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Characterizing Approximate Global Minimizers of the Difference of two Abstract Convex Functions with Applications

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Abstract. In this paper, we first investigate characterizations of maximal elements of abstract convex functions under a mild condition. Also, we give various characterizations for global ε-minimum of the difference of two abstract convex functions and, by using the abstract Rockafellar's antiderivative, we present the abstract ε-subdifferential of abstract convex functions in terms of their abstract subdifferential. Finally, as an application, a necessary and sufficient condition for global ε-minimum of the difference of two increasing and positively homogeneous (IPH) functions is presented.

1. Introduction

Abstract convexity opens the way for extending some main ideas and results from classical convex analysis to much more general classes of functions, mappings and sets [6, 14–18]. It is well known that every lower semi-continuous convex function is the upper envelope of a set of affine functions. Therefore, affine functions play a crucial role in classical convex analysis. In abstract convexity, the role of affine functions is taken by an alternative set H of functions, and their upper envelopes constitute the set of abstract convex functions. Different choices of the set H generate variants of the classical concepts, and have shown important applications, especially, in global optimization. Moreover, if a family of functions is abstract convex for a specific choice of H, then we can use some key ideas of convex analysis in order to gain new insight on these functions. On the other hand, by replacing H with families which are more general than the set of affine functions, we identify the crucial features in classical convex analysis. Abstract convexity has found many applications in the study of mathematical analysis and optimization problems [15–17]. Functions which can be represented as upper envelopes of subsets of a set H of sufficiently simple (elementary) functions, are studied in this theory (for more details, see [6, 14, 15, 18]).

Minimum and ε-minimum of the difference of two convex functions is one of the most important global optimization problems. In a general case, the difference of two convex functions can be replaced by DAC-functions (difference of two abstract convex functions). In particular, minimizing of the difference of two

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increasing and co-radiant (ICR) functions [3], and also, minimizing of the difference of two increasing co-radiant and quasi-concave functions (see; for example, [4, 9]). A formula for ε-subdifferential of the difference of two lower semi-continuous convex functions has been given in [8]. Also, ε-subdifferentials in terms of subdifferential have been given in [7]. We point out that, one of the main questions in abstract convexity, is the representation of subgradients and ε-subgradients. Because of the abstract versions of the subdifferential and the ε-subdifferential might be very large, we work with the affine counterparts of the subdifferential and the ε-subdifferential.

In this paper, we first obtain a formula for the conjugate of the difference of two abstract functions, and give a formula for the abstract ε- subdifferential of the difference of two abstract functions. Next, by using this, we present a necessary and sufficient condition for global ε-minimum of the difference of two abstract convex functions, and give abstract ε-subdifferentials in terms of abstract subdifferential.

The paper has the following structure: In Section 2, we provide some preliminaries, definitions and results related to abstract convexity. In Section 3, we first obtain a formula for abstract conjugate and abstract ε-subdifferential of the difference of two abstract functions, and give a necessary and sufficient condition for global ε-minimum of the difference of two abstract convex functions. Next, we present characterizations of maximal elements of the support set of abstract convex functions under a mild condition. Abstract ε-subdifferentials in terms of abstract subdifferential, and another characterization of (global) ε-minimum of the difference of two abstract convex functions are represented in Section 4. In Section 5, as an application, we give a necessary and sufficient condition for global ε-minimum of the difference of two increasing and positively homogeneous (IPH) functions. In Section 6, we present our conclusions and discuss applications.

2. Preliminaries

Let X be a set and $L := \{\ell : X \longrightarrow \mathbb{R} : \ell \textit{ is an abstract linear function}\}$ be a set of real valued abstract linear functions defined on X. Throughout this paper, we assume that L contains the function 0_X, which is defined by $0_X(x) := 0$ for all $x \in X$. For each $\ell \in L$ and $c \in \mathbb{R}$, consider the shift $h_{\ell,c}$ of ℓ on the constant c :

$$h_{\ell,c}(x) := \ell(x) - c, \quad (x \in X).$$

The function $h_{\ell,c}$ is called L-affine, and the set L is called a set of abstract linear functions if $h_{l,c} \notin L$ for all $l \in L$ and all $c \in \mathbb{R} \setminus \{0\}$ [15]. The set of all L-affine functions will be denoted by H_L. There exists a one-to-one correspondence between $L \times \mathbb{R}$ and H_L, given by, $(\ell, c) \longrightarrow h_{\ell,c}$.

Recall [15] that a function $f : X \longrightarrow (-\infty, +\infty] =: \mathbb{R}_{+\infty}$ is called H-convex ($H = L$, *or* $H = H_L$) if

$$f(x) = \sup\{h(x) : h \in \text{supp }(f, H)\}, \text{ for all } x \in X,$$

where

$$\text{supp }(f, H) := \{h \in H : h \leq f\}$$

is called the support set of the function f. Also, if $h = (\ell, c) \in H_L$ and $h' = (\ell', c') \in H_L$. Then,

$$h' \geq h \quad \text{if and only if} \quad \ell' \geq \ell \quad \text{and} \quad c \geq c'. \tag{1}$$

Let $U \subseteq H$ be a set of functions. Recall [15] that a function $f \in U$ is called a maximal element of the set U, if $\tilde{f} \in U$ is such that $\tilde{f}(x) \geq f(x)$ for all $x \in X$, then, $\tilde{f}(x) = f(x)$ for all $x \in X$. The function $\text{co}_{H_L} f$ is defined by

$$\text{co}_{H_L} f(x) := \sup\{h_{\ell,c}(x) := \ell(x) - c : h_{\ell,c} = (\ell, c) \in \text{supp }(f, H_L)\}, \quad (x \in X)$$

is called the H_L-convex hull of the function f [15].

It is clear, by the definition, that $\text{co}_{H_L} f \leq f$. Clearly, f is H_L-convex if and only if $f = \text{co}_{H_L} f$ (see [15]).

For a function $f : X \longrightarrow (-\infty, +\infty]$, define [15] the Fenchel-Moreau L-conjugate f_L^* of f by

$$f_L^*(\ell) := \sup_{x\in X}(\ell(x) - f(x)), \quad \ell \in L.$$

The function $f_{L,X}^{**} := (f_L^*)_X^*$ is called the second L-conjugate (or L-biconjugate) of f and, by the definition, we have

$$f_{L,X}^{**}(x) := \sup_{\ell\in L}(\ell(x) - f^*(\ell)), \quad x \in X.$$

It is well known that $f = f_{L,X}^{**}$ if and only if f is an H_L-convex function [15].

Let $f : X \longrightarrow (-\infty, +\infty]$ be a function and $x_0 \in dom f$. Recall [6, 15] that the L-subdifferential of f is the set valued mapping $\partial_L f : X \rightrightarrows L$ is defined by

$$\partial_L f(x_0) := \{\ell \in L : \ell(x) - \ell(x_0) \leq f(x) - f(x_0), \text{ for all } x \in X\},$$

and for given $\varepsilon \geq 0$, the L-ε-subdifferential of f is the set valued mapping $\partial_{L,\varepsilon} f : X \rightrightarrows L$ is defined by

$$\partial_{L,\varepsilon} f(x_0) := \{\ell \in L : \ell(x) - \ell(x_0) - \varepsilon \leq f(x) - f(x_0), \text{ for all } x \in X\}.$$

Also, for $x_0 \notin \operatorname{dom} f$, we define $\partial_L f(x_0) = \partial_{L,\varepsilon} f(x_0) := \emptyset$. It is well known that we can characterize the L-subdifferential of f and the L-ε-subdifferential of f, as follows

$$\partial_L f(x_0) = \{\ell \in L : f(x_0) + f_L^*(\ell) = \ell(x_0)\}, \tag{2}$$

and

$$\partial_{L,\varepsilon} f(x_0) = \{\ell \in L : f(x_0) + f_L^*(\ell) \leq \ell(x_0) + \varepsilon\}. \tag{3}$$

