpha_1 < \alpha_2 < 1.$$

(iv) It is well-known from [12] that the class $\mathcal{P}_k(\alpha)$ is a convex set.

We recall the differential operator $\mathcal{D}^m_{\lambda,p} : \Sigma(p,n) \to \Sigma(p,n)$, defined as follows:

$$\begin{aligned} &\mathcal{D}^0_{\lambda,p} f(z) = f(z), \\ &\mathcal{D}^m_{\lambda,p} f(z) = (1-\lambda)\,\mathcal{D}^{m-1}_{\lambda,p} f(z) + \lambda \frac{\left(z^{p+1}\mathcal{D}^{m-1}_{\lambda,p} f(z)\right)'}{z^p} = \\ &\frac{1}{z^p} + \sum_{j=n}^{\infty} [1+\lambda(j+p)]^m a_j z^j, \quad (\lambda \ge 0,\ p \in \mathbb{N},\ m \in \mathbb{N}), \end{aligned} \tag{4}$$

where the function $f \in \Sigma(p,n)$ is given by (1). This operator could be written by using the *Hadamard (convolution) product*, like

$$\mathcal{D}^m_{\lambda,p} = \varphi_{p,n}(\lambda, m; z) * f(z), \tag{5}$$

where

$$\varphi_{p,n}(\lambda, m; z) = \frac{1}{z^p} + \sum_{j=n}^{\infty} [1+\lambda(j+p)]^m z^j.$$

From the expansion formula (4) it is easy to verify the differentiation relation

$$\lambda z \left(\mathcal{D}^m_{\lambda,p} f(z)\right)' = \mathcal{D}^{m+1}_{\lambda,p} f(z) - (1+\lambda p)\,\mathcal{D}^m_{\lambda,p} f(z). \tag{6}$$

Remark 1.2. *The operator $\mathcal{D}^m_{\lambda,p}$ was defined and studied by Aouf et al. [2] and Aouf and Seoudy [3], and we note that:*

(i) The operator $\mathcal{D}^m_{1,p} = \mathcal{D}^m_p$ was introduced and studied by Aouf and Hossen [1], Liu and Owa [8], Liu and Srivastava [9], and Srivastava and Patel [20].

(ii) The operator $\mathcal{D}^m_{1,1} = \mathcal{D}^m$ was introduced and studied by Uralegaddi and Somanatha [21]. More general results than the work [21], with a different notation for convolution (to distinguish from the analytic case) were obtained in [17].

Next, by using the convolution operator $\mathcal{D}^m_{\lambda,p}$ we will introduce the *subclass of p–valent Bazilević functions* of $\Sigma(p,n)$ as follows:

Definition 1.3. *A function $f \in \Sigma(p,n)$ is said to be in the class $\Sigma\mathcal{B}^m_k(p, \lambda; \gamma, \mu, \alpha)$ if it satisfies the condition*

$$(1-\gamma)\left(z^p \mathcal{D}^m_{\lambda,p} f(z)\right)^{\mu} + \gamma \frac{\mathcal{D}^{m+1}_{\lambda,p} f(z)}{\mathcal{D}^m_{\lambda,p} f(z)} \left(z^p \mathcal{D}^m_{\lambda,p} f(z)\right)^{\mu} \in \mathcal{P}_k(\alpha),$$

$$(k \ge 2,\ \gamma \ge 0,\ \mu > 0,\ 0 \le \alpha < 1),$$

where all the powers represent the principal branches, i.e. $\log 1 = 0$.

We need to remark that, since the left-hand side function from the above definition need to be analytic in U, we implicitly assumed that $\mathcal{D}^m_{\lambda,p}f(z) \neq 0$ for all $z \in \dot{\mathrm{U}}$.

To prove our main results, the following lemma will be required in our investigation. We emphasize that slightly general situation than the above lemma is covered in [18], which might be useful to cover the case of nonlinear differential subordination.

