Filomat 33:8 (2019), 2447–2455
https://doi.org/10.2298/FIL1908447L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Grüss-Landau Inequalities for Elementary Operators and Inner Product Type Transformers in Q and Q* Norm Ideals of Compact Operators

Milan Lazarević[a]

[a] *University of Belgrade, Department of Mathematics, Studentski trg 16, P.O.box 550, 11000 Belgrade, Serbia*

Abstract. For a probability measure μ on Ω and square integrable (Hilbert space) operator valued functions $\{A_t^*\}_{t\in\Omega}$, $\{B_t\}_{t\in\Omega}$, we prove Grüss-Landau type operator inequality for inner product type transformers

$$\left|\int_\Omega A_t X B_t \, d\mu(t) - \int_\Omega A_t \, d\mu(t) X \int_\Omega B_t \, d\mu(t)\right|^{2\eta} \leqslant \left\|\int_\Omega A_t A_t^* \, d\mu(t) - \left|\int_\Omega A_t^* \, d\mu(t)\right|^2\right\|^\eta \left(\int_\Omega B_t^* X^* X B_t \, d\mu(t) - \left|X \int_\Omega B_t \, d\mu(t)\right|^2\right)^\eta,$$

for all $X \in \mathcal{B}(\mathcal{H})$ and for all $\eta \in [0,1]$.

Let $p \geqslant 2$, Φ to be a symmetrically norming (s.n.) function, $\Phi^{(p)}$ to be its p-modification, $\Phi^{(p)*}$ is a s.n. function adjoint to $\Phi^{(p)}$ and $\|\cdot\|_{\Phi^{(p)*}}$ to be a norm on its associated ideal $\mathcal{C}_{\Phi^{(p)*}}(\mathcal{H})$ of compact operators. If $X \in \mathcal{C}_{\Phi^{(p)*}}(\mathcal{H})$ and $\{\alpha_n\}_{n=1}^\infty$ is a sequence in $(0,1]$, such that $\sum_{n=1}^\infty \alpha_n = 1$ and $\sum_{n=1}^\infty \|\alpha_n^{-1/2} A_n f\|^2 + \|\alpha_n^{-1/2} B_n^* f\|^2 < +\infty$ for some families $\{A_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ of bounded operators on Hilbert space $\mathcal{H}$ and for all $f \in \mathcal{H}$, then

$$\left\|\sum_{n=1}^\infty \alpha_n^{-1} A_n X B_n - \sum_{n=1}^\infty A_n X \sum_{n=1}^\infty B_n\right\|_{\Phi^{(p)*}} \leqslant \left\|\sqrt{\sum_{n=1}^\infty \alpha_n^{-1}|A_n|^2 - \left|\sum_{n=1}^\infty A_n\right|^2}\, X \sqrt{\sum_{n=1}^\infty \alpha_n^{-1}|B_n^*|^2 - \left|\sum_{n=1}^\infty B_n^*\right|^2}\right\|_{\Phi^{(p)*}},$$

if at least one of those operator families consists of mutually commuting normal operators.

The related Grüss-Landau type $\|\cdot\|_{\Phi^{(p)}}$ norm inequalities for inner product type transformers are also provided.

1. Introduction

A well known Grüss-Landau inequality says that for a probability measure μ on Ω and measurable complex functions f and g on Ω

$$\left|\int_\Omega fg \, d\mu - \int_\Omega f \, d\mu \int_\Omega g \, d\mu\right| \leqslant \sqrt{\int_\Omega |f|^2 \, d\mu - \left|\int_\Omega f \, d\mu\right|^2} \sqrt{\int_\Omega |g|^2 \, d\mu - \left|\int_\Omega g \, d\mu\right|^2} =: R. \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 47A30, 47A63; Secondary 47B10, 47B15, 46L05
Keywords. Unitarily invariant norms, symmetrically norming functions, p-modified norms and their dual norms
Received: 17 August 2018; Accepted: 21 October 2018
Communicated by Dragan S. Djordjević
Research supported by MPNTR grant No. 174017, Serbia
Email address: lazarevic@matf.bg.ac.rs (Milan Lazarević)

Specially, if f and g are real bounded functions on Ω, then the rightmost side of (1) can be further estimated by

$$R \leqslant \left(\left(\Phi - \int_\Omega f\,d\mu\right)\left(\int_\Omega f\,d\mu - \varphi\right)\left(\Gamma - \int_\Omega g\,d\mu\right)\left(\int_\Omega g\,d\mu - \gamma\right)\right)^{1/2} \leqslant \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma), \tag{2}$$

where $\varphi := \inf\operatorname{ess}_\Omega f (\overset{\text{def}}{=} -\sup\operatorname{ess}_\Omega(-f))$, $\Phi := \sup\operatorname{ess}_\Omega f$, $\gamma := \inf\operatorname{ess}_\Omega g$ and $\Gamma := \sup\operatorname{ess}_\Omega g$.

The first special case of inequalities (1) and (2), for the normalized Lebesgue measure on $\Omega := [a,b]$ (i.e. $d\mu(t) := \frac{dt}{b-a}$), was essentially proved by G. Grüss in [4]. E. Landau in his paper [11] reformulated those results in the above presented form, also providing an explicit application of Cauchy-Schwarz inequality to prove (1). An alternative application of Cauchy-Schwarz inequality, based on Korkine type identities given in [13, (7.1) p. 243], was used in [13] to prove (1), which immediately implies (2). A refined form of (1) was given in [9, Lemma 2.1] in the case of finite set Ω, including the case of operator valued functions f and g, while some other generalization of Grüss-Landau inequalities (1) and (2) where presented in [7, 8, 12] and the references therein.

Let $\mathcal{H}$ be a separable, complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}_\infty(\mathcal{H})$ denote the spaces of all bounded and all compact linear operators, respectively. Each "symmetric gauge"or "symmetrically norming" (s.n.) function Φ, defined on sequences of complex numbers, gives rise to a symmetric or a unitary invariant (u.i.) norm $\|\cdot\|_\Phi$ on operators. Basic examples of s.n. functions are trace s.n. function ℓ^1, defined by $\ell^1((\lambda_n)_{n=1}^\infty) \overset{\text{def}}{=} \sum_{n=1}^\infty |\lambda_n|$ and ($\mathcal{B}(\mathcal{H})$ or) operator norm ℓ^∞ defined by $\ell^\infty((\lambda_n)_{n=1}^\infty) \overset{\text{def}}{=} \sup_{n\in\mathbb{N}} |\lambda_n|$. Any such norm is unitarily invariant (u.i.) and it is defined on the naturally associated norm ideal $\mathcal{C}_\Phi(\mathcal{H})$ of $\mathcal{C}_\infty(\mathcal{H})$. If Φ is a s.n. function, then its adjoint s.n. function will be denoted by Φ^*. For any $p > 0$ a s.n. function Φ could be p-modified and its modification $\Phi^{(p)}$ represent a new s.n. functions (only) for $p \geqslant 1$. The proof of the triangle inequality for norms induced by this type of s.n. functions, and other properties, can be seen in preliminary section in [6]. Also, the corresponding ideals of compact operators will be denoted by $\mathcal{C}_{\Phi^{(p)}}(\mathcal{H})$ and its dual by $\mathcal{C}_{\Phi^{(p)*}}(\mathcal{H})$.