If f is an H_L convex function, then, from [15, Page 268], we observe that $\partial_{L,\varepsilon} f(x_0) \neq \emptyset$ for all $x_0 \in dom f$ and all $\varepsilon \gneqq 0$. Also, it is easy to see that

$$\partial_L f(x_0) = \bigcap_{\varepsilon\geq 0} \partial_{L,\varepsilon} f(x_0), \ (x_0 \in \operatorname{dom} f). \tag{4}$$

Definition 2.1. *Let $\varepsilon \geq 0$ be given. A point $x_0 \in X$ is said to be a global ε-minimum (approximate global minimum) of the proper function $f : X \to \mathbb{R}_{+\infty} := [0, +\infty]$, if*

$$f(x_0) \leq \inf_{x\in X} f(x) + \varepsilon.$$

Lemma 2.1. *Let $f : X \longrightarrow \mathbb{R}_{+\infty}$ be an H_L-convex function, and let $\varepsilon \geq 0$ be given. A point $x_0 \in X$ is an ε-minimum of the function f if and only if $0_X \in \partial_{L,\varepsilon} f(x_0)$.*

Proof: This is an immediate consequence of the definitions of ε- minimum and L-ε-subdifferential. ■

Let X and Y be sets. For a set valued mapping $F : X \rightrightarrows Y$, we define [2] the domain and the graph of F by

$$Dom(F) := \{x \in X : F(x) \neq \emptyset\},$$

and

$$G(F) := \{(x, y) \in X \times Y : y \in F(x)\},$$

respectively.

3. ε-Subdifferential and ε-Minimum of the Difference of Abstract Convex Functions

In this section, we first obtain a formula for conjugate of the difference of two abstract functions. Next, by using (3), we characterize abstract ε-subdifferential of the difference of two abstract functions. Finally, we give a necessary and sufficient condition for global ε-minimum of the difference of two abstract convex functions.

Proposition 3.1. *Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be proper functions such that $Dom(\partial_L g) = X$.Then,*

$$(f-g)^*_L(\ell) = \sup_{w\in dom(g^*_L)} (f^*_L(\ell+w) - g^*_L(w)), \text{ for all } \ell \in L. \tag{5}$$

Proof: Let $\ell, w \in L$ be arbitrary. Consider

$$\begin{aligned}(f-g)^*_L(\ell) \quad &= \sup_{x\in X}\{\ell(x) - (f-g)(x)\} \\ &\geq (\ell+w)(x) - f(x) - (w(x) - g(x)) \\ &\geq (\ell+w)(x) - f(x) - g^*_L(w), \text{ for all } x \in X, \text{ for all } w \in L.\end{aligned}$$

Therefore,

$$(f-g)^*_L(\ell) + g^*_L(w) \geq \sup_{x\in X}\{(\ell+w)(x) - f(x)\} = f^*_L(\ell+w), \text{ for all } w \in L.$$

By taking supremum over all $w \in dom(g^*_L)$ (note that, since $Dom(\partial_L g) = X$ and g is a proper function, by (2) we conclude that $dom(g^*_L) \neq \emptyset$), one has

$$(f-g)^*_L(\ell) \geq \sup_{w\in dom(g^*_L)} \{f^*_L(\ell+w) - g^*_L(w)\}. \tag{6}$$

For the converse of (6), let $x \in X$ be arbitrary. Since by the hypothesis $Dom(\partial_L g) = X$, we choose $\tilde{\ell} \in \partial_L g(x)$. Therefore, it follows from (1) that

$$\begin{aligned}&\ell(x) - (f-g)(x) = (\ell+\tilde{\ell})(x) - f(x) - (\tilde{\ell}(x) + g(x)) \\ &= (\ell+\tilde{\ell})(x) - f(x) - g^*_L(\tilde{\ell}) \leq f^*_L(\ell+\tilde{\ell}) - g^*_L(\tilde{\ell}) \\ &\leq \sup_{w\in dom(g^*_L)} \{f^*_L(\ell+w) - g^*_L(w)\}.\end{aligned}$$

Taking supremum over all $x \in X$, we conclude that

$$(f-g)^*_L(\ell) \leq \sup_{w\in dom(g^*_L)} \{f^*_L(\ell+w) - g^*_L(w)\},$$

which completes the proof. ∎

The Toland-Singer duality [18] is an immediate consequence of Proposition 3.1.

Corollary 3.1. (*Toland-Singer duality [18]*) *Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be functions such that $Dom(\partial_L g) = X$. Then,*

$$\inf_{x\in \mathrm{dom}\, f} \{f(x) - g(x)\} = \inf_{\ell\in \mathrm{dom}\,(g^*_L)} \{g^*_L(\ell) - f^*_L(\ell)\}.$$

Proof: From the definition of the L-conjugate and Proposition 3.1, one has

$$\inf_{x\in \mathrm{dom}\, f} \{f(x) - g(x)\} = -(f-g)^*_L(0_X) = \inf_{\ell\in \mathrm{dom}\,(g^*_L)} \{g^*_L(\ell) - f^*_L(\ell)\}.$$

∎

By using Toland-Singer duality, we can state the following results on ε-minimum of the difference of two abstract functions.

Theorem 3.1. *Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be functions such that $Dom(\partial_L g) = X$. Suppose that $x_0 \in dom f$ is a global ε-minimum of $f - g$. Let $\ell_0 \in \partial_L f(x_0) \cap \partial_L g(x_0)$, then, ℓ_0 is a global ε-minimum of $g^*_L - f^*_L$.*

Proof: Since $\ell_0 \in \partial_L f(x_0) \cap \partial_L g(x_0)$, in view of (2), $f(x_0) + f^*_L(\ell_0) = \ell_0(x_0)$ and $g(x_0) + g^*_L(\ell_0) = \ell_0(x_0)$. Hence, $f(x_0) - g(x_0) = g^*_L(\ell_0) - f^*_L(\ell_0)$. Since x_0 is a global ε- minimum of $f - g$, we have

$$(f - g)(x_0) - \varepsilon \leq \inf_{x \in X}(f - g)(x).$$

Therefore, by Toland-Singer duality, one has

$$(g^*_L - f^*_L)(\ell_0) - \varepsilon \leq \inf_{\ell \in L}(g^*_L - f^*_L)(\ell),$$

and the proof is complete. ■

Corollary 3.2. *Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be H_L-convex functions. Suppose that $\ell_0 \in dom(g^*_L)$ is a global ε-minimum of $g^*_L - f^*_L$. Let $x_0 \in \partial_L f^*_L(\ell_0) \cap \partial_L g^*_L(\ell_0)$, then, x_0 is a global ε-minimum of $f - g$.*

Proof: The result follows by replacing f by f^*_L in Theorem 3.1 and the fact that $f = f^{**}_L$. ■

Let U and W be subsets of L. We use the following notation.

$$U \ominus W := \{\ell \in L : \ell + W \subseteq U\}. \tag{7}$$

Theorem 3.2. *Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be functions such that $Dom(\partial_L g) = X$. Let $x_0 \in \mathrm{dom}\, f$ and $\varepsilon \geq 0$ be given. Then,*

$$\partial_{L,\varepsilon}(f - g)(x_0) = \bigcap_{\delta \geq 0}\{\partial_{L,\varepsilon+\delta} f(x_0) \ominus \partial_{L,\delta} g(x_0)\}.$$

Proof: It follows from (3) that $\ell \in \partial_{L,\varepsilon}(f - g)(x_0)$ if and only if

$$(f - g)(x_0) + (f - g)^*_L(\ell) \leq \ell(x_0) + \varepsilon.$$

By Proposition 3.1,

$$(f - g)(x_0) + f^*_L(\ell + w) - g^*_L(w) \leq \ell(x_0) + \varepsilon, \text{ for all } w \in \mathrm{dom}\,(g^*_L). \tag{8}$$