Lemma 1.4. *[19] If g is an analytic function in* U, *with* $g(0) = 1$, *and if λ_1 is a complex number satisfying* $\operatorname{Re}\lambda_1 \geq 0$, $\lambda_1 \neq 0$, *then*

$$\operatorname{Re}\left[g(z) + \lambda_1 z g'(z)\right] > \alpha,\ z \in \mathrm{U}, \quad (0 \leq \alpha < 1)$$

implies

$$\operatorname{Re} g(z) > \beta,\ z \in \mathrm{U},$$

where β is given by

$$\beta = \alpha + (1-\alpha)(2\beta_1 - 1), \quad \beta_1 = \int_0^1 \left(1 + t^{\operatorname{Re}\lambda_1}\right)^{-1} dt, \tag{7}$$

and β_1 is an increasing function of $\operatorname{Re}\lambda_1$, *and* $\frac{1}{2} \leq \beta_1 < 1$. *The estimate is sharp in the sense that the bound cannot be improved.*

In this paper we investigate several properties of the class $\Sigma\mathcal{B}^m_k(p,\lambda;\gamma,\mu,\alpha)$ associated with the operator $\mathcal{D}^m_{\lambda,p}$, like inclusion results, radius problems, and some connections with the generalized Bernardi–Libera–Livingston integral operator introduced in [6].

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $k \geq 2$, $\gamma \geq 0$, $\mu > 0$, $0 \leq \alpha < 1$, and all the powers represent *the principal branches*, i.e. $\log 1 = 0$.

Theorem 2.1. *If* $f \in \Sigma\mathcal{B}^m_k(p,\lambda;\gamma,\mu,\alpha)$, *then*

$$\left(z^p \mathcal{D}^m_{\lambda,p} f(z)\right)^{\mu} \in \mathcal{P}_k(\beta), \tag{8}$$

where β is given by (7), *with* $\lambda_1 = \dfrac{\gamma\lambda}{\mu}$.

Proof. Since the implication is obvious for $\gamma = 0$, suppose that $\gamma > 0$. Let f be an arbitrary function in $\Sigma\mathcal{B}^m_k(p,\lambda;\gamma,\mu,\alpha)$, and denote

$$g(z) := \left(z^p \mathcal{D}^m_{\lambda,p} f(z)\right)^{\mu}. \tag{9}$$

It follows that g is analytic in U, with $g(0) = 1$, and according to the part (*ii*) of Remarks 1.1 the function g can be written in the form

$$g(z) = \left(\frac{k}{4} + \frac{1}{2}\right) g_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) g_2(z), \tag{10}$$

where g_1 and g_2 are analytic in U, with $g_1(z) = g_2(z) = 1$.

From the part (*i*) of Remarks 1.1 we have that $g \in \mathcal{P}_k(\beta)$, if and only if the function g has the representation given by the above relation, where $g_1, g_2 \in \mathcal{P}(\alpha)$. Consequently, supposing that g is of the form (10), we will prove that $g_1, g_2 \in \mathcal{P}(\alpha)$.

Using the differentiation formula (6) and the notation (9), after an elementary computation we obtain

$$(1-\gamma)\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^\mu+\gamma\,\frac{\mathcal{D}^{m+1}_{\lambda,p}f(z)}{\mathcal{D}^m_{\lambda,p}f(z)}\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^\mu=$$

$$g(z)+\frac{\gamma\lambda}{\mu}zg'(z). \tag{11}$$

Now, using the representation formula (3), we have

$$g(z)+\frac{\gamma\lambda}{\mu}zg'(z)= \tag{12}$$

$$\left(\frac{k}{4}+\frac{1}{2}\right)\left[g_1(z)+\frac{\gamma\lambda}{\mu}zg_1'(z)\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[g_2(z)+\frac{\gamma\lambda}{\mu}zg_2'(z)\right].$$

Since $f\in\Sigma\mathcal{B}^m_k(p,\lambda;\gamma,\mu,\alpha)$, from the relations (11) and (12) it follows that

$$g_i(z)+\frac{\gamma\lambda}{\mu}zg_i'(z)\in\mathcal{P}(\alpha),\ i=1,2. \tag{13}$$

To prove our result we need to show that (13) implies $g_i\in\mathcal{P}(\beta)$, $i=1,2$. Thus, the conditions (13) are equivalent to

$$\operatorname{Re}\left[g_i(z)+\lambda_1zg_i'(z)\right]>\alpha,\ z\in\mathrm{U},$$

with $\lambda_1=\dfrac{\gamma\lambda}{\mu}$. According to Lemma 1.4, it follows that $g_i\in\mathcal{P}(\beta)$, where β is given by (7), with $\lambda_1=\dfrac{\gamma\lambda}{\mu}$. Thus, according to the part (*i*) of Remarks 1.1 and to the representation formula (3) we obtain the desired result. □