Schatten-von Neumann trace classes $\mathcal{C}_p(\mathcal{H}) \overset{\text{def}}{=} \mathcal{C}_{\ell^{(p)}}(\mathcal{H})$ represent classical examples of norm ideals associated to degree p-modified (i.e. its s.n. function ℓ^1) norms. $\mathcal{C}_1(\mathcal{H})$ is also known as the class of nuclear operators, while $\mathcal{C}_2(\mathcal{H})$ is known as the Hilbert-Schmidt class. Norm in $\mathcal{C}_p(\mathcal{H})$ will be denoted simply by $\|\cdot\|_p$. For $p \geqslant 2$, all norms $\|\cdot\|_{\Phi^{(p)}}$ are also known as Q-norms, as $\Phi^{(p)} = (\Phi^{(\frac{p}{2})})^{(2)}$ and $\Phi^{(\frac{p}{2})}$ is also a s.n. function, while its dual norms $\|\cdot\|_{\Phi^{(p)*}}$ are commonly known as Q*-norms. Norm dual to some classes of p-modified ones are characterized in [6, Th. 2.1].

If $(\Omega, \mathfrak{M}, \mu)$ is a space Ω with a measure μ on σ-algebra $\mathfrak{M}$, then we will refer to a function $A\colon \Omega \to \mathcal{B}(\mathcal{H})\colon t \mapsto A_t$ as to a weakly*-measurable if $t \mapsto \langle A_t g, h\rangle$ is a measurable for all $g, h \in \mathcal{H}$. If, in addition, those functions are integrable, then there is the unique (known as Gel'fand or weak*-integral and denoted by $\int_\Omega A_t\,d\mu(t)$) operator in $\mathcal{B}(\mathcal{H})$, satisfying

$$\left\langle \int_\Omega A_t\,d\mu(t) h, k\right\rangle = \int_\Omega \langle A_t h, k\rangle\,d\mu(t) \qquad \text{for all } h, k \in \mathcal{H}. \tag{3}$$

Thus, it also complies with the definition of Pettis integral. For a more complete account about weak*-integrals the reader is referred to [2, p.53], [5, p.320] and [7, Lemma 1.2]. For every $h \in \mathcal{H}$, the function $t \mapsto \|A_t h\|$ is also measurable, and, if additionally $\int_\Omega \|A_t h\|^2\,d\mu(t) < +\infty$ for all $h \in \mathcal{H}$, then there exists weak*-integral $\int_\Omega A_t^* A_t\,d\mu(t) \in \mathcal{B}(\mathcal{H})$, satisfying $\left\langle \int_\Omega A_t^* A_t\,d\mu(t)\,h, h\right\rangle = \int_\Omega \|A_t h\|^2\,d\mu(t)$ for all $h \in \mathcal{H}$, as shown in [5, Ex. 2]. Such families $\{A_t\}_{t\in\Omega}$ will be simple called square integrable (s.i.). By $L^2_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ will be denoted the Banach space of all weakly*-measurable functions $A\colon \Omega \to \mathcal{B}(\mathcal{H})\colon t \mapsto A_t$ such that $\int_\Omega \|A_t h\|^2\,d\mu(t) < +\infty$ for all $h \in \mathcal{H}$, endowed by the norm $\|A\|_{L^2(\mu,\mathcal{B}(\mathcal{H}))} \overset{\text{def}}{=} \left\|\int_\Omega A_t^* A_t\,d\mu(t)\right\|^{1/2}$ for any $A \in L^2_G(\Omega, \mu, \mathcal{B}(\mathcal{H}))$. For a more general class of norms and its associated Banach spaces of weakly*-measurable operator valued (o.v.) functions see [5, Th. 2.1].

In a discrete case, a family $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{B}(\mathcal{H})$ will be called a strongly square summable (s.s.s.) if $\sum_{n=1}^{\infty}\|A_nh\|^2<+\infty$ for all $h\in\mathcal{H}$. If a family $\{A_t\}_{t\in\Omega}$ (resp. $\{A_n\}_{n=1}^{\infty}$) consists of mutually commuting normal operators, i.e., those satisfying $A_s^*A_t=A_tA_s^*$ for all $s,t\in\Omega$ (resp. $A_mA_n^*=A_nA_m^*$ for all $m,n\in\mathbb{N}$), we will refer to it as to a m.c.n.o. family. The terminology used in this paper is closely related to the that one used in [10], and a more detailed introduction therein may contribute to the comfort of the reader of this article itself. For a more complete account on the theory of norm ideals, the reader is referred to [1, 3, 14].

We also need to emphasize that throughout this paper we will treat (address to) every unnumbered line in a multline formula as (to) a part of the consequent numbered one.

2. Main results

The next theorem extends operator Grüss-Landau inequality (2.14) in [9, Cor. 2.1] from elementary operators to the settings of i.p.t. transformers, by taking $\Omega:=\{1,\dots,N\}$ and $\mu(\{n\}):=\alpha_n$ for $n=1,\dots,N$, with $\sum_{n=1}^{N}\alpha_n=1$ for some $N\in\mathbb{N}$ and $\alpha_1,\dots,\alpha_N\in(0,1]$.

Theorem 2.1. *Let μ be a probability measure on Ω, $\{A_t^*\}_{t\in\Omega}$, $\{B_t\}_{t\in\Omega}$ to be in $L^2_G(\Omega,\mu,\mathcal{B}(\mathcal{H}))$, $f,g\in\mathcal{H}$, $X\in\mathcal{B}(\mathcal{H})$ and $\eta\in[0,1]$. Then*

$$\left|\int_\Omega\langle A_tXB_tf,g\rangle\,d\mu(t)-\left\langle\int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)f,g\right\rangle\right|^2$$
$$\leqslant\left(\int_\Omega\langle A_tA_t^*g,g\rangle\,d\mu(t)-\left\langle\left|\int_\Omega A_t^*\,d\mu(t)\right|^2g,g\right\rangle\right)\left(\int_\Omega\langle B_t^*X^*XB_tf,f\rangle\,d\mu(t)-\left\langle\left|X\int_\Omega B_t\,d\mu(t)\right|^2f,f\right\rangle\right),\tag{4}$$

$$\left|\int_\Omega A_tXB_t\,d\mu(t)-\int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)\right|^{2\eta}$$
$$\leqslant\left\|\int_\Omega A_tA_t^*\,d\mu(t)-\left|\int_\Omega A_t^*\,d\mu(t)\right|^2\right\|^\eta\left(\int_\Omega B_t^*X^*XB_t\,d\mu(t)-\left|X\int_\Omega B_t\,d\mu(t)\right|^2\right)^\eta,\tag{5}$$

$$\left|\left(\int_\Omega A_tXB_t\,d\mu(t)-\int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)\right)^*\right|^{2\eta}$$
$$\leqslant\left\|\int_\Omega B_t^*B_t\,d\mu(t)-\left|\int_\Omega B_t\,d\mu(t)\right|^2\right\|^\eta\left(\int_\Omega A_tXX^*A_t^*\,d\mu(t)-\left|X^*\int_\Omega A_t^*\,d\mu(t)\right|^2\right)^\eta.\tag{6}$$