Put $\ell + w := u$ in (8), so,

$$f(x_0) + f^*_L(u) - u(x_0) \leq g(x_0) + g^*_L(w) - w(x_0) + \varepsilon, \text{ for all } w \in L. \tag{9}$$

Let $\delta \geq 0$ be arbitrary. Hence, in view of (3) and (9), for each $w \in \partial_{L,\delta} g(x_0)$, one has $u \in \partial_{L,\delta+\varepsilon} f(x_0)$. Since $u = \ell + w$, we conclude that

$$\ell + \partial_{L,\delta} g(x_0) \subseteq \partial_{L,\delta+\varepsilon} f(x_0), \text{ for all } \delta \geq 0. \tag{10}$$

Therefore, in view of (7) and (10), we deduce that $\ell \in \partial_{L,\varepsilon}(f - g)(x_0)$ if and only if $\ell \in \bigcap_{\delta \geq 0}\{\partial_{L,\varepsilon+\delta} f(x_0) \ominus \partial_{L,\delta} g(x_0)\}$. Hence, the proof is complete. ■

Theorem 3.3. *Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be H_L-convex functions such that $Dom(\partial_L g) = X$. Then, x_0 is a global ε-minimum of $f - g$ if and only if*

$$\partial_{L,\delta} g(x_0) \subseteq \partial_{L,\varepsilon+\delta} f(x_0), \textit{ for all } \delta \geq 0. \tag{11}$$

Proof: It follows from Lemma 2.1 that x_0 is a global ε-minimum of $f - g$ if and only if $0_X \in \partial_{L,\varepsilon}(f - g)(x_0)$. In view of Theorem 3.2, we conclude that (11) holds. ■

Corollary 3.3. *Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be H_L-convex functions such that $Dom(\partial_L g) = X$. Then, x_0 is a global ε-minimum of $f - g$ if and only if $\partial_L g(x_0) \subseteq \partial_{L,\varepsilon} f(x_0)$.*

Proof: In view of Theorem 3.3, x_0 is a global ε-minimum of $f - g$ if and only if

$$\bigcap_{\delta \geq 0} \partial_{L,\delta} g(x_0) \subseteq \bigcap_{\delta \geq 0} \partial_{L,\delta+\varepsilon} f(x_0).$$

Therefore, by applying (4), the result follows. ■

Let $f : X \longrightarrow \mathbb{R}_{+\infty}$ be a function. Let $\varepsilon \geq 0$ and $x_0 \in X$ be given. We consider the set $\mathcal{S}_\varepsilon f(x_0)$ is defined as follows

$$\mathcal{S}_\varepsilon f(x_0) := \{h \in \text{supp } (f; H_L) : h(x_0) = f(x_0) - \varepsilon\}.$$

In the following, we present the relation between $\partial_{L,\varepsilon} f(x_0)$ and $\mathcal{S}_\varepsilon f(x_0)$.

Proposition 3.2. *Let $f : X \longrightarrow \mathbb{R}_{+\infty}$ be an H_L-convex function. Let $x_0 \in X$ and $\varepsilon \geq 0$ be given. Then, the following assertions are true.*
(1) $\ell \in \partial_{L,\varepsilon} f(x_0)$ if and only if $h = (\ell, \ell(x_0) - f(x_0) + \varepsilon) \in \mathcal{S}_\varepsilon f(x_0)$.
(2) An element $\ell \in L$ is a maximal element of $\partial_{L,\varepsilon} f(x_0)$ if and only if $h = (\ell, \ell(x_0) - f(x_0) + \varepsilon)$ is a maximal element of $\mathcal{S}_\varepsilon f(x_0)$.

Proof: (1). Let $\ell \in \partial_{L,\varepsilon} f(x_0)$ be arbitrary. Consider $h(x) = \ell(x) - (\ell(x_0) - f(x_0) + \varepsilon)$ for all $x \in X$. It follows from the definition of $\partial_{L,\varepsilon} f(x_0)$ that $h \in supp(f, H_L)$. Moreover, $h(x_0) = f(x_0) - \varepsilon$. Hence, $h = (\ell, \ell(x_0) - f(x_0) + \varepsilon) \in \mathcal{S}_\varepsilon f(x_0)$. Conversely, let $h = (\ell, \ell(x_0) - f(x_0) + \varepsilon) \in \mathcal{S}_\varepsilon f(x_0)$. In view of the definition of $\mathcal{S}_\varepsilon f(x_0)$, one has $h \in supp(f, H_L)$. Therefore, $h(x) = \ell(x) - (\ell(x_0) - f(x_0) + \varepsilon) \leq f(x)$ for all $x \in X$. Hence, $\ell \in \partial_{L,\varepsilon} f(x_0)$.
(2). Let $\ell \in L$ be a maximal element of $\partial_{L,\varepsilon} f(x_0)$. By (1), we have $h = (\ell, \ell(x_0) - f(x_0) + \varepsilon) \in \mathcal{S}_\varepsilon f(x_0)$. Suppose that $h' = (\ell', \ell'(x_0) - f(x_0) + \varepsilon) \in \mathcal{S}_\varepsilon f(x_0)$ is such that $h' \geq h$ on X. In view of (1), $\ell' \geq \ell$ on X and $\ell(x_0) \geq \ell'(x_0)$. It follows from (1) that $\ell' \in \partial_{L,\varepsilon} f(x_0)$. Since $\ell \in L$ is a maximal element $\partial_{L,\varepsilon} f(x_0)$, so, $\ell' = \ell$ on X, which implies that $h' = h$ on X, and so, h is a maximal element of $\mathcal{S}_\varepsilon f(x_0)$. Conversely, let $(\ell, \ell(x_0) - f(x_0) + \varepsilon)$ be a maximal element of $\mathcal{S}_\varepsilon f(x_0)$. By (1), we have $\ell \in \partial_{L,\varepsilon} f(x_0)$. Suppose that $\ell' \in \partial_{L,\varepsilon} f(x_0)$ is such that $\ell' \geq \ell$ on X. In view of (1), one has $(\ell', \ell'(x_0) - f(x_0) + \varepsilon) \in \mathcal{S}_\varepsilon f(x_0)$. Since $(\ell, \ell(x_0) - f(x_0) + \varepsilon)$ is a maximal element of $\mathcal{S}_\varepsilon f(x_0)$, hence, $\ell = \ell'$ on X, which completes the proof. ■

We now give a necessary and sufficient condition for global ε- minimum of $f - g$ in terms of elements of $\mathcal{S}_\varepsilon f(x_0)$ and $\mathcal{S}_\varepsilon g(x_0)$.

Lemma 3.1. *Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be H_L-convex functions such that $Dom(\partial_L g) = X$. Let $\varepsilon \geq 0$ and $x_0 \in X$ be given. Then, x_0 is a global ε-minimum of $f - g$ if and only if, for each $\delta \geq 0$, $(\ell, \ell(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$ whenever $(\ell, \ell(x_0) - g(x_0) + \delta) \in \mathcal{S}_\delta g(x_0)$.*

Proof: Suppose that x_0 is a global ε-minimum of $f - g$. Let $\delta \geq 0$ and $(\ell, \ell(x_0) - g(x_0) + \delta) \in \mathcal{S}_\varepsilon g(x_0)$. By Proposition 3.2(1), $\ell \in \partial_{L,\delta} g(x_0)$. Now, in view of Theorem 3.3, we conclude that $\ell \in \partial_{L,\varepsilon+\delta} f(x_0)$. Again, Proposition 3.2(1) implies that $(\ell, \ell(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$. For the converse, it is enough to show that (11) holds. Let $\delta \geq 0$ be arbitrary and $\ell \in \partial_{L,\delta} g(x_0)$. It follows from Proposition 3.2(1) that $(\ell, \ell(x_0) - g(x_0) + \delta) \in \mathcal{S}_\delta g(x_0)$. Thus, by the hypothesis, $(\ell, \ell(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$. In view of Proposition 3.2(1), one has $\ell \in \partial_{L,\varepsilon+\delta} f(x_0)$. Hence, (11) holds. ■

The following condition has been presented in [15, Page 367].