Theorem 2.2. *If* $0\le\gamma_1<\gamma_2$, *then*

$$\Sigma\mathcal{B}^m_k(p,\lambda;\gamma_2,\mu,\alpha)\subset\Sigma\mathcal{B}^m_k(p,\lambda;\gamma_1,\mu,\alpha).$$

Proof. If we consider an arbitrary function $f\in\Sigma\mathcal{B}^m_k(p,\lambda;\gamma_2,\mu,\alpha)$, then $\varphi_2\in\mathcal{P}_k(\alpha)$, where

$$\varphi_2(z):=(1-\gamma_2)\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^\mu+\gamma_2\,\frac{\mathcal{D}^{m+1}_{\lambda,p}f(z)}{\mathcal{D}^m_{\lambda,p}f(z)}\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^\mu.$$

According to Theorem 2.1 we have

$$\varphi_1(z):=\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^\mu\in\mathcal{P}_k(\beta),$$

where β is given by (7), with $\lambda_1=\dfrac{\gamma\lambda}{\mu}$. Since $\beta=\alpha+(1-\alpha)(2\beta_1-1)$ and $\frac{1}{2}\le\beta_1<1$, it follows that $\beta\ge\alpha$, and from the part (*ii*) of Remarks 1.1 we conclude that $\mathcal{P}_k(\beta)\subset\mathcal{P}_k(\alpha)$, hence $\varphi_1\in\mathcal{P}_k(\alpha)$.

A simple computation shows that

$$(1-\gamma_1)\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^\mu+\gamma_1\,\frac{\mathcal{D}^{m+1}_{\lambda,p}f(z)}{\mathcal{D}^m_{\lambda,p}f(z)}\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^\mu=$$

$$\left(1-\frac{\gamma_1}{\gamma_2}\right)\varphi_1(z)+\frac{\gamma_1}{\gamma_2}\varphi_2(z). \tag{14}$$

Since the class $\mathcal{P}_k(\alpha)$ is a convex set (see the part (*iv*) of Remarks 1.1), it follows that right-hand side of (14) belongs to $\mathcal{P}_k(\alpha)$ for $0\le\gamma_1<\gamma_2$, which implies that $f\in\Sigma\mathcal{B}^m_k(p,\lambda;\gamma_1,\mu,\alpha)$. □

Let us define the integral operator $J_{c,p} : \Sigma(p,n) \to \Sigma(p,n)$ by

$$J_{c,p}f(z) = \frac{c+1}{z^{c+p+1}} \int_0^z t^{c+p} f(t)dt \quad (c > -1). \tag{15}$$

We will give a short proof that this operator is well-defined, as follows. If the function $f \in \Sigma(p,n)$ is of the form (1), then the definition (15) can be written

$$J_{c,p}f(z) = \frac{1}{z^p}\frac{c+1}{z^{c+1}} \int_0^z t^c \left(t^p f(t)\right) dt =$$
$$\frac{1}{z^p}\frac{c+1}{z^{c+1}} \int_0^z t^c \varphi(t)dt = \frac{c+1}{z^p} I_{c,p}\varphi(z),$$

where

$$I_{c,p}\varphi(z) = \frac{1}{z^{c+1}} \int_0^z t^c \varphi(t)dt$$

and

$$\varphi(z) = z^p f(z) = 1 + \sum_{j=n}^{\infty} a_j z^{j+p},\ z \in \mathrm{U}, \tag{16}$$

is analytic in U. We see that integral operator $I_{c,p}$ defined above is similar to that of Lemma 1.2c. of [11]. According to this lemma, it follows that $I_{c,p}$ is an analytic integral operator for any function φ of the form (16) whenever $\operatorname{Re} c > -1$, and $J_{c,p}f \in \Sigma(p,n)$ has the form

$$J_{c,p}f(z) = \frac{1}{z^p} + (c+1)\sum_{j=n}^{\infty} \frac{a_j}{j+p+c+1} z^j,\ z \in \mathrm{U}.$$

The operator $J_{c,p}$ was introduced by Kumar and Shukla [6], connected with the Bernardi–Libera–Livingston integral operators (see [4], [7] and [10]).