Proof. To prove (4), we note that

$$\begin{vmatrix}\int_\Omega\langle A_tA_t^*g,g\rangle\,d\mu(t)-\left\langle\left|\int_\Omega A_t^*\,d\mu(t)\right|^2g,g\right\rangle & \int_\Omega\langle A_tXB_tf,g\rangle\,d\mu(t)-\left\langle\int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)f,g\right\rangle\\ \int_\Omega\langle(A_tXB_t)^*g,f\rangle\,d\mu(t)-\left\langle\left(\int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)\right)^*g,f\right\rangle & \int_\Omega\langle B_t^*X^*XB_tf,f\rangle\,d\mu(t)-\left\langle\left|X\int_\Omega B_t\,d\mu(t)\right|^2f,f\right\rangle\end{vmatrix}\geqslant 0.$$

This is a direct consequence of the fact that

$$\begin{aligned}&\begin{bmatrix}\int_\Omega A_tA_t^*\,d\mu(t)-\left|\int_\Omega A_t^*\,d\mu(t)\right|^2 & \int_\Omega A_tXB_t\,d\mu(t)-\int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)\\ \int_\Omega(A_tXB_t)^*\,d\mu(t)-\left(\int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)\right)^* & \int_\Omega B_t^*X^*XB_t\,d\mu(t)-\left|X\int_\Omega B_t\,d\mu(t)\right|^2\end{bmatrix}\\
&=\frac12\int_{\Omega^2}\begin{bmatrix}(A_s-A_t)(A_s-A_t)^* & (A_s-A_t)X(B_s-B_t)\\ (B_s-B_t)^*X^*(A_s-A_t)^* & (B_s-B_t)^*X^*X(B_s-B_t)\end{bmatrix}d(\mu\times\mu)(s,t)\\
&=\frac12\int_{\Omega^2}\begin{bmatrix}A_s-A_t & 0\\ (XB_s-XB_t)^* & 0\end{bmatrix}\begin{bmatrix}(A_s-A_t)^* & XB_s-XB_t\\ 0 & 0\end{bmatrix}d(\mu\times\mu)(s,t)\\
&=\frac12\int_{\Omega^2}\left|\begin{bmatrix}(A_s-A_t)^* & XB_s-XB_t\\ 0 & 0\end{bmatrix}\right|^2d(\mu\times\mu)(s,t)\geqslant 0\end{aligned}\tag{7}$$

on $\mathcal{H}\oplus\mathcal{H}$, if we take $\mathcal{C}_{s,t} := (A_s - A_t)/\sqrt{2}$, $\mathcal{D}_{s,t} := (B_s - B_t)/\sqrt{2}$, $\mathcal{X}_{s,t} := X$ for all $s, t \in \Omega$ and $\theta := 1$ in [5, Th. 3.1(a)]. To justify equality in (7), we rely on the following Korkine type identity for i.t.p. transformers:

$$\int_\Omega A_t X B_t \,d\mu(t) - \int_\Omega A_t \,d\mu(t) X \int_\Omega B_t \,d\mu(t) = \int_\Omega d\mu(s) \int_\Omega A_t X B_t \,d\mu(t) - \int_\Omega\int_\Omega A_t X B_s \,d\mu(s) d\mu(t)$$
$$= \frac{1}{2}\int_{\Omega^2} (A_s - A_t) X (B_s - B_t) d(\mu\times\mu)(s,t), \tag{8}$$

presented in [7, (2.2)]. Specially, by taking $X := I$ and $B_t := A_t^*$ (resp. X^*X instead of X and $A_t := B_t^*$) for $t \in \Omega$ in (8), we get $\int_\Omega A_t A_t^* d\mu(t) - \int_\Omega A_t \,d\mu(t) \int_\Omega A_t^* \,d\mu(t) = \frac{1}{2}\int_{\Omega^2}(A_s - A_t)(A_s - A_t)^* d(\mu\times\mu)(s,t)$ (resp. $\int_\Omega B_t^* X^* X B_t d\mu(t) - \int_\Omega B_t^* \,d\mu(t) X^* X \int_\Omega B_t \,d\mu(t) = \frac{1}{2}\int_{\Omega^2}(B_s - B_t)^* X^* X (B_s - B_t) d(\mu\times\mu)(s,t)$), which based on (8), implies (7) and completes the proof of (4).

First, to prove the case $\eta := 1$ in (5), we start from (4) to get

$$\begin{aligned}
&\left|\left\langle\left(\int_\Omega A_t X B_t \,d\mu(t) - \int_\Omega A_t \,d\mu(t) X \int_\Omega B_t \,d\mu(t)\right) f, g\right\rangle\right|^2 \\
&\quad\leqslant \left\langle\left(\int_\Omega A_t A_t^* \,d\mu(t) - \left|\int_\Omega A_t^* \,d\mu(t)\right|^2\right) g, g\right\rangle \left\langle\left(\int_\Omega B_t^* X^* X B_t \,d\mu(t) - \left|X\int_\Omega B_t \,d\mu(t)\right|^2\right) f, f\right\rangle \\
&\quad\leqslant \left\|\int_\Omega A_t A_t^* \,d\mu(t) - \left|\int_\Omega A_t^* \,d\mu(t)\right|^2\right\| \left\langle\left(\int_\Omega B_t^* X^* X B_t \,d\mu(t) - \left|X\int_\Omega B_t \,d\mu(t)\right|^2\right) f, f\right\rangle \|g\|^2,
\end{aligned} \tag{9}$$

where we used the very definition (3) of weak* (or Gel'fand) integral to estimate the middle expression in (4), to justify the last inequality in (9). Now, by taking $g := \left(\int_\Omega A_t X B_t \,d\mu(t) - \int_\Omega A_t \,d\mu(t) X \int_\Omega B_t \,d\mu(t)\right) f$, we actually obtain (5) for $\eta := 1$. So it suffices to consider the remaining case $\eta \in (0,1)$, which follows immediately from the operator monotonicity of the function $t \mapsto t^\eta$ on $[0,\infty)$, when applied to the already proven case $\eta := 1$.

