



The Mehler-Fock-Clifford Transform and Pseudo-Differential Operator on Function Spaces

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Abstract. In this article, we introduce a new index transform associated with the cone function $P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})$, named as Mehler-Fock-Clifford transform and study its some basic properties. Convolution and translation operators are defined and obtained their estimates under $L^p(I; x^{-\frac{1}{2}}dx)$ norm. The test function spaces \mathcal{G}_α and \mathcal{F}_α are introduced and discussed the continuity of the differential operator and MFC-transform on these spaces. Moreover, the pseudo-differential operator (p.d.o.) involving MFC-transform is defined and studied its continuity between \mathcal{G}_α and \mathcal{F}_α .

1. Introduction

The transform of integrable function f is defined first by F. G. Mehler [9] in 1881 as:

$$F(\tau) = (\mathfrak{M}f)(\tau) = \int_1^\infty f(x)P_{i\tau-\frac{1}{2}}(x)dx, \quad \tau > 0, \quad (1)$$

and its inversion defined by V. A. Fock [2] in 1943 as:

$$f(x) = (\mathfrak{M}^{-1}F)(x) = \int_0^\infty \tau \tanh(\pi\tau)P_{i\tau-\frac{1}{2}}(x)F(\tau)d\tau, \quad x > 1, \quad (2)$$

where $P_{i\tau-\frac{1}{2}}(x)$ is cone function (associated Legendre function of zero order) and it is represented in terms of Gaussian hypergeometric function ${}_2F_1$ as:

$$P_{i\tau-\frac{1}{2}}(x) = {}_2F_1\left(1/2 + i\tau, 1/2 - i\tau; 1; (1-x)/2\right).$$

Therefore the (1) is known as Mehler-Fock transform and (2) is as its inversion. The theory and properties of Mehler-Fock transform have been studied by C. Nasim [10], Srivastava et al. [19], Yakubovich et al.[21–23], Lebedev [7, 8], Sneddon [18] and Prasad et al.[16] etc. In this paper we modified the cone function obtained a new integral transform and we named it as Mehler-Fock-Clifford (MFC) transform.

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Initially the idea of Clifford type integral transform was first evolved by English Mathematician W. K. Clifford on taking the kernel Bessel-Clifford function

$$C_{\vartheta}(x) = x^{-\vartheta/2} J_{\vartheta}(2\sqrt{x}),$$

where J_{ϑ} is the Bessel function of order ϑ , which satisfies the differential equation

$$(xD_x^2 + (\vartheta + 1)D_x + 1)u(x) = 0.$$

The theory and properties of Hankel-Clifford transform have already been studied by the several researchers viz [12, 14, 15] etc. As per this argument the Legendre-Clifford function according to [1, p.156] is defined as

$$P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) = \frac{\sqrt{2}}{\pi} \cosh(\pi\sqrt{\tau}) \int_0^{\infty} \frac{\cos(\sqrt{\tau}t)}{\sqrt{(2\sqrt{x} + \cosh t)}} dt, \tag{3}$$

and it satisfies the differential equation

$$(x(4x - 1)D_x^2 + \frac{12x - 1}{2}D_x + (\tau + \frac{1}{4}))u(x) = 0.$$

Now we define the Mehler-Fock-Clifford (MFC) transform as

$$F(\tau) = (\mathfrak{M}_{\mathbb{C}}f)(\tau) = \int_{\frac{1}{4}}^{\infty} f(x)P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})\frac{dx}{\sqrt{x}}, \quad \tau > 0 \tag{4}$$

and its inversion

$$f(x) = (\mathfrak{M}_{\mathbb{C}}^{-1}F)(x) = \frac{1}{2} \int_0^{\infty} \tanh(\pi\sqrt{\tau})P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})F(\tau)d\tau, \quad x > \frac{1}{4}. \tag{5}$$

The kernel $P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})$ is an eigen function of the operator

$$A_x = x(4x - 1)D_x^2 + \frac{(12x - 1)}{2}D_x, \tag{6}$$

and satisfies the property

$$A_x P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) = (-1)\left(\tau + \frac{1}{4}\right)P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}). \tag{7}$$

The series representation of the differential operator A_x^k is as follows:

$$A_x^k = \sum_{j=1}^{2k} q_j^k(x)D_x^j, \tag{8}$$

where $q_{2k}^k(x) = (x(4x - 1))^k$ and the intermediate terms $q_j^k(x)$, for $1 \leq j < 2k$, are polynomials of j^{th} degree. Also, adjoint of the operator A_x is obtained as:

$$A_x^* = D_x^2 x(4x - 1) - D_x\left(\frac{12x - 1}{2}\right).$$

We recall from [11, pp. 171-173], the asymptotic behaviours of $P_{-\frac{1}{2}}(2\sqrt{x})$ near to $x = \frac{1}{4}$ and infinity are as

$$P_{-\frac{1}{2}}(2\sqrt{x}) \sim 1 \text{ as } x \rightarrow \frac{1}{4}, \tag{9}$$

and

$$P_{-\frac{1}{2}}(2\sqrt{x}) \sim \frac{1}{\sqrt{2\pi}} \frac{\ln(2\sqrt{x})}{x^{1/4}} \text{ as } x \rightarrow \infty. \tag{10}$$