Theorem 2.3. *If $f \in \Sigma(p,n)$, the integral operator $J_{c,p}$ is given by* (15), *$\gamma \geq 0$ and $\mu > 0$, then*

$$(1-\gamma)\left(z^p \mathcal{D}^m_{\lambda,p} J_{c,p} f(z)\right)^{\mu} + \gamma\, z^p \mathcal{D}^m_{\lambda,p} f(z) \left(z^p \mathcal{D}^m_{\lambda,p} J_{c,p} f(z)\right)^{\mu-1} \in \mathcal{P}_k(\alpha),$$

implies that

$$\left(z^p \mathcal{D}^m_{\lambda,p} J_{c,p} f(z)\right)^{\mu} \in \mathcal{P}_k(\beta),$$

where β is given by (7), *with $\lambda_1 = \dfrac{\gamma}{\mu(c+1)}$.*

Proof. Like in the remark mentioned after the Definition 1.3, since the left-hand side function from the above definition need to be analytic in U, we implicitly assumed that $\mathcal{D}^m_{\lambda,p} J_{c,p} f(z) \neq 0$ for all $z \in \dot{\mathrm{U}}$.

The implication is obvious for $\gamma = 0$, hence suppose that $\gamma > 0$. Differentiating the relation (15) we have

$$z\left(J_{c,p}f(z)\right)' = (c+1)f(z) - (c+p+1)J_{c,p}f(z),$$

and using the fact that $\mathcal{D}^m_{\lambda,p}$ and $J_{c,p}$ commute, this implies

$$z\left(\mathcal{D}^m_{\lambda,p} J_{c,p} f(z)\right)' = (c+1)\mathcal{D}^m_{\lambda,p} f(z) - (c+p+1)\mathcal{D}^m_{\lambda,p} J_{c,p} f(z). \tag{17}$$

If we let

$$g(z) := \left(z^p \mathcal{D}^m_{\lambda,p} J_{c,p} f(z)\right)^{\mu},$$

then by part (*ii*) of Remarks 1.1 the function g can be written in the form (10), where g_1 and g_2 are analytic in U, with $g_1(0) = g_2(0) = 1$. According to the the part (*i*) of Remarks 1.1 we need to prove that $g_1, g_2 \in \mathcal{P}(\beta)$.

Using (17), from the above relation we have

$$\begin{aligned}(1-\gamma)\left(z^p \mathcal{D}^m_{\lambda,p} J_{c,p} f(z)\right)^{\mu} + \gamma\, z^p \mathcal{D}^m_{\lambda,p} f(z) \left(z^p \mathcal{D}^m_{\lambda,p} J_{c,p} f(z)\right)^{\mu-1} = \\ g(z) + \frac{\gamma}{\mu(c+1)} zg'(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\left[g_1(z) + \frac{\gamma}{\mu(c+1)} zg_1'(z)\right] - \\ \left(\frac{k}{4}-\frac{1}{2}\right)\left[g_2(z) + \frac{\gamma}{\mu(c+1)} zg_2'(z)\right] \in \mathcal{P}_k(\alpha).\end{aligned}$$

Now, from the part (*i*) of Remarks 1.1 it follows that

$$g_i(z) + \frac{\gamma}{\mu(c+1)} zg_i'(z) \in \mathcal{P}(\alpha),\ i = 1, 2,$$

and from Lemma 1.4 we conclude that $g_i \in \mathcal{P}(\beta)$, $i = 1, 2$, with β given by (7) and $\lambda_1 = \dfrac{\gamma}{\mu(c+1)}$. □

The following result represents the converse of Theorem 2.1.

Theorem 2.4. *If $f \in \Sigma(p, n)$ such that $\left(z^p \mathcal{D}^m_{\lambda,p} f(z)\right)^{\mu} \in \mathcal{P}_k(\alpha)$, then $\rho^p f(\rho z) \in \Sigma\mathcal{B}^m_k(p, \lambda; \gamma, \mu, \alpha)$, with*

$$\rho = \min\left\{\left(\frac{-n\gamma\lambda + \sqrt{\mu^2 + n^2\gamma^2\lambda^2}}{\mu}\right)^{\frac{1}{n}}; r_0\right\} \tag{18}$$

where

$$r_0 = \begin{cases} \min\{r > 0 : \varphi(r) = 0\}, & \text{if } \ \exists\, r > 0 : \varphi(r) = 0 \\ 1, & \text{if } \ \nexists\, r > 0 : \varphi(r) = 0, \end{cases} \tag{19}$$

and

$$\varphi(r) = (2\alpha - 1)r^{2n} + 2\left[2\alpha - 1 - n(1-\alpha)\frac{\gamma\lambda}{\mu}\right] r^n + 1.$$

Proof. For an arbitrary $f \in \Sigma(p, n)$ such that $\left(z^p \mathcal{D}^m_{\lambda,p} f(z)\right)^{\mu} \in \mathcal{P}_k(\alpha)$, let g be defined as in (9), i.e.

$$\left(z^p \mathcal{D}^m_{\lambda,p} f(z)\right)^{\mu} = g(z) \in \mathcal{P}_k(\alpha). \tag{20}$$

From the part (*i*) of Remarks 1.1 we have that (20) holds if and only if

$$g(z) = \left(\frac{k}{4}+\frac{1}{2}\right) g_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) g_2(z),$$

where $g_1, g_2 \in \mathcal{P}(\alpha)$.