Condition (B): Let $H = L$, or $H = H_L$. Let $h(x) := \lim_{\alpha \in \Delta} h_\alpha(x)$ $(x \in X)$, where Δ is a directed set and $(h_\alpha)_{\alpha \in \Delta} \subset H$ is a net which is bounded from below. Then, either $h \in H$, or $h \equiv +\infty$.

We now describe some sets of functions, which enjoys or does not enjoy the property (B).

Example 3.1. *Let $X = L := \mathbb{R}^2$. For each $\ell \in L$, define $\ell : X \longrightarrow \mathbb{R}$ by*

$$\ell(x) := \langle \ell, x\rangle, \text{ for all } x \in X,$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product of $\mathbb{R}^2$. Note that L is the set of all linear functions defined on X. Therefore, for each $\ell \in L$ and each $c \in \mathbb{R}$, we have

$$h_{\ell,c}(x) := \ell(x) - c, \text{ for all } x \in X,$$

and

$$H_L := \{h_{\ell,c} : \ell \in L,\ c \in \mathbb{R}\}.$$

We show that H_L enjoys the property (B). Now, let $\{h_{\ell_k,c_k}\}_{k\geq 1}$ be a sequence in H_L which is bounded from below and $h(x) := \lim_{k\longrightarrow+\infty} h_{\ell_k,c_k}(x)$ for each $x \in X$. We show that $h \in H_L$ or $h \equiv +\infty$. Since $h_{\ell_k,c_k} \in H_L$ $(k = 1, 2, \cdots)$, it follows that there exist sequences $\{\ell_k\}_{k\geq1} \subset L$ and $\{c_k\}_{k\geq1} \subset \mathbb{R}$ such that

$$h_{\ell_k,c_k}(x) = \ell_k(x) - c_k, \text{ for all } x \in X,\ k = 1, 2, \cdots. \tag{12}$$

If $h \equiv +\infty$, we are done. Assume that there exists $x_0 \in X$ such that $h(x_0) < +\infty$. This together with the fact that the sequence $\{h_{\ell_k,c_k}\}_{k\geq1}$ is bounded from below and $h(x_0) := \lim_{k\longrightarrow+\infty} h_{\ell_k,c_k}(x_0) = \lim_{k\longrightarrow+\infty}[\ell_k(x_0) - c_k]$ implies that the sequence $\{c_k\}_{k\geq1}$ is bounded. Then, by passing a subsequence, there exists $c \in \mathbb{R}$ such that $c_k \longrightarrow c$. Since $\ell_k(x) - c_k = h_{\ell_k,c_k}(x) \longrightarrow h(x)$ for each $x \in X$, we conclude that $\ell_k(x) \longrightarrow h(x) + c$ for each $x \in X$. Put $\ell(x) := h(x) + c$ for each $x \in X$. So, $h(x) = \ell(x) - c$ for each $x \in X$ (note that $\ell \in L$ because $\langle \ell_k, x\rangle \longrightarrow \langle \ell, x\rangle$ for each $x \in X$, and so $\langle \ell_k - \ell, x\rangle \longrightarrow 0$ for each $x \in X$. This implies that $\|\ell_k - \ell\| \longrightarrow 0$. Since $\ell_k \in \mathbb{R}^2 = L$ for all $k \geq 1$, it follows that $\ell \in \mathbb{R}^2 = L$). Hence, $h \in H_L$. Thus, H_L enjoys the property (B).

Example 3.2. *Let $X = L := \mathbb{R}^3_+$. For each $\ell = (\ell_1, \ell_2, \ell_3) \in L$, define $\ell : X \longrightarrow \mathbb{R}$ by*

$$\ell(x) := \min_{i\in I_+(\ell)} \ell_i x_i, \text{ for all } x = (x_1, x_2, x_3) \in X,$$

where $I_+(\ell) := \{i \in \{1, 2, 3\} : \ell_i > 0\}$. We show that $H := L$ does not enjoys the property (B). Note that H is a set of min-type functions defined on X. Let $\ell_k := (k, 1, 1) \in H$, $k = 1, 2, \cdots$, and let

$$h(x) := \begin{cases} \min\{x_2, x_3\}, & x_1 > 0, \\ 0, & x_1 = 0, \end{cases} \quad \text{for each } x = (x_1, x_2, x_3) \in X.$$

It is easy to see that $\ell_k(x) \longrightarrow h(x)$ for each $x \in X$. Now, we show that $h \notin H$, i.e., h is not a min-type function. Assume if possible that there exists $\ell = (\ell_1, \ell_2, \ell_3) \in L$ such that $h(x) = \min_{i\in I_+(\ell)} \ell_i x_i$ for each $x = (x_1, x_2, x_3) \in X$. Clearly, $\ell \neq 0$, and so $I_+(\ell) \neq \emptyset$. We claim that $1 \in I_+(\ell)$. If $1 \notin I_+(\ell)$, then for the vector $x = (0, 1, 1) \in X$, we have

$$h(x) = 0, \text{ and } \ \ell(x) > 0,$$

which is a contradiction. Hence, $1 \in I_+(\ell)$. Let $x := (\varepsilon, 1, 1) \in X$ with $\varepsilon > 0$. Therefore, for sufficiently small $\varepsilon > 0$, one has

$$\min_{i\in I_+(\ell)} \ell_i x_i < \min\{x_2, x_3\} = 1 = h(x),$$

and we arrive at a contradiction. Thus, h is not a min-type function, i.e., $h \notin H$. Hence, H does not enjoy the property (B).

Theorem 3.4. *Suppose that H_L enjoys the condition (B). Let $f : X \longrightarrow \mathbb{R}_{+\infty}$ be an H_L- convex function. Let $x_0 \in X$ and $\varepsilon \geq 0$ be given. Then, for each element $h := (\ell, \ell(x_0) - f(x_0) + \varepsilon) \in \mathcal{S}_\varepsilon f(x_0)$, there exists a maximal element $\tilde{h} = (\tilde{\ell}, \tilde{\ell} - f(x_0) + \varepsilon) \in \mathcal{S}_\varepsilon f(x_0)$ such that $\ell \leq \tilde{\ell}$ on X and $\ell(x_0) = \tilde{\ell}(x_0)$.*

Proof: Let $h := (\ell, \ell(x_0) - f(x_0) + \varepsilon) \in \mathcal{S}_\varepsilon f(x_0)$, and let $\mathcal{U} := \{h' \in \mathcal{S}_\varepsilon f(x_0) : h' \geq h \text{ on } X\}$. Let C be any chain in $\mathcal{U}$. We can consider C as a net $(h_\alpha)_{\alpha \in I}$. It is clear that this net is bounded from below. Let $\bar{h}(x) := \sup\{h'(x) : h' \in C\}$ $(x \in X)$. Since f is proper and $\bar{h} \leq f$ on X, it follows that $\bar{h}$ is proper, and hence, it follows from the condition (B) that $\bar{h} \in H_L$. Since $h \leq h' \leq f$ on X for all $h' \in C$, so, $\bar{h} \in supp(f, H_L)$ and $\bar{h} \geq h$ on X. Also, one has

$$\bar{h}(x_0) = \sup_{h' \in C} h'(x_0) = f(x_0) - \varepsilon.$$

Therefore, $\bar{h} \in \mathcal{S}_\varepsilon f(x_0)$, and so, $\bar{h} \in \mathcal{U}$. Then, it follows from Zorn's lemma that there exists a maximal element $\tilde{h}$ of the set $\mathcal{U}$. Since $\tilde{h} \in \mathcal{S}_\varepsilon f(x_0)$, we have $\tilde{h} = (\tilde{\ell}, \tilde{\ell}(x_0) - f(x_0) + \varepsilon)$. So, in view of (1), $\ell \leq \tilde{\ell}$ on X and $\tilde{\ell}(x_0) \leq \ell(x_0)$, which completes the proof. ∎

In the following, we characterize global ε-minimum of $f - g$ in terms of maximal elements of $\mathcal{S}_\varepsilon f(x_0)$ and $\mathcal{S}_\varepsilon g(x_0)$.