From [3], we define the symmetric function $D(x, y, z) \geq 0$ as:

$$D(x, y, z) = \int_0^\infty \tanh(\pi\sqrt{\tau}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{z}) d\tau, \tag{11}$$

where

$$D(x, y, z) = \begin{cases} \frac{1}{\pi}(16\sqrt{xyz} + 1 - 4x - 4y - 4z)^{-\frac{1}{2}} & \text{for } z \in I_{x,y}, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$I_{x,y} =: (4\sqrt{xy} - [(4x-1)(4y-1)]^{\frac{1}{2}}, 4\sqrt{xy} + [(4x-1)(4y-1)]^{\frac{1}{2}}).$$

Now using the inversion of MFC-transform, then we have

$$P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y}) = \int_{\frac{1}{4}}^\infty D(x, y, z) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{z}) \frac{dz}{\sqrt{z}} \tag{12}$$

and

$$\int_{\frac{1}{4}}^\infty D(x, y, z) \frac{dz}{\sqrt{z}} = 1. \tag{13}$$

The translation operator is defined as:

$$(\mathcal{T}_x f)(y) = \int_{\frac{1}{4}}^\infty D(x, y, z) f(z) \frac{dz}{\sqrt{z}}. \tag{14}$$

From (12) and (14), we see that

$$P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y}) = (\mathcal{T}_x P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{z}))(y).$$

Simultaneously convolution operator is defined as:

$$\begin{aligned} (f * g)(x) &= \int_{\frac{1}{4}}^\infty (\mathcal{T}_x f)(z) g(z) \frac{dz}{\sqrt{z}}, \\ &= \int_{\frac{1}{4}}^\infty \int_{\frac{1}{4}}^\infty D(x, y, z) f(y) g(z) \frac{dy}{\sqrt{y}} \frac{dz}{\sqrt{z}}. \end{aligned} \tag{15}$$

By $L^p(I; \omega(x)dx)$, $I = (1/4, \infty)$, $1 \leq p \leq \infty$ we denote the weighted L^p -space with the norm

$$\|f\|_{L^p(I; \omega(x)dx)} = \begin{cases} \left(\int_{\frac{1}{4}}^\infty |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty, \\ \text{ess. sup}_{x \in I} |f(x)|, & \text{for } p = \infty. \end{cases}$$

Plancherel's and Parseval's relations have been obtained as:

$$\int_{\frac{1}{4}}^\infty f(x) \overline{g(x)} \frac{dx}{\sqrt{x}} = \frac{1}{2} \int_0^\infty \tanh(\pi\sqrt{\tau}) (\mathfrak{M}_\mathbb{C} f)(\tau) \overline{(\mathfrak{M}_\mathbb{C} g)(\tau)} d\tau,$$

and

$$\int_{\frac{1}{4}}^{\infty} |f(x)|^2 \frac{dx}{\sqrt{x}} = \frac{1}{2} \int_0^{\infty} \tanh(\pi \sqrt{\tau}) |(\mathfrak{M}_{\mathbb{C}} f)(\tau)|^2 d\tau,$$

or

$$\|f\|_{L^2(I; x^{-\frac{1}{2}} dx)} = \|(\mathfrak{M}_{\mathbb{C}} f)\|_{L^2(\mathbb{R}_+; \frac{1}{2} \tanh(\pi \sqrt{\tau}) d\tau)}.$$

Thus, the MFC-transform is isometrically-isomorphism operator from $L^2(I; x^{-\frac{1}{2}} dx)$ onto $L^2(\mathbb{R}_+; \frac{1}{2} \tanh(\pi \sqrt{\tau}) d\tau)$.

The article is organized as follows: Section 1 is introductory, in which Mehler-Fock-Clifford (MFC) transform is introduced with cone function $P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})$. The convolution and translation operators are defined and Parseval’s as well as Plancherel’s relations are obtained. Section 2 consists of some useful results like MFC-transform of translation and convolution operators, relation between differential operator A_x and its adjoint A_x^* , some estimates of the kernel of MFC-transform. Moreover estimates of translation and convolution operators are also obtained in Lebesgue space. In Section 3, the test function spaces \mathcal{G}_α and \mathcal{F}_α are defined and discussed the continuity of MFC-transform on these spaces. Section 4 includes the pseudo-differential operator (p.d.o.) associated with the MFC-transform. Moreover, continuity of the p.d.o. is discussed between the spaces \mathcal{G}_α and \mathcal{F}_α . An another integral representation of p.d.o. is obtained. Further an estimate of the p.d.o. is also discussed.