Using the above representation formula, like in the proof of Theorem 2.1 we deduce that

$$(1-\gamma)\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^{\mu}+\gamma\,\frac{\mathcal{D}^{m+1}_{\lambda,p}f(z)}{\mathcal{D}^m_{\lambda,p}f(z)}\,\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^{\mu}=$$
$$\left(\frac{k}{4}+\frac{1}{2}\right)\left[g_1(z)+\frac{\gamma\lambda}{\mu}\,zg_1'(z)\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[g_2(z)+\frac{\gamma\lambda}{\mu}\,zg_2'(z)\right],$$

and substituting $G_i(z):=\dfrac{g_i(z)-\alpha}{1-\alpha}$, $i=1,2$, we finally obtain

$$(1-\gamma)\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^{\mu}+\gamma\,\frac{\mathcal{D}^{m+1}_{\lambda,p}f(z)}{\mathcal{D}^m_{\lambda,p}f(z)}\,\left(z^p\mathcal{D}^m_{\lambda,p}f(z)\right)^{\mu}=$$
$$\left(\frac{k}{4}+\frac{1}{2}\right)\left[(1-\alpha)\left(G_1(z)+\frac{\alpha}{1-\alpha}+\frac{\gamma\lambda}{\mu}\,zG_1'(z)\right)\right]-$$
$$\left(\frac{k}{4}-\frac{1}{2}\right)\left[(1-\alpha)\left(G_2(z)+\frac{\alpha}{1-\alpha}+\frac{\gamma\lambda}{\mu}\,zG_2'(z)\right)\right],$$

where $G_1,G_2\in\mathcal{P}(0)$.

To prove our result we need to determine the value of ρ, such that

$$\operatorname{Re}\left[G_i(z)+\frac{\alpha}{1-\alpha}+\frac{\gamma\lambda}{\mu}\,zG_i'(z)\right]>0,\ \text{for } |z|<\rho,\ i=1,2,$$

whenever $G_1,G_2\in\mathcal{P}(0)$.

Since $f\in\Sigma(p,n)$, using the well-known estimates [5] for the class $\mathcal{P}(0)$, i.e.

$$\left|zG_i'(z)\right|\leq\frac{2nr^n\operatorname{Re}G_i(z)}{1-r^{2n}},\ |z|\leq r<1,\ i=1,2,$$
$$\operatorname{Re}G_i(z)\geq\frac{1-r^n}{1+r^n},\ |z|\leq r<1,\ i=1,2,$$

we conclude that

$$\operatorname{Re}\left[G_i(z)+\frac{\alpha}{1-\alpha}+\frac{\gamma\lambda}{\mu}\,zG_i'(z)\right]\geq\frac{\alpha}{1-\alpha}+\operatorname{Re}G_i(z)-\frac{\gamma\lambda}{\mu}\,\left|zG_i'(z)\right|\geq$$
$$\frac{\alpha}{1-\alpha}+\operatorname{Re}G_i(z)\left[1-\frac{\gamma\lambda}{\mu}\frac{2nr^n}{1-r^{2n}}\right],\tag{21}$$

for all $|z|\leq r<1$ and $i=1,2$.