As $\left(\int_\Omega A_t X B_t \,d\mu(t) - \int_\Omega A_t \,d\mu(t) X \int_\Omega B_t \,d\mu(t)\right)^* = \int_\Omega B_t^* X^* A_t^* \,d\mu(t) - \int_\Omega B_t^* \,d\mu(t) X^* \int_\Omega A_t^* \,d\mu(t)$, it is suffices to take $A_t := B_t^*$, $B_t := A_t^*$, for $t \in \Omega$, and X^* instead of X in (5), to get (6). □

In the case of bounded self-adjoint families $\{A_t\}_{t\in\Omega}$ and $\{B_t\}_{t\in\Omega}$, inequality (5) can be upgraded to the more widely known form of Grüss-Landau inequality.

Theorem 2.2. *Let under conditions of Theorem 2.1, $\{A_t\}_{t\in\Omega}$ and $\{B_t\}_{t\in\Omega}$ be families of self-adjoint operators, which satisfy $\varphi \leqslant A_t \leqslant \Phi$ for some self-adjoint $\varphi, \Phi \in \mathcal{B}(\mathcal{H})$ commuting with A_t for every $t \in \Omega$ and satisfying $\varphi\Phi = \Phi\varphi$, as well as $\gamma \leqslant B_t \leqslant \Gamma$ for some self-adjoint $\gamma, \Gamma \in \mathcal{B}(\mathcal{H})$ commuting with B_t for every $t \in \Omega$ and satisfying $\gamma\Gamma = \Gamma\gamma$. Then*

$$\begin{aligned}
&\left|\int_\Omega A_t B_t \,d\mu(t) - \int_\Omega A_t \,d\mu(t) \int_\Omega B_t \,d\mu(t)\right|^{2\eta} \\
&\quad\leqslant \left\|\int_\Omega A_t^2 \,d\mu(t) - \left|\int_\Omega A_t \,d\mu(t)\right|^2\right\|^\eta \left(\int_\Omega B_t^2 \,d\mu(t) - \left|\int_\Omega B_t \,d\mu(t)\right|^2\right)^\eta && (10)\\
&\quad\leqslant \left\|\left(\Phi - \int_\Omega A_t \,d\mu(t)\right)\left(\int_\Omega A_t \,d\mu(t) - \varphi\right)\right\|^\eta \left(\Gamma - \int_\Omega B_t \,d\mu(t)\right)^\eta \left(\int_\Omega B_t \,d\mu(t) - \gamma\right)^\eta \\
&\quad\leqslant \frac{1}{4^{2\eta}}\|\Phi - \varphi\|^{2\eta}(\Gamma - \gamma)^{2\eta}. && (11)
\end{aligned}$$

Proof. We start the proof by the following identities:

$$\int_\Omega A_t^2\,d\mu(t) - \left|\int_\Omega A_t\,d\mu(t)\right|^2 = \frac{1}{2}\int_{\Omega^2}(A_s - A_t)^2 d(\mu\times\mu)(s,t) \tag{12}$$

$$= \frac{1}{2}\int_{\Omega^2}\left(\left(A_s - \frac{\Phi+\varphi}{2}\right) - \left(A_t - \frac{\Phi+\varphi}{2}\right)\right)^2 d(\mu\times\mu)(s,t)$$

$$= \int_\Omega\left(A_t - \frac{\Phi+\varphi}{2}\right)^2 d\mu(t) - \left(\int_\Omega\left(A_t - \frac{\Phi+\varphi}{2}\right)d\mu(t)\right)^2 \tag{13}$$

$$= \int_\Omega (A_t-\Phi)(A_t-\varphi)\,d\mu(t) + \left(\frac{\Phi-\varphi}{2}\right)^2 - \int_\Omega (A_t-\Phi)\,d\mu(t)\int_\Omega (A_t-\varphi)\,d\mu(t) - \left(\frac{\Phi-\varphi}{2}\right)^2 \tag{14}$$

$$= \left(\Phi - \int_\Omega A_t\,d\mu(t)\right)\left(\int_\Omega A_t\,d\mu(t) - \varphi\right) - \int_\Omega(\Phi - A_t)(A_t-\varphi)\,d\mu(t), \tag{15}$$

where (12) and the second equality in (13) are based on Korkine type equality (8) applied this time on the family $\left\{A_t - \frac{\Phi+\varphi}{2}\right\}_{t\in\Omega}$ instead of $\{A_t\}_{t\in\Omega}$, while (14) and (15) checks directly. As $\Phi - A_t$ and $A_t - \varphi$ are positive, mutually commuting operators, then $(\Phi - A_t)(A_t - \varphi) \geqslant 0$ and consequently $\int_\Omega(\Phi - A_t)(A_t - \varphi)\,d\mu(t) \geqslant 0$. Therefore, a straightforward calculation shows

$$\int_\Omega A_t^2\,d\mu(t) - \left|\int_\Omega A_t\,d\mu(t)\right|^2 \leqslant \left(\Phi - \int_\Omega A_t\,d\mu(t)\right)\left(\int_\Omega A_t\,d\mu(t) - \varphi\right)$$
$$= -\left(\int_\Omega\left(A_t - \frac{\Phi+\varphi}{2}\right)d\mu(t)\right)^2 + \left(\frac{\Phi-\varphi}{2}\right)^2 \leqslant \frac{(\Phi-\varphi)^2}{4} \leqslant \frac{\|\Phi-\varphi\|^2}{4}I,$$

and similarly,

$$\int_\Omega B_t^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2 \leqslant \left(\Gamma - \int_\Omega B_t\,d\mu(t)\right)\left(\int_\Omega B_t\,d\mu(t) - \gamma\right) \leqslant \frac{(\Gamma-\gamma)^2}{4}.$$

As (10) is just a special case $X := I$ in (5) of Theorem 2.1, the final inequalities in (11) follows by the monotonicity of operator norm on positive operators and operator monotonicity of function $t \mapsto t^\eta$ on $[0,\infty)$. □

Remark 2.3. *The commutativity requirement that φ and Φ (resp. γ and Γ) both commute with all A_t (resp. B_t) for $t \in \Omega$ and that $\varphi\Phi = \Phi\varphi$ (resp. $\gamma\Gamma = \Gamma\gamma$), in Theorem 2.2, is obviously satisfied if $\varphi, \Phi, \gamma, \Gamma \in \mathcal{B}(\mathcal{H})$ are of the form $\varphi := dI, \Phi := DI, \gamma := cI$ and $\Gamma := CI$ for some $c, C, d, D \in \mathbb{R}$.*

Now, we are in a position to complement [7, Th. 2.6].