2. Preliminary Results and Some Estimate

Theorem 2.1. *If $f, g \in L^1(I; x^{-\frac{1}{2}} dx)$, then the MFC-transform of the translation and convolution operators are respectively as follows:*

$$\begin{aligned} (i) \quad (\mathfrak{M}_{\mathbb{C}}(\mathcal{T}_x f))(\tau) &= P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})(\mathfrak{M}_{\mathbb{C}} f)(\tau) \\ (ii) \quad (\mathfrak{M}_{\mathbb{C}}(f * g))(\tau) &= (\mathfrak{M}_{\mathbb{C}} f)(\tau)(\mathfrak{M}_{\mathbb{C}} g)(\tau). \end{aligned} \tag{16}$$

Proof. (i) From (4) and (14), we have

$$(\mathfrak{M}_{\mathbb{C}}(\mathcal{T}_x f))(\tau) = \int_{\frac{1}{4}}^{\infty} \int_{\frac{1}{4}}^{\infty} P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y}) D(x, y, z) f(z) \frac{dy}{\sqrt{y}} \frac{dz}{\sqrt{z}},$$

using (12), we readily yield

$$\begin{aligned} (\mathfrak{M}_{\mathbb{C}}(\mathcal{T}_x f))(\tau) &= P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) \int_{\frac{1}{4}}^{\infty} P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{z}) f(z) \frac{dz}{\sqrt{z}}, \\ &= P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})(\mathfrak{M}_{\mathbb{C}} f)(\tau). \end{aligned}$$

Hence (i) is proved.

(ii) From (4), (15) and by Fubini’s theorem, we have

$$(\mathfrak{M}_{\mathbb{C}}(f * g))(\tau) = \int_{\frac{1}{4}}^{\infty} \int_{\frac{1}{4}}^{\infty} \left(\int_{\frac{1}{4}}^{\infty} P_{i\sqrt{\tau-\frac{1}{2}}}(x) D(x, y, z) \frac{dx}{\sqrt{x}} \right) f(y) g(z) \frac{dy}{\sqrt{y}} \frac{dz}{\sqrt{z}},$$

using (12), we get

$$(\mathfrak{M}_{\mathbb{C}}(f * g))(\tau) = (\mathfrak{M}_{\mathbb{C}} f)(\tau)(\mathfrak{M}_{\mathbb{C}} g)(\tau).$$

This completes the proof of (ii). \square

The relation between the differential operator A_x and its adjoint A_x^* is as:

$$A_x^*(x^{-\frac{1}{2}}f(x)) = x^{-\frac{1}{2}}A_x f(x), \tag{17}$$

thus

$$A_x^*(x^{-\frac{1}{2}}P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})) = x^{-\frac{1}{2}}A_x P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}). \tag{18}$$

Further it can be extended upto finite times and we obtained

$$(A_x^*)^n(x^{-\frac{1}{2}}P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})) = x^{-\frac{1}{2}}A_x^n P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}).$$

Applying operator A_x on (11) and using (7), we have

$$\begin{aligned} A_x D(x, y, z) &= \int_0^\infty \tanh(\pi\sqrt{\tau}) \left(-1\right) \left(\tau + \frac{1}{4}\right) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{z}) d\tau \\ &= \int_0^\infty \tanh(\pi\sqrt{\tau}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) \left(-1\right) \left(\tau + \frac{1}{4}\right) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{z}) d\tau \\ &= A_y D(x, y, z). \end{aligned}$$

Similarly,

$$A_x D(x, y, z) = A_z D(x, y, z).$$

Therefore

$$A_x D(x, y, z) = A_y D(x, y, z) = A_z D(x, y, z).$$

On repeating k times, we have

$$A_x^k D(x, y, z) = A_y^k D(x, y, z) = A_z^k D(x, y, z).$$

Lemma 2.2. If $f, g \in L^1(I; x^{-\frac{1}{2}}dx)$, then

- (i) $(\mathfrak{M}_{\mathbb{C}}(A_x^r f))(\tau) = (-1)^r \left(\tau + \frac{1}{4}\right)^r (\mathfrak{M}_{\mathbb{C}} f)(\tau), \quad \forall r \in \mathbb{N}_0,$
- (ii) $A_x(f * g) = A_x f * g = f * A_x g.$

Proof. From (4), we have

$$(\mathfrak{M}_{\mathbb{C}}(A_x f))(\tau) = \int_{\frac{1}{4}}^\infty P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) A_x f(x) \frac{dx}{\sqrt{x}}.$$

Using (18) and (7), we get

$$\begin{aligned} (\mathfrak{M}_{\mathbb{C}}(A_x f))(\tau) &= \int_{\frac{1}{4}}^\infty (-1) \left(\tau + \frac{1}{4}\right) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) f(x) \frac{dx}{\sqrt{x}}, \\ &= (-1) \left(\tau + \frac{1}{4}\right) (\mathfrak{M}_{\mathbb{C}} f)(\tau). \end{aligned}$$

Continuing in this way, we have the desired result. This completes the proof (i).