Theorem 3.5. *Suppose that H_L enjoys the condition (B). Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be H_L-convex functions such that $Dom(\partial_L g) = X$. Let $\varepsilon \geq 0$ and $x_0 \in X$ be given. Then, x_0 is a global ε-minimum of $f - g$ if and only if, for each $\delta \geq 0$ and each maximal element $(\ell, \ell(x_0) - g(x_0) + \delta) \in \mathcal{S}_\delta g(x_0)$, there exists a maximal element $(\ell', \ell'(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$ such that $\ell \leq \ell'$ on X and $\ell(x_0) = \ell'(x_0)$.*

Proof: Suppose that x_0 is a global ε-minimum of $f - g$. Let $\delta \geq 0$ and $(\ell, \ell(x_0) - g(x_0) + \delta)$ be a maximal element of $\mathcal{S}_\delta g(x_0)$. Thus, by Lemma 3.1, one has $(\ell, \ell(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$. Now, it follows from Theorem 3.4 that there exists a maximal element $(\ell', \ell'(x_0) - f(x_0) + \varepsilon + \delta)$ of $\mathcal{S}_{\varepsilon+\delta} f(x_0)$ such that $\ell \leq \ell'$ on X and $\ell'(x_0) = \ell(x_0)$. Conversely, let $\delta \geq 0$. By Lemma 3.1, it is enough to show that $(\ell, \ell(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$ whenever $(\ell, \ell(x_0) - g(x_0) + \delta) \in \mathcal{S}_\delta g(x_0)$. Let $(\ell, \ell(x_0) - g(x_0) + \delta) \in \mathcal{S}_\delta g(x_0)$ be arbitrary. In view of Theorem 3.4, there exists a maximal element $(\tilde{\ell}, \tilde{\ell}(x_0) - g(x_0) + \delta) \in \mathcal{S}_\delta g(x_0)$ such that $\ell \leq \tilde{\ell}$ on X and $\ell(x_0) = \tilde{\ell}(x_0)$. Hence, by the hypothesis, there exists a maximal element $(\ell', \ell'(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$ such that $\tilde{\ell} \leq \ell'$ on X and $\ell'(x_0) = \tilde{\ell}(x_0)$. Therefore, $\ell \leq \ell'$ on X and $\ell(x_0) = \ell'(x_0)$. This implies that $(\ell, \ell(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$. ∎

We now illustrate Theorem 3.5 by the following example.

Example 3.3. *Let X, L and H_L be as in Example 3.1. In view of Example 3.1, H_L enjoys the property (B). Note that H_L is the set of affine functions defined on X. Define the functions $f, g : X \longrightarrow \mathbb{R}$ by*

$$f(x_1, x_2) := x_1^2 + x_2^2 + 1, \text{ for all } (x_1, x_2) \in X,$$

and

$$g(x_1, x_2) := x_1^2, \text{ for all } (x_1, x_2) \in X.$$

It is clear that f and g are continuous convex functions. It is well known that every lower semicontinuous proper convex function is the pointwise supremum of a subset U of H_L [19]. So, the functions f and g are H_L-convex functions. It is not difficult to show that

$$\partial_L g(x) = \{(2x_1, 0)\} \text{ for each } x = (x_1, x_2) \in X.$$

Therefore,

$$Dom(\partial_L g) := \{x \in X : \partial_L g(x) \neq \emptyset\} = X.$$

We have

$$f(x) - g(x) = x_1^2 + x_2^2 + 1 - x_1^2 = x_2^2 + 1, \text{ for all } x = (x_1, x_2) \in X.$$

Now, let $\varepsilon \geq 0$ be given, and let $x_0 := (0, 0) \in X$. Then, $f(x_0) = 1$ and $g(x_0) = 0$. In view of Theorem 3.5, if we show that for each $\delta \geq 0$ and each maximal element $(\ell, \ell(x_0) - g(x_0) + \delta) \in \mathcal{S}_\delta g(x_0)$ there exists a maximal element

$(\ell', \ell'(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$ *such that* $\ell \leq \ell'$ *on* X *and* $\ell(x_0) = \ell'(x_0)$, *then* x_0 *is a global* ε*-minimum of the function* $f - g$. *To this end, let* $\delta \geq 0$, *and let* $(\ell, \ell(x_0) - g(x_0) + \delta) \in \mathcal{S}_\delta g(x_0)$ *be a maximal element. Put* $c := \ell(x_0) - g(x_0) + \delta = \ell(0) - g(0) + \delta = \delta$, *and so* $h_{\ell,c}(x) := \ell(x) - c = \ell(x) - \delta$ *for each* $x \in X$. *Since*

$$h_{\ell,c} = (\ell, \ell(x_0) - g(x_0) + \delta) \in \mathcal{S}_\delta g(x_0),$$

it follows from the definition of $\mathcal{S}_\delta g(x_0)$ *that*

$$\ell(x) - \delta = h_{\ell,c}(x) \leq g(x), \text{ for all } x \in X. \tag{13}$$

Now, let $\ell' := \ell$ *and*

$$c' := \ell'(x_0) - f(x_0) + \varepsilon + \delta = \ell'(0) - f(0) + \varepsilon + \delta = -1 + \varepsilon + \delta. \tag{14}$$

We first show that

$$h_{\ell',c'} = (\ell', \ell'(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0).$$

By (14), *we have*

$$h_{\ell',c'}(x_0) = \ell'(x_0) - c' = -c' = 1 - \varepsilon - \delta = f(x_0) - \varepsilon - \delta. \tag{15}$$

Moreover, in view of (13) *and* (14), *one has*

$$\begin{aligned} h_{\ell',c'}(x) &= \ell'(x) - c' \\ &= \ell(x) + 1 - \varepsilon - \delta \\ &= [\ell(x) - \delta] + 1 - \varepsilon \\ &\leq g(x) + 1 - \varepsilon \\ &= x_1^2 + 1 - \varepsilon \\ &\leq x_1^2 + x_2^2 + 1 - \varepsilon \\ &= f(x) - \varepsilon \\ &\leq f(x), \text{ for all } x \in X. \end{aligned} \tag{16}$$

Then, by (15) *and* (16), *we conclude that*

$$h_{\ell',c'} = (\ell', \ell'(x_0) - f(x_0) + \varepsilon + \delta) \in \mathcal{S}_{\varepsilon+\delta} f(x_0).$$

Now, we show that $h_{\ell',c'}$ *is a maximal element of* $\mathcal{S}_{\varepsilon+\delta} f(x_0)$. *Let* $h_{\bar{\ell},\bar{c}} \in \mathcal{S}_{\varepsilon+\delta} f(x_0)$ *be such that*

$$h_{\bar{\ell},\bar{c}}(x) \geq h_{\ell',c'}(x), \text{ for all } x \in X. \tag{17}$$

By the definition of $\mathcal{S}_{\varepsilon+\delta} f(x_0)$, *we have*

$$\bar{\ell}(x_0) - \bar{c} = h_{\bar{\ell},\bar{c}}(x_0) = f(x_0) - \varepsilon - \delta = 1 - \varepsilon - \delta.$$

This together with the fact that $\bar{\ell}(x_0) = \bar{\ell}(0) = 0$ *implies that* $\bar{c} = -1 + \varepsilon + \delta$. *So, in view of* (14), *we obtain* $c' = \bar{c}$. *On the other hand, by* (17) *and the fact that* $c' = \bar{c}$, *one has*

$$\bar{\ell}(x) - \bar{c} \geq \ell'(x) - c' = \ell'(x) - \bar{c}, \text{ for all } x \in X.$$

This implies that $\bar{\ell}(x) \geq \ell'(x)$ *for all* $x \in X$. *If replace* x *by* $-x$, *then* $\bar{\ell}(x) \leq \ell'(x)$ *for all* $x \in X$. *Thus,* $\bar{\ell}(x) = \ell'(x)$ *for all* $x \in X$, *and so* $\bar{\ell} = \ell'$. *Hence,* $h_{\bar{\ell},\bar{c}} = h_{\ell',c'}$. *Then* $h_{\ell',c'}$ *is a maximal element of* $\mathcal{S}_{\varepsilon+\delta} f(x_0)$ *and, we also have* $\ell' = \ell$ *on* X *and* $\ell'(x_0) = 0 = \ell(x_0)$. *Therefore, by Theorem 3.5,* x_0 *is a global* ε*-minimum of the function* $f - g$.