Corollary 2.4. *If Φ is a s.n. function and $\theta > 0$, then, under conditions of Theorem 2.1, for all $X \in \mathfrak{C}_{\Phi^{(\theta)}}(\mathcal{H})$*

$$\left\|\int_\Omega A_tXB_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)\right\|_{\Phi^{(\theta)}}^2$$
$$\leqslant \left\|\int_\Omega A_tA_t^*\,d\mu(t) - \left|\int_\Omega A_t^*\,d\mu(t)\right|^2\right\| \left\|\int_\Omega B_t^*X^*XB_t\,d\mu(t) - \left|X\int_\Omega B_t\,d\mu(t)\right|^2\right\|_{\Phi^{(\theta/2)}}, \tag{16}$$

$$\left\|\int_\Omega A_tXB_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)\right\|_{\Phi^{(\theta)}}^2$$
$$\leqslant \left\|\int_\Omega B_t^*B_t\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right\| \left\|\int_\Omega A_tXX^*A_t^*\,d\mu(t) - \left|X^*\int_\Omega A_t^*\,d\mu(t)\right|^2\right\|_{\Phi^{(\theta/2)}}. \tag{17}$$

Proof. We will rely on the monotonicity of singular values combined by the monotonicity of all θ modifications of u.i. norms, which says that if $0 \leqslant A \leqslant B$ for $A, B \in \mathcal{C}_\Phi(\mathcal{H})$, then $s_n^\theta(A) \leqslant s_n^\theta(B)$ for all $n \in \mathbb{N}$, as well as $\|A\|_{\Phi^{(\theta)}} \leqslant \|B\|_{\Phi^{(\theta)}}$ for all $\theta > 0$. So (16) follows from the case $\eta := 1$ in (5), as

$$\left\|\int_\Omega A_t X B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t) X \int_\Omega B_t\,d\mu(t)\right\|_{\Phi^{(\theta)}}^2 = \left\|\left|\int_\Omega A_t X B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t) X \int_\Omega B_t\,d\mu(t)\right|^2\right\|_{\Phi^{(\theta/2)}}$$
$$\leqslant \left\|\int_\Omega A_t A_t^*\,d\mu(t) - \left|\int_\Omega A_t^*\,d\mu(t)\right|^2\right\| \left\|\int_\Omega B_t^* X^* X B_t\,d\mu(t) - \left|X\int_\Omega B_t\,d\mu(t)\right|^2\right\|_{\Phi^{(\theta/2)}}. \quad (18)$$

Here, the equality in (18) is based on the very definition of $\theta/2$-modification $\|\cdot\|_{\Phi^{(\theta/2)}}$ of the norm $\|\cdot\|_\Phi$. The proof of (17) follows immediately from the already proven inequality (16), as

$$\left\|\int_\Omega A_t X B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t) X \int_\Omega B_t\,d\mu(t)\right\|_{\Phi^{(\theta)}}^2 = \left\|\int_\Omega B_t^* X^* A_t^*\,d\mu(t) - \int_\Omega B_t^*\,d\mu(t) X^* \int_\Omega A_t^*\,d\mu(t)\right\|_{\Phi^{(\theta)}}^2$$
$$\leqslant \left\|\int_\Omega B_t^* B_t\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right\| \left\|\int_\Omega A_t X X^* A_t^*\,d\mu(t) - \left|X^*\int_\Omega A_t^*\,d\mu(t)\right|^2\right\|_{\Phi^{(\theta/2)}}. \quad (19)$$

Equality in (19) is a simple consequence of B* property of u.i. norms, combined with the weak* integral property $\left(\int_\Omega C_t\,d\mu(t)\right)^* = \int_\Omega C_t^*\,d\mu(t)$, for weak* integrable families $\{C_t\}_{t\in\Omega}$. □

Remark 2.5. *Previous Corollary 2.4 extends inequality (2.17) in [9, Th. 2.2] in the settings of i.t.p. transformers, when they act on ideals of compact operators. Namely, it is enough to take* $\theta := p$, $\Omega := \{1, \ldots, N\}$, $\mu(\{n\}) := \alpha_n$ *for* $n = 1, \ldots, N$, *where* $N \in \mathbb{N}$ *and* $\sum_{n=1}^N \alpha_n = 1$ *for some* $\alpha_n \in (0, 1]$, *to get inequality (2.17) in [9, Th. 2.2] from inequality (16).*

In the case of Hilbert-Schmidt norm (i.e. if $\Phi := \ell^1$ and $\theta := 2$), then the lefthand side in (16) and (17) can be written in the more transparent form.

Theorem 2.6. *Let* $X \in \mathcal{C}_2(\mathcal{H})$ *and* μ *to be a probability measure on* Ω. *If* $\{A_t^*\}_{t\in\Omega}$, $\{B_t^*\}_{t\in\Omega}$ *are in* $L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ *then*

$$\left\|\int_\Omega A_t X B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t) X \int_\Omega B_t\,d\mu(t)\right\|_2$$
$$\leqslant \left\|\int_\Omega |A_t^*|^2\,d\mu(t) - \left|\int_\Omega A_t^*\,d\mu(t)\right|^2\right\|^{1/2} \left\|X\sqrt{\int_\Omega |B_t^*|^2\,d\mu(t) - \left|\int_\Omega B_t^*\,d\mu(t)\right|^2}\right\|_2, \quad (20)$$

while, if $\{A_t\}_{t\in\Omega}$, $\{B_t\}_{t\in\Omega}$ *are in* $L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$, *then*

$$\left\|\int_\Omega A_t X B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t) X \int_\Omega B_t\,d\mu(t)\right\|_2$$
$$\leqslant \left\|\sqrt{\int_\Omega |A_t|^2\,d\mu(t) - \left|\int_\Omega A_t\,d\mu(t)\right|^2}\,X\right\|_2 \left\|\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right\|^{1/2}. \quad (21)$$

Proof. As $\|\cdot\|_{\Phi^{(\theta/2)}}$ is a nuclear norm $\|\cdot\|_1$ for $\Phi := \ell^1$ and $\theta := 2$, then it will suffice to recognize that in the righthand side of (16) we have

$$\left\|\int_\Omega B_t^* X^* X B_t\,d\mu(t) - \left|X\int_\Omega B_t\,d\mu(t)\right|^2\right\|_1 = \operatorname{tr}\left(\int_\Omega B_t^* X^* X B_t\,d\mu(t) - \left(\int_\Omega B_t\,d\mu(t)\right)^* X^* X \int_\Omega B_t\,d\mu(t)\right) \tag{22}$$

$$= \int_\Omega \operatorname{tr}(B_t^* X^* X B_t)\,d\mu(t) - \operatorname{tr}\left(X^* X \int_\Omega B_t\,d\mu(t)\int_\Omega B_t^*\,d\mu(t)\right)$$

$$= \int_\Omega \operatorname{tr}(X^* X B_t B_t^*)\,d\mu(t) - \operatorname{tr}\left(X^* X \left|\int_\Omega B_t^*\,d\mu(t)\right|^2\right) \tag{23}$$