Now using (16) and (19), we have

$$\begin{aligned} (\mathfrak{M}_{\mathbb{C}} A_x(f * g))(\tau) &= (-1) \left(\tau + \frac{1}{4}\right) (\mathfrak{M}_{\mathbb{C}} f)(\tau) (\mathfrak{M}_{\mathbb{C}} g)(\tau), \\ &= (\mathfrak{M}_{\mathbb{C}}(A_x f))(\tau) (\mathfrak{M}_{\mathbb{C}} g)(\tau), \\ &= (\mathfrak{M}_{\mathbb{C}}(A_x f * g))(\tau). \end{aligned}$$

Applying inverse MFC-transform (5), we get

$$A_x(f * g) = A_x f * g.$$

Similarly

$$A_x(f * g) = f * A_x g.$$

Therefore

$$A_x(f * g) = A_x f * g = f * A_x g.$$

Hence (ii) is proved. \square

Some properties of kernel $P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x})$:

(i) For every positive integer m there exists $M > 0$ such that

$$\left| \frac{d^m}{dx^m} P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) \right| \leq M \cosh(\pi\sqrt{\tau}). \tag{20}$$

Proof. Differentiating (3) m times with respect to x , we have

$$\frac{d^m}{dx^m} P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) = (-1)^m \frac{\sqrt{2}}{\pi} \cosh(\pi\sqrt{\tau}) \int_0^\infty \sum_{k=1}^m \frac{C_k}{(2\sqrt{x} + \cosh(t))^{\frac{2k+1}{2}} x^{\frac{2m-k}{2}}} \cos(\sqrt{\tau}t) dt,$$

where C_k are positive constants. Now

$$\begin{aligned} \left| \frac{d^m}{dx^m} P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) \right| &\leq C \cosh(\pi\sqrt{\tau}) \sum_{k=1}^m \frac{1}{x^{\frac{2m-k}{2}}} \int_0^\infty 2^{\frac{2k+1}{2}} e^{-\frac{2k+1}{2}t} dt \\ &\leq M \cosh(\pi\sqrt{\tau}), \end{aligned}$$

where $C > 0$ and $M > 0$ are constants. \square

(ii) As per [1, (6), p. 155], $P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x})$ can also be represented as:

$$P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) = \frac{1}{\pi} \int_0^\pi [2\sqrt{x} + (4x - 1)^{1/2} \cos(\xi)]^{-\frac{1}{2} + i\sqrt{\tau}} d\xi,$$

and

$$\left| P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) \right| \leq P_{-\frac{1}{2}}(2\sqrt{x}). \tag{21}$$

Using asymptotic behaviours of $P_{-\frac{1}{2}}(2\sqrt{x})$, (9), (10) and (21), we have

$$\left| P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) \right| \leq C, \tag{22}$$

where $C > 0$ is a constant.

(iii) The function $P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x})$ satisfies the following estimate

$$\left| \frac{d^m}{d\tau^m} P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) \right| \leq e^{\pi\sqrt{\tau}} m! \frac{\sqrt{2}}{\pi} \sum_{l=0}^m \sum_{s=0}^{m-l} C_s \tau^{-(2(m-l)-s)/2} \sum_{r=0}^l C'_r \tau^{-(2l-r)/2}, \tag{23}$$

where C_s and C'_r are positive constants.

Proof. Differentiating (3) m times with respect to τ , we have

$$\frac{d^m}{d\tau^m} P_{i\sqrt{\tau} - \frac{1}{2}}(2\sqrt{x}) = \frac{\sqrt{2}}{\pi} \sum_{l=0}^m \binom{m}{l} \frac{d^{m-l}}{d\tau^{m-l}} \cosh(\pi\sqrt{\tau}) \int_0^\infty \frac{d^l}{d\tau^l} \cos(\sqrt{\tau}t) \frac{1}{\sqrt{(2\sqrt{x} + \cosh t)}} dt,$$

Now, assuming $p(\tau) = \cos(t\sqrt{\tau})$ and $q(\tau) = \sqrt{\tau}$ and invoking Faá di Bruno’s formula [5] for the l th derivatives of the composite function $p(q(\tau)) = \cos(t\sqrt{\tau})$, we have

$$\begin{aligned} \frac{d^l}{d\tau^l} \cos(t\sqrt{\tau}) &= \sum \frac{l!}{b_1!b_2! \cdots b_