4. L-ε-Subdifferential in terms of L-Subdifferential

In this section, we present L-ε-subdifferentials in terms of L-subdifferential for an abstract convex function by using Rockafellar's antiderivative. In Theorem 4.4, below, we will give another characterization of global ε-minimum of the difference of abstract convex functions. We now give the following definitions in the abstract sense (for the classical sense, see [1]). Assume that X and L are as in Section 2.

Definition 4.1. *Let $M : X \rightrightarrows L$ be a set valued mapping, and let $n \in \mathbb{N}$. We say that M is n-L-monotone, if for any set of n pairs $\{(x_i, \ell_i)\}_{i=1}^n \subseteq G(M)$ with $x_{n+1} := x_1$, we have*

$$\sum_{i=1}^{n} [\ell_i(x_i) - \ell_i(x_{i+1})] \geq 0. \tag{18}$$

A mapping M is said to be L-cyclically monotone, if it is n-L-monotone for all $n \in \mathbb{N}$. An 2-L-monotone mapping is simply called L-monotone. The mapping M is said to be maximal n-L-monotone, if $G(M)$ has no proper n-L-cyclically monotone extension in $X \times L$.

It is worth noting that the properties and characterizations of L-monotone (abstract monotone) operators have been investigated in [5, 11].

Lemma 4.1. *Let $f : X \longrightarrow \mathbb{R}_{+\infty}$ be a proper function. Then, $\partial_L f$ is L-cyclically monotone.*

Proof: Fix an integer $n \geq 2$. For every $1 \leq i \leq n$, take $(x_i, \ell_i) \in G(\partial_L f)$. Set $x_{n+1} := x_1$. Then, by the definition of subdifferential,

$$\ell_i(x_{i+1}) - \ell_i(x_i) \leq f(x_{i+1}) - f(x_i), \text{ for all } i \in \{1, ..., n\}.$$

By adding the above inequalities, we obtain (18). ∎

In the sequel, we give some definitions and results in the abstract sense. The proofs are similar to the classical case (for more details, see [1]).

Definition 4.2. *Let $M : X \rightrightarrows L$ be a set valued mapping. Let $f : X \longrightarrow \mathbb{R}_{+\infty}$ be a proper function. We say that f is an L-antiderivative of M whenever $G(M) \subseteq G(\partial_L f)$.*

Definition 4.3. *Let $M : X \rightrightarrows L$ be a set valued mapping and $s \in Dom(M)$. Define L-Rockafellar's function $R_{[M,s]} : X \longrightarrow \mathbb{R}_{+\infty}$ associated with M by*

$$R_{[M,s]}(x) := \sup_{\substack{\{(x_i, \ell_i)\}_{i=1}^n \subseteq G(M) \\ x_1 := s,\, x_{n+1} := x}} \sum_{i=1}^{n} [\ell_i(x_{i+1}) - \ell_i(x_i)]. \tag{19}$$

Theorem 4.1. *A set valued mapping $M : X \rightrightarrows L$ is L-cyclically monotone if and only if for any $s \in Dom(M)$, the function $R_{[M,s]}$ is a proper H_L-convex L-antiderivative of M satisfying $R_{[M,s]}(s) = 0$.*

Proof: The proof is similar to that one [1, Theorem 2.5]. ∎

Theorem 4.2. *Suppose that the function $f : X \longrightarrow \mathbb{R}_{+\infty}$ is the unique L-antiderivative (up to an additive constant) of the mapping $M : X \rightrightarrows L$. Then, f is H_L-convex and*

$$f(x) = f(s) + R_{[M,s]}(x), \textit{ for all } (s, x) \in Dom(M) \times X. \tag{20}$$

Proof: The proof is similar to that one [1, Proposition 5.4]. ∎

Remark 4.1. *Let $f : X \longrightarrow \mathbb{R}_{+\infty}$ be a proper function. If $\partial_L f$ admits a unique (up to an additive constant) L-antiderivative, then, since f is an L-antiderivative of $\partial_L f$, it follows from Theorem 4.2 that*

$$f(x) = f(s) + R_{[\partial_L f,s]}(x), \text{ for all } (s,x) \in Dom(\partial_L f) \times X. \tag{21}$$

We now present L-ε-subdifferentials in terms of L-subdifferential for an abstract convex function by using Rockafellar's antiderivative.

Theorem 4.3. *Let $f : X \longrightarrow \mathbb{R}_{+\infty}$ be a proper H_L-convex function such that $Dom(\partial_L f) = dom f$. Suppose that $\partial_L f$ admits a unique (up to an additive constant) L-antiderivative. Let $x \in dom f$ and $\varepsilon \geq 0$ be given. Then,*

$$\begin{aligned} &\partial_{L,\varepsilon} f(x) \\ = &\left\{ \begin{array}{l} \ell \in L : \ell(x) - \ell(x_0) + \ell_m(x_m) - \ell_m(x) + \sum_{i=0}^{m-1}[\ell_i(x_i) - \ell_i(x_{i+1})] \\ \geq -\varepsilon, \text{ for all } (x_i, \ell_i) \in G(\partial_L f),\ (i = 0, ..., m) \end{array} \right\}. \end{aligned} \tag{22}$$

Proof: Let $\ell \in \partial_{L,\varepsilon} f(x)$ and $(x_i, \ell_i) \in G(\partial_L f)$ $(i = 0, ..., m)$ be arbitrary. Hence,

$$\begin{aligned} &\ell(x_0) - \ell(x) - \varepsilon \leq f(x_0) - f(x), \\ &\ell_i(x_{i+1}) - \ell_i(x_i) \leq f(x_{i+1}) - f(x_i) \ (i = 1, ..., m-1), \\ &\ell_m(x) - \ell_m(x_m) \leq f(x) - f(x_m). \end{aligned}$$

By adding the above inequalities, we conclude that

$$\ell(x) - \ell(x_0) + \ell_m(x_m) - \ell_m(x) + \sum_{i=0}^{m-1}[\ell_i(x_i) - \ell_i(x_{i+1})] \geq -\varepsilon.$$

Conversely, let $\ell \in L$ belongs to the right-hand side of (22). Let $x_0 \in Dom(\partial_L f)$ and $\gamma \in \mathbb{R}$ be arbitrary such that $\gamma > f(x_0) - f(x)$. It follows from (21) and Remark 4.1 that $f(x) - f(x_0) = R_{[\partial_L f, x_0]}(x)$. In view of (19), there exists $(x_i, \ell_i) \in G(\partial_L f)$ $(i = 1, ..., m)$ such that

$$\sum_{i=0}^{m-1}[\ell_i(x_{i+1}) - \ell_i(x_i)] + \ell_m(x) - \ell_m(x_m) > -\gamma.$$

Now, by (22) and the choice of ℓ, one has

$$\begin{aligned} \varepsilon \quad &\geq \ell(x_0) - \ell(x) + \ell_m(x) - \ell_m(x_m) + \sum_{i=0}^{m-1}[\ell_i(x_{i+1}) - \ell_i(x_i)] \\ &> \ell(x_0) - \ell(x) - \gamma. \end{aligned}$$

Since $\gamma > f(x_0) - f(x)$ was arbitrary and $x_0 \in Dom(\partial_L f)$, so, as $\alpha \longrightarrow (f(x_0) - f(x))$, we obtain

$$\varepsilon + f(x_0) - f(x) \geq \ell(x_0) - \ell(x), \text{ for all } x_0 \in Dom(\partial_L f). \tag{23}$$

Since $Dom(\partial_L f) = dom f$, In view of (23), we conclude $\ell \in \partial_{L,\varepsilon} f(x)$, which completes the proof. ∎

In the following, by Theorem 4.3, we give a characterization of global ε-minimizers of $f - g$, where f and g are H_L-convex functions.