$$= \operatorname{tr}\left(\int_\Omega X^* X B_t B_t^*\,d\mu(t)\right) - \operatorname{tr}\left(X^* X \left|\int_\Omega B_t^*\,d\mu(t)\right|^2\right) = \operatorname{tr}\left(X^* X\left(\int_\Omega B_t B_t^*\,d\mu(t) - \left|\int_\Omega B_t^*\,d\mu(t)\right|^2\right)\right) \tag{24}$$

$$= \operatorname{tr}\left(\left(\int_\Omega |B_t^*|^2\,d\mu(t) - \left|\int_\Omega B_t^*\,d\mu(t)\right|^2\right)^{1/2} X^* X \left(\int_\Omega B_t B_t^*\,d\mu(t) - \left|\int_\Omega B_t^*\,d\mu(t)\right|^2\right)^{1/2}\right) \tag{25}$$

$$= \left\|X\left(\int_\Omega |B_t^*|^2\,d\mu(t) - \left|\int_\Omega B_t^*\,d\mu(t)\right|^2\right)^{1/2}\right\|_2^2. \tag{26}$$

Equality in (22) justifies by the positivity of $\int_\Omega B_t^* X^* X B_t\,d\mu(t) - \left|X\int_\Omega B_t\,d\mu(t)\right|^2 = \frac{1}{2}\int_{\Omega^2} |X(B_s - B_t)|^2 d(\mu\times\mu)(s,t)$, based on the already used Korkine type identity (8). First equalities in (23) and (24) follow by the alternative definition of weak* (Gel'fand) integrals, given in [7, Lemma 1.2], while the second equality in (23) is a consequence of the operator's commutativity under trace, as $X^* X B_t \in \mathfrak{C}_1(\mathcal{H})$ and $B_t^* \in \mathfrak{B}(\mathcal{H})$ for all $t \in \Omega$. Equality in (25) is again based on the commutativity under trace, while equality in (26) is due to the basic Hilbert-Schmidt norm property $\|Y\|_2^2 = \operatorname{tr}(Y^* Y)$ for all $Y \in \mathfrak{C}_2(\mathcal{H})$.

To prove inequality (21), we use again B* property of u.i. norms and weak* integrals, combined with already proven inequality (20), to get

$$\begin{aligned}\left\|\int_\Omega A_t X B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t) X \int_\Omega B_t\,d\mu(t)\right\|_2 &= \left\|\int_\Omega B_t^* X^* A_t^*\,d\mu(t) - \int_\Omega B_t^*\,d\mu(t) X^* \int_\Omega A_t^*\,d\mu(t)\right\|_2 \\ &\leqslant \left\|\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right\|^{1/2} \left\|X^*\left(\int_\Omega |A_t|^2\,d\mu(t) - \left|\int_\Omega A_t\,d\mu(t)\right|^2\right)^{1/2}\right\|_2 \\ &= \left\|\left(\int_\Omega |A_t|^2\,d\mu(t) - \left|\int_\Omega A_t\,d\mu(t)\right|^2\right)^{1/2} X\right\|_2 \left\|\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right\|^{1/2}. \quad \square\end{aligned}$$

Inequalities (20) and (21) are still true for arbitrary $\|\cdot\|_{\Phi^{(p)}}$ norms, whenever $p \geqslant 2$ and at least one of families $\{A_t\}_{t\in\Omega}, \{B_t\}_{t\in\Omega}$ is m.c.n.o. family, as we show in the next theorem, which also complements (the case $p \geqslant 2$ of) the Grüss-Landau type inequality [7, Th. 2.4] for Schatten norms $\|\cdot\|_p$.

Theorem 2.7. *Let μ be a probability measure on Ω, Φ to be a s.n. function, $p \geqslant 2$, $X \in \mathfrak{B}(\mathcal{H})$ and let both families $\{A_t^*\}_{t\in\Omega}$ and $\{B_t\}_{t\in\Omega}$ be in $L_G^2(\Omega,\mu,\mathfrak{B}(\mathcal{H}))$. If, in addition, $\{B_t\}_{t\in\Omega}$ is a m.c.n.o. family, such that $X\sqrt{\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2} \in \mathfrak{C}_{\Phi^{(p)}}(\mathcal{H})$, then $\int_\Omega A_t X B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t) X \int_\Omega B_t\,d\mu(t) \in \mathfrak{C}_{\Phi^{(p)}}(\mathcal{H})$ and*

$$\begin{aligned}&\left\|\int_\Omega A_t X B_t d\mu(t) - \int_\Omega A_t\,d\mu(t) X \int_\Omega B_t\,d\mu(t)\right\|_{\Phi^{(p)}} \\ &\qquad \leqslant \left\|\int_\Omega |A_t^*|^2\,d\mu(t) - \left|\int_\Omega A_t^*\,d\mu(t)\right|^2\right\|^{1/2} \left\|X\sqrt{\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2}\right\|_{\Phi^{(p)}}.\end{aligned} \tag{27}$$

Alternatively, if $\{A_t\}_{t\in\Omega}$ is a m.c.n.o. family, such that $\sqrt{\int_\Omega |A_t|^2\,d\mu(t) - \left|\int_\Omega A_t\,d\mu(t)\right|^2}\,X \in \mathfrak{C}_{\Phi^{(p)}}(\mathcal{H})$, then it also follows that $\int_\Omega A_t X B_t\,d\mu(t) - \int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t) \in \mathfrak{C}_{\Phi^{(p)}}(\mathcal{H})$ and

$$\left\|\int_\Omega A_t X B_t d\mu(t) - \int_\Omega A_t\,d\mu(t)X\int_\Omega B_t\,d\mu(t)\right\|_{\Phi^{(p)}} \leqslant \left\|\sqrt{\int_\Omega |A_t^*|^2\,d\mu(t) - \left|\int_\Omega A_t^*\,d\mu(t)\right|^2}\,X\right\|_{\Phi^{(p)}}\left\|\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right\|^{1/2}. \quad (28)$$

Proof. To prove (27), we first apply (16) case $\theta := p$, and we proceed by an application of [7, Th. 2.1] to X^*X instead of X, $\|\cdot\|_{\Phi^{(\frac{p}{2})}}$ instead of $|||\cdot|||$, in the special case $\mathscr{A}_t^* := \mathscr{B}_t := B_t$, for all $t \in \Omega$, to get the estimate

$$\begin{aligned}&\left\|\int_\Omega B_t^* X^* X B_t\,d\mu(t) - \int_\Omega B_t^*\,d\mu(t)X^*X\int_\Omega B_t\,d\mu(t)\right\|_{\Phi^{(\frac{p}{2})}}\\ &\leqslant \left\|\left(\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right)^{1/2} X^*X\left(\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right)^{1/2}\right\|_{\Phi^{(\frac{p}{2})}}\\ &= \left\|\left|X\left(\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right)^{1/2}\right|^2\right\|_{\Phi^{(\frac{p}{2})}} = \left\|X\left(\int_\Omega |B_t|^2\,d\mu(t) - \left|\int_\Omega B_t\,d\mu(t)\right|^2\right)^{1/2}\right\|^2_{\Phi^{(p)}}.\end{aligned}$$