A simple calculation shows that $1-\dfrac{\gamma\lambda}{\mu}\dfrac{2nr^n}{1-r^{2n}}\geq 0$ $(0\leq r<1)$ if and only if

$$r\in\left[0,\left(\frac{-n\gamma\lambda+\sqrt{\mu^2+n^2\gamma^2\lambda^2}}{\mu}\right)^{\frac{1}{n}}\right],\tag{22}$$

and assuming that (22) holds, from (21) we obtain

$$\operatorname{Re}\left[G_i(z)+\frac{\alpha}{1-\alpha}+\frac{\gamma\lambda}{\mu}\,zG_i'(z)\right]\geq\frac{\alpha}{1-\alpha}+\frac{1-r^n}{1+r^n}\left[1-\frac{\gamma\lambda}{\mu}\frac{2nr^n}{1-r^{2n}}\right],$$
$$|z|\leq r<1,\ \text{for } i=1,2.$$

It is easy to check that the right-hand side of the above inequality is greater or equal than zero if and only if

$$r \in [0, \min\{1; r_0\}],$$

where r_0 is given by (19), and combining this with (22) we obtain our result. □

Remark 2.5. *(i) For the special case $n = 1$, it follows that if $f \in \Sigma(p, 1)$ then*

$$\rho = \min\left\{\frac{-\gamma\lambda + \sqrt{\mu^2 + \gamma^2\lambda^2}}{\mu}; r_0\right\}.$$

(ii) We remark that for the special case $n = 1$ and $\alpha = 0$, the formula (18) *reduces to*

$$\rho = -\left(1 + \frac{\gamma\lambda}{\mu}\right) + \sqrt{\left(1 + \frac{\gamma\lambda}{\mu}\right)^2 + 1}.$$

(iii) Putting $\lambda = 1$ in the above results, we obtain the similar results associated with the operator $\mathcal{D}_p^m$.
(iv) Taking $\lambda = p = 1$ in the above results, we obtain the similar results involving the operator $\mathcal{D}^m$.

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References

[1] M. K. Aouf, H. M. Hossen, New criteria for meromorphic p–valent starlike functions, Tsukuba J. Math., 17(1993), 481–486.
[2] M. K. Aouf, A. Shamandy, A. O. Mostafa and S. M. Madian, Properties of some families of meromorphic p–valent functions involving certain differential operator, Acta Univ. Apulensis Math. Inform., 20(2009), 7–16.
[3] M. K. Aouf, T. M. Seoudy, Subordination and superordination of certain linear operator on meromorphic functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A, 64(1)(2010), 1–16.
[4] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135(1969), 429–446.
[5] S. D. Bernardi, New distortion theorems for functions of positive real part and applications to the partial sums of univalent convex functions, Proc. Amer. Math. Soc. 45(1)(1974), 113–118.
[6] V. Kumar and S. L. Shukla, Certain integrals for classes of p–valent meromorphic functions, Bull. Austral. Math. Soc., 25(1982), 85–97.
[7] R. J. Libera, Some radius of convexity problems, Duke Math. J., 31(1964), 143–158.
[8] J.-L. Liu, S. Owa, Certain meromorphic p–valent functions, Taiwanese J. Math., 2(1)(1998), 107–110.
[9] J.-L. Liu, H. M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized hypergeometric functions, Math. Comput. Modelling, 39(2004), 21–34.
[10] A. E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc., 17(1966), 352–357.
[11] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.
[12] K. I. Noor, On subclasses of close-to-convex functions of higher order, Int. J. Math. Math. Sci., 15(1992) 279–290.
[13] K. I. Noor, On certain classes of analytic functions, J. Ineq. Pure Appl. Math., 7(2)(2006), Art. 49.
[14] K. I. Noor, On analytic functions related to certain family of integral operators, J. Ineq. Pure Appl. Math., 7(2)(2006), Art. 69.
[15] K. S. Padmanabhan, R. Parvatham, Properties of a class of functions with bounded boundary rotations, Ann. Polon. Math., 31(3)(1975), 311–323.
[16] B. Pinchuk, Functions of bounded boundary rotations, Israel J. Math., 10(1971), 6–16.
[17] S. Ponnusamy, Convolution properties of some classes of meromorphic univalent functions, Proc. Indian Acad. Sci. (Math. Sci.) 103(1)(1993), 73-89.
[18] S. Ponnusamy, Differential subordination concerning starlike functions, Proc. Ind. Acad. Sci. (Math. Sci.) 104(2)(1994), 397-411.
[19] S. Ponnusamy, Differential subordination and Bazilević functions, Proc. Indian Acad. Sci., 105(1995), 169–186.
[20] H. M. Srivastava, J. Patel, Applications of differential subordinations to certain classes of meromorphically multivalent functions, J. Ineq. Pure Appl. Math., 6(3)(2005), Art. 88.
[21] B. A. Uralegaddi, C. Somanatha, New criteria for meromorphic starlike univalent functions, Bull. Aust. Math. Soc., 43(1991), 137–140.