Theorem 4.4. *Let $f, g : X \longrightarrow \mathbb{R}_{+\infty}$ be H_L-convex functions such that $Dom(\partial_L f) = dom f$ and $\partial_L f$ admits a unique (up to an additive constant) L-antiderivative. Let $y_0 \in Dom(\partial_L g)$ and $\ell'_0 \in \partial_L g(y_0)$. Then, y_0 is a global ε-minimum of $f - g$ if and only if*

$$\begin{aligned}
&\sum_{i=0}^{r}[\ell_i(x_i) - \ell_i(x_{i+1})] + \sum_{j=0}^{s}[\ell'_j(y_j) - \ell'_j(y_{j+1})] \\
&+\ell_r(x_0) - \ell_r(y_0) + \ell'_s(y_0) - \ell'_s(x_0) \geq -\varepsilon, \\
&\textit{for all } (x_i, \ell_i) \in G(\partial_L f),\ (i = 0, ..., r),\ x_{r+1} := x_0, \\
&\textit{for all } (y_j, \ell'_j) \in G(\partial_L g),\ (j = 0, ..., s),\ y_{s+1} := y_0.
\end{aligned} \tag{24}$$

Proof: Suppose that y_0 is a global ε-minimum of $f - g$. Let $(x_i, \ell_i) \in G(\partial_L f)$ $(i = 0, ..., r)$ and $(y_j, \ell'_j) \in G(\partial_L g)$ $(j = 0, ..., s)$ be arbitrary. Hence, the following inequalities holds,

$$\begin{aligned}
&f(x_0) - g(x_0) \geq f(y_0) - g(y_0) - \varepsilon, \\
&f(x_{i+1}) - f(x_i) \geq \ell_i(x_{i+1}) - \ell_i(x_i)\ (i = 0, ..., r-1), \\
&f(y_0) - f(x_r) \geq \ell_r(y_0) - \ell_r(x_r) = \ell_r(y_0) - \ell_r(x_{r+1}) + \ell_r(x_0) - \ell_r(x_r), \\
&g(y_{j+1}) - g(y_j) \geq \ell'_j(y_{j+1}) - \ell'_j(y_j)\ (j = 0, ..., s-1), \\
&g(x_0) - g(y_s) \geq \ell'_s(x_0) - \ell'(y_s) = \ell'_s(x_0) - \ell'_s(x_{s+1}) + \ell'_s(x_0) - \ell'(y_s).
\end{aligned}$$

By adding the above inequalities, we get (24). Conversely, suppose that (24) holds. We show that y_0 is a global ε-minimum of $f - g$. In view of Corollary 3.3, it is enough to show that $\partial_L g(y_0) \subseteq \partial_{L,\varepsilon} f(y_0)$. Let $\ell' \in \partial_L g(y_0)$ be arbitrary. Assume that $(x_i, \ell_i) \in G(\partial_L f)$ $(i = 1, ..., r)$ is arbitrary and $x_{r+1} := x_0$. Set $(y_j, \ell'_j) := (y_0, \ell'_0)$ $(j = 1, ..., s)$ in (24), so, it follows from (24) that

$$\sum_{i=0}^{r}[\ell_i(x_i) - \ell_i(x_{i+1})] + \ell_r(x_0) - \ell_r(y_0) + \ell'_0(y_0) - \ell'_0(x_0) \geq -\varepsilon.$$

Therefore,

$$\sum_{i=0}^{r-1}[\ell_i(x_i) - \ell_i(x_{i+1})] + \ell_r(x_r) - \ell_r(y_0) + \ell'_0(y_0) - \ell'_0(x_0) \geq -\varepsilon. \tag{25}$$

Hence, (25) and Theorem 4.3 implies that $\ell' \in \partial_{L,\varepsilon} f(y_0)$. ■

5. ε-Minimum of the Difference of Increasing and Positively Homogeneous Functions

In this section, as an application of the obtained results, we give a necessary and sufficient condition for global ε- minimum of the difference of increasing and positively homogeneous (IPH) functions (also, see [13]). We first state some properties of IPH functions over topological vector spaces, which were obtained in [12]. Moreover, for more details, see [10, 12].

In the sequel, let X be a real topological vector space. We assume that X is equipped with a closed convex pointed cone $S \subseteq X$ (the latter means that $S \cap (-S) = \{0\}$). We say $x \leq y$ or $y \geq x$ if and only if $y - x \in S$.

Recall that [15] the function $p : X \longrightarrow [-\infty, +\infty]$ is IPH if p is an increasing and positively homogeneous function (the latter means that $p(\lambda x) = \lambda p(x)$ for all $x \in X$ and all $\lambda > 0$). We say that p is increasing if $(x \leq y \Longrightarrow p(x) \leq p(y))$.

Now, consider the function $l : X \times X \longrightarrow [0, +\infty]$ is defined by

$$l(x, y) := max\{\lambda \geq 0 : \lambda y \leq x\},\ \text{for all } x, y \in X,$$

(with the convention $\max\emptyset := 0$). The properties of the function l have been given in [12, Proposition 3.1].

Define $L_S := \{l_y : y \in X \setminus (-S)\}$, where $l_y(x) := l(x, y)$ for all $x \in X$ and all $y \in X$. Note that l_y is an IPH function for each $y \in X$, and every non-negative IPH function is L_S-convex [12]. Consider the set $L := L_S \cup \{0\}$. Clearly, if $p : X \longrightarrow [0, +\infty]$ is an IPH function, then, p is also L-convex. Moreover, one has [12]

$$\text{supp}\ (p, L) = \{l_y \in L : p(y) \geq 1\}. \tag{26}$$

If $p : X \longrightarrow [0, +\infty]$ is an IPH function, it is easy to see that $p^*_L(l_y) = \delta_{\text{supp}\ (p,L)}(l_y)$ for all $l_y \in L$, where $\delta_{\text{supp}\ (p,L)}$ is the indicator function of supp (p, L) and p^*_L is the L-conjugate of p. For a subset U of L, the indicator function of U is denoted by δ_U.

Let $x_0 \in X$ and $p(x_0) \neq 0, +\infty$. In view of (2), we can characterize $\partial_L p(x_0)$ as follows,

$$\begin{aligned} \partial_L p(x_0) &= \{l_y \in L : p(x_0) + p^*_L(l_y) = l_y(x_0)\} \\ &= \{l_y \in \text{supp}\ (p, L) : p(x_0) = l_y(x_0)\}. \end{aligned} \tag{27}$$

Also, the following characterization has been proved in [10, Theorem 3.3].

$$\partial_L p(x_0) = \{l_y \in L_S : l_y(x_0) = p(x_0),\ p(y) = 1\} \cup \{0\}. \tag{28}$$

Moreover, if $x_0 \in domp$ and $\varepsilon \geq 0$, in view of (3), we have

$$\begin{aligned} \partial_{L,\varepsilon} p(x_0) &= \{l_y \in L : p(x_0) + p^*_L(l_y) \leq l_y(x_0) + \varepsilon\} \\ &= \{l_y \in \text{supp}\ (p, L) : p(x_0) \leq l_y(x_0) + \varepsilon\}. \end{aligned} \tag{29}$$