(28) can be proved by analogy, by the use of (17) instead of (16). □

A special case of the previous Theorem 2.7 in the discrete setting says:

Corollary 2.8. *Let $\alpha_n \in (0,1]$ for $n \in \mathbb{N}$ such that $\sum_{n=1}^\infty \alpha_n = 1$, Φ to be a s.n. function, $p \geqslant 2$, $X \in \mathcal{B}(\mathcal{H})$ and let $\{\alpha_n^{-1/2}C_n^*\}_{n=1}^\infty$ and $\{\alpha_n^{-1/2}D_n\}_{n=1}^\infty$ be s.s.i. families. If $\{D_n\}_{n=1}^\infty$ is additionally a m.c.n.o. family, such that $X\left(\sum_{n=1}^\infty \alpha_n^{-1}|D_n|^2 - \left|\sum_{n=1}^\infty D_n\right|^2\right)^{1/2} \in \mathfrak{C}_{\Phi^{(p)}}(\mathcal{H})$, then $\sum_{n=1}^\infty \alpha_n^{-1}C_nXD_n - \sum_{n=1}^\infty C_nX\sum_{n=1}^\infty D_n \in \mathfrak{C}_{\Phi^{(p)}}(\mathcal{H})$ and*

$$\left\|\sum_{n=1}^\infty \alpha_n^{-1}C_nXD_n - \sum_{n=1}^\infty C_nX\sum_{n=1}^\infty D_n\right\|_{\Phi^{(p)}} \leqslant \left\|\sum_{n=1}^\infty \alpha_n^{-1}|C_n^*|^2 - \left|\sum_{n=1}^\infty C_n^*\right|^2\right\|^{1/2}\left\|X\sqrt{\sum_{n=1}^\infty \alpha_n^{-1}|D_n|^2 - \left|\sum_{n=1}^\infty D_n\right|^2}\right\|_{\Phi^{(p)}}. \quad (29)$$

Alternatively, if $\{C_n\}_{n=1}^\infty$ is a m.c.n.o. family, such that $\left(\sum_{n=1}^\infty \alpha_n^{-1}|C_n^*|^2 - \left|\sum_{n=1}^\infty C_n^*\right|^2\right)^{1/2}X \in \mathfrak{C}_{\Phi^{(p)}}(\mathcal{H})$, then also $\sum_{n=1}^\infty \alpha_n^{-1}C_nXD_n - \sum_{n=1}^\infty C_nX\sum_{n=1}^\infty D_n \in \mathfrak{C}_{\Phi^{(p)}}(\mathcal{H})$ and

$$\left\|\sum_{n=1}^\infty \alpha_n^{-1}C_nXD_n - \sum_{n=1}^\infty C_nX\sum_{n=1}^\infty D_n\right\|_{\Phi^{(p)}} \leqslant \left\|\sqrt{\sum_{n=1}^\infty \alpha_n^{-1}|C_n^*|^2 - \left|\sum_{n=1}^\infty C_n^*\right|^2}\,X\right\|_{\Phi^{(p)}}\left\|\sum_{n=1}^\infty \alpha_n^{-1}|D_n|^2 - \left|\sum_{n=1}^\infty D_n\right|^2\right\|^{1/2}. \quad (30)$$

Proof. It is enough to apply Theorem 2.7 special case $\Omega := \mathbb{N}$, $\mu(\{n\}) := \alpha_n$, $A_n := \alpha_n^{-1}C_n$ and $B_n := \alpha_n^{-1}D_n$, for all $n \in \mathbb{N}$, to get the proclaimed inequalities (29) and (30). □

Remark 2.9. *Similarly to the situation discussed in Remark 2.5, the previous Corollary 2.8 extends inequalities (2.21) (in the case $c := \left\|\sum_{n=1}^N \alpha_n^{-1}A_n^*A_n - \left|\sum_{n=1}^N A_n\right|^2\right\|^{1/2}$) and (2.22), when $q := p$, in [9, Th. 2.2], in the settings of elementary operators.*

To complement the case $1 \leqslant p \leqslant 2$ of [7, Th. 2.4], let us first note that $\mathfrak{C}_1(\mathcal{H}) \subset \mathfrak{C}_\Phi(\mathcal{H})$ for all s.n. function Φ, as the nuclear norm is the maximal one amongst all u.i. norms. So, if $2 \leqslant p < +\infty$, then $\mathfrak{C}_2(\mathcal{H}) \subset \mathfrak{C}_p(\mathcal{H}) \subset \mathfrak{C}_{\Phi^{(p)}}(\mathcal{H})$ and $\mathfrak{C}_{\Phi^{(p)*}}(\mathcal{H}) \subset \mathfrak{C}_{p/(p-1)}(\mathcal{H}) \subset \mathfrak{C}_2(\mathcal{H})$, following the duality argument. Thus, we are now in a position to further complement [7, Th. 2.4] for $\mathfrak{C}_{\Phi^{(p)*}}(\mathcal{H})$ ideals, as follows.

Theorem 2.10. *Let $\alpha_n \in (0,1]$ for $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$, Φ to be a s.n. function, $p \geqslant 2$ and let $\{\alpha_n^{-1/2} A_n\}_{n=1}^{\infty}$ and $\{\alpha_n^{-1/2} B_n^*\}_{n=1}^{\infty}$ be s.s.i. families such that one of them is a m.c.n.o. family. If $X \in \mathcal{C}_{\Phi^{(p)*}}(\mathcal{H})$ then $\sum_{n=1}^{\infty} \alpha_n^{-1} A_n X B_n - \sum_{n=1}^{\infty} A_n X \sum_{n=1}^{\infty} B_n \in \mathcal{C}_{\Phi^{(p)*}}(\mathcal{H})$ and we have*

$$\left\| \sum_{n=1}^{\infty} \alpha_n^{-1} A_n X B_n - \sum_{n=1}^{\infty} A_n X \sum_{n=1}^{\infty} B_n \right\|_{\Phi^{(p)*}} \leqslant \left\| \sqrt{\sum_{n=1}^{\infty} \alpha_n^{-1} |A_n|^2 - \left|\sum_{n=1}^{\infty} A_n\right|^2} \, X \sqrt{\sum_{n=1}^{\infty} \alpha_n^{-1} |B_n^*|^2 - \left|\sum_{n=1}^{\infty} B_n^*\right|^2} \right\|_{\Phi^{(p)*}}. \quad (31)$$