Remark 5.1. *Let $p : X \longrightarrow [0, +\infty)$ be an IPH function. Then, $Dom(\partial_L p) = X$. Indeed, assume that $p(x) \neq 0$ (note that in this case, $p(x) > 0$, and hence, $x \notin -S$). Thus, by the properties of the function l and (27), one has $l_{\frac{x}{p(x)}} \in \partial_L p(x)$. If $p(x) = 0$, then, $0 \in \partial_L p(x)$. Also, since p is L-convex, it follows from [15, Proposition 7.15] that p is H_L-convex, so, $\partial_{L,\varepsilon} p(x_0) \neq \emptyset$ for all $x_0 \in dom(p)$ and all $\varepsilon \geq 0$.*

Theorem 5.1. *[11, Theorem 3.1] Let $p : X \longrightarrow [0, +\infty)$ be an IPH function. Then, $\partial_L p$ is a maximal L-monotone operator.*

Proposition 5.1. *Let $p : X \longrightarrow [0, +\infty)$ be an IPH function. Then, $\partial_L p$ admits a unique (up to an additive constant) IPH L-antiderivative.*

Proof: Let $f : X \longrightarrow [0, +\infty)$ be an IPH L-antiderivative of $\partial_L p$. Thus, $G(\partial_L p) \subseteq G(\partial_L f)$, and so, by Theorem 5.1, we have $G(\partial_L p) = G(\partial_L f)$. Let $x \in X$ be arbitrary. If $l_y \in \partial_L p(x)$, then, by (28) one has $l_y(x) = p(x)$. On the other hand, since $\partial_L p(x) = \partial_L f(x)$, again, it follows from (28) that $l_y(x) = f(x)$. Therefore, $p(x) = f(x)$ for all $x \in X$. Since p is an IPH L-antiderivative of $\partial_L p$ and $\partial_L p(x) = \partial_L (p + \alpha)(x)$ for all $x \in X$ and all $\alpha \in \mathbb{R}$, we conclude that $\partial_L p$ admits a unique (up to an additive constant) IPH L-antiderivative. ■

Theorem 5.2. *Let $p, q \longrightarrow [0, +\infty)$ be IPH functions. Let $\varepsilon \geq 0$ and $x \in X$. Consider the following systems of inequalities:*

$$\begin{cases} l_y(x) = q(x), \\ q(y) = 1,\ y \in X. \end{cases} \quad (Q_q) \qquad\qquad \begin{cases} p(x) \leq l_y(x) + \varepsilon, \\ p(y) \geq 1,\ y \in X. \end{cases} \quad (Q_p)$$

Then, x is a global ε-minimum of $p - q$ if and only if the solution set of (Q_p) contains the solution set of (Q_q).

Proof: In view of Corollary 3.3, x is a global ε-minimum of $p - q$ if and only if $\partial_L q(x) \subseteq \partial_{L,\varepsilon} p(x)$. The proof is complete if we apply (26), (28) and (29). ■

Theorem 5.3. *Let $p, q : X \longrightarrow [0, +\infty)$ be IPH functions. Let $y \in X$ and $\varepsilon \geq 0$ be given. Then, y is a global ε-minimum of $p - q$ if and only if*

$$p(x) + q(y) - [l_z(y) + l_t(x)] \geq -\varepsilon, \text{ for all } l_z \in \partial_L p(x) \text{ and all } l_t \in \partial_L q(y),\ (x \in X).$$

Proof: The result follows from (27), Remark 5.1, Proposition 5.1 and Theorem 4.4. ■

We now give an example to illustrate Theorem 5.3.

Example 5.1. *Let $X = \mathbb{R}^2_{++} := \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$. Define the function $l : X \times X \longrightarrow [0, +\infty)$ by*

$$l(x, y) := \max\{\lambda \geq 0 : \lambda y \leq x\}, \text{ for all } x, y \in X.$$

Let $x, y \in X$. Define an order on X by:

$$x \leq y \Longleftrightarrow x_i \leq y_i,\ i = 1, 2.$$

It is clear that $'' \leq ''$ is a partial order on X. Now, for each $y \in X$, define the function $l_y : X \longrightarrow [0, +\infty)$ by $l_y(x) := l(x, y)$ for all $x \in X$. It is easy to see that l_y is an IPH function for each $y \in X$. Let

$$L := \{l_y : y \in X\}.$$

Note that in this case $S = X$. Now, let $y = (y_1, y_2) \in X$ be fixed and arbitrary. Then we have

$$\begin{aligned} l_y(x) &= \max\{\lambda \geq 0 : \lambda y \leq x\} \\ &= \max\{\lambda \geq 0 : \lambda y_i \leq x_i,\ i = 1, 2\} \\ &= \max\{\lambda \geq 0 : \lambda \leq \frac{x_i}{y_i},\ i = 1, 2\} \\ &= \max\{\lambda \geq 0 : \lambda \leq \min_{1\leq i\leq 2} \frac{x_i}{y_i}\} \\ &= \min_{1\leq i\leq 2} \frac{x_i}{y_i}, \text{ for all } x = (x_1, x_2) \in X. \end{aligned} \tag{30}$$

Define the functions $p, q : X \longrightarrow [0, +\infty)$ by

$$p(x) := \max_{1\leq i\leq 2} x_i, \text{ and } q(x) := \min_{1\leq i\leq 2} x_i, \text{ for all } x = (x_1, x_2) \in X.$$

Clearly, p and q are IPH functions. In view of (28) *and* (30)*, for each $y = (y_1, y_2) \in X$ and each IPH function $p : X \longrightarrow [0, +\infty)$, we have*

$$\partial_L p(x) = \{l_y \in L : \max_{1\leq i\leq 2} y_i = 1,\ \min_{1\leq i\leq 2} \frac{x_i}{y_i} = \max_{1\leq i\leq 2} x_i\}, \text{ for all } x = (x_1, x_2) \in X. \tag{31}$$

Now, let $\varepsilon \geq 0$, and let $y_0 = (1, 1) \in X$. Therefore, by (30), (31) *and the definitions p and q, we obtain*

$$\begin{aligned} p(x) + q(y_0) - [l_z(y_0) + l_t(x)] &= \max_{1\leq i\leq 2} x_i + \min_{1\leq i\leq 2} y_{0_i} - [\min_{1\leq i\leq 2} \frac{y_{0_i}}{z_i} + \min_{1\leq i\leq 2} \frac{x_i}{t_i}] \\ &= \max_{1\leq i\leq 2} x_i + 1 - [\max_{1\leq i\leq 2} y_{0_i} + \max_{1\leq i\leq 2} x_i] \\ &= \max_{1\leq i\leq 2} x_i + 1 - [1 + \max_{1\leq i\leq 2} x_i] \\ &= 0 \geq -\varepsilon, \end{aligned}$$

for all $l_z \in \partial_L p(x)$ and all $l_t \in \partial_L q(y_0)$ with $x = (x_1, x_2), z = (z_1, z_2), t = (t_1, t_2) \in X$. Then

$$p(x) + q(y_0) - [l_z(y_0) + l_t(x)] \geq -\varepsilon,$$
$$\text{for all } l_z \in \partial_L p(x) \text{ and all } l_t \in \partial_L q(y_0),\ (x = (x_1, x_2), z = (z_1, z_2), t = (t_1, t_2) \in X).$$

Thus, by Theorem 5.3, $y_0 = (1, 1) \in X$ is a global ε-minimum of the function $p - q$. It is worth noting that each point $(r, r) \in X$ $(r > 0)$ is a global ε-minimum of the function $p - q$.

6. Conclusion

This paper presented various characterizations for global ε-minimum of the difference of two abstract convex functions. Next, characterizations of maximal elements of abstract convex functions were given. By using the abstract Rockafellar's antiderivative, the abstract ε-subdifferential of abstract convex functions in terms of their abstract subdifferentials presented. Finally, as an application, a necessary and sufficient condition for global ε-minimum of the difference of two increasing and positively homogeneous (IPH) functions obtained. These results have many applications in microeconomic analysis.

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