Proof. First, note that using identity (8) for $\Omega := \mathbb{N}$, $\mu(\{n\}) := \alpha_n$ for $n \in \mathbb{N}$, applied to $\alpha_n^{-1} A_n$ and $\alpha_n^{-1} B_n$, we obtain

$$\sum_{n=1}^{\infty} \alpha_n^{-1} A_n X B_n - \sum_{n=1}^{\infty} A_n X \sum_{n=1}^{\infty} B_n = \frac{1}{2} \sum_{m,n=1}^{\infty} \alpha_m \alpha_n (\alpha_m^{-1} A_m - \alpha_n^{-1} A_n) X (\alpha_m^{-1} B_m - \alpha_n^{-1} B_n).$$

As the above identity also implies $\sum_{n=1}^{\infty} |\alpha_n^{-1/2} A_n|^2 - \left|\sum_{n=1}^{\infty} A_n\right|^2 = \frac{1}{2} \sum_{m,n=1}^{\infty} \alpha_m \alpha_n |\alpha_m^{-1} A_m - \alpha_n^{-1} A_n|^2$ and $\{\alpha_n^{-1/2} A_n\}_{n=1}^{\infty}$ is s.s.s. family, then we also have that $\{\sqrt{\alpha_m \alpha_n}(\alpha_m^{-1} A_m - \alpha_n^{-1} A_n)\}_{m,n=1}^{\infty}$ is s.s.s. family, i.e. for every $f \in \mathcal{H}$ we have $\sum_{m,n=1}^{\infty} \alpha_m \alpha_n \|(\alpha_m^{-1} A_m - \alpha_n^{-1} A_n) f\|^2 < +\infty$. Similarly, we have that $\{\sqrt{\alpha_m \alpha_n}(\alpha_m^{-1} B_m^* - \alpha_n^{-1} B_n^*)\}_{m,n=1}^{\infty}$ is s.s.s. family.

Therefore, we can apply the first inequality in (6) in [10, Lemma 2.1] to s.s.s. families $\{\sqrt{\alpha_m \alpha_n}(\alpha_m^{-1} A_m - \alpha_n^{-1} A_n)\}_{m,n=1}^{\infty}$ and $\{\sqrt{\alpha_m \alpha_n}(\alpha_m^{-1} B_m^* - \alpha_n^{-1} B_n^*)\}_{m,n=1}^{\infty}$, to obtain inequality in

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} \alpha_n^{-1} A_n X B_n - \sum_{n=1}^{\infty} A_n X \sum_{n=1}^{\infty} B_n \right\|_{\Phi^{(p)*}} &= \frac{1}{2} \left\| \sum_{m,n=1}^{\infty} \alpha_m \alpha_n (\alpha_m^{-1} A_m - \alpha_n^{-1} A_n) X (\alpha_m^{-1} B_m - \alpha_n^{-1} B_n) \right\|_{\Phi^{(p)*}} \\ &\leqslant \frac{1}{2} \left\| \left(\sum_{m,n=1}^{\infty} \alpha_m \alpha_n |\alpha_m^{-1} A_m - \alpha_n^{-1} A_n|^2 \right)^{1/2} X \left(\sum_{m,n=1}^{\infty} \alpha_m \alpha_n |\alpha_m^{-1} B_m^* - \alpha_n^{-1} B_n^*|^2 \right)^{1/2} \right\|_{\Phi^{(p)*}} \\ &= \left\| \left(\sum_{n=1}^{\infty} \alpha_n^{-1} |A_n|^2 - \left|\sum_{n=1}^{\infty} A_n\right|^2 \right)^{1/2} X \left(\sum_{n=1}^{\infty} \alpha_n^{-1} |B_n^*|^2 - \left|\sum_{n=1}^{\infty} B_n^*\right|^2 \right)^{1/2} \right\|_{\Phi^{(p)*}}, \end{aligned} \quad (32)$$

which proves (31). As $\sum_{n=1}^{\infty} \alpha_n^{-1} A_n X B_n - \sum_{n=1}^{\infty} A_n X \sum_{n=1}^{\infty} B_n = \frac{1}{2} \sum_{m,n=1}^{\infty} \alpha_m \alpha_n (\alpha_m^{-1} A_m - \alpha_n^{-1} A_n) X (\alpha_m^{-1} B_m - \alpha_n^{-1} B_n) \in \mathcal{C}_{\Phi^{(p)*}}(\mathcal{H})$ can also be concluded from the part of the proof presented in (32), this altogether ends the proof. □

References

[1] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.

[2] J. Diestel, J.J. Uhl, Vector measures, Mathematical Surways, Vol. 15, American Mathematical Society, Providence, RI, 1977, MR56:12216.

[3] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Non-selfadjoint Operators, Transl. Math. Monographs, Vol. 18, Amer. Math. Soc. Providence, R.I. 1969.

[4] G. Grüss, *Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$*, Math. Z. **39** (1935), 215-226.

[5] D.R. Jocić, *Cauchy-Schwarz norm inequalities for weak*-integrals of operator valued functions*, J. Funct. Anal. **218** (2005), 318-346.

[6] D.R. Jocić, *Multipliers of elementary operators and comparison of row and column space Schatten p norms*, Linear Algebra Appl. **431** (2009), 2062-2070.

[7] D.R. Jocić, Đ. Krtinić, M. Sal Moslehian, *Landau and Grüss type inequalities for inner product type integral transformers in norm ideals*, Math. Ineq. Appl. **16** (2013), 109-125.

[8] D.R. Jocić, *Clarkson-McCarthy inequalities for several operators and other norm inequalities for operators and operator matrices related trough sums of its squares*, Complex Anal. Oper. Theory **13** (2019), 583-613.

[9] D.R. Jocić, Đ. Krtinić, M. Lazarević, P. Melentijević, S. Milošević, *Refinements of inequalities related to Landau-Grüss inequalities for elementary operators acting on ideals associated to p-modified unitarily invariant norms*, Complex Anal. Oper. Theory **12** (2018), 195-205.

[10] D.R. Jocić, M. Lazarević, S. Milošević, *Norm inequalities for a class of elemetary operators generated by analytic functions with non-negative Taylor coeficients in ideals of compact operators related to p-modified unitarily invariant norms*, Linear Algebra Appl. **540** (2018), 60-83.

[11] E. Landau, *Über einige Ungleichungen von Herrn G. Grüss*, Math. Z. **39** (1935), 742-744.

[12] J.S. Matharu, M. Sal Moslehian, *Grüss inequality for some types of positive linear maps*, J. Oper. Theory **73** (2015), 265-278.

[13] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic, Dordrecht, 1993.

[14] B. Simon, Trace Ideals and Their Applications, Amer. Math. Soc., Mat. Surveys and Monographs, Vol. 120, 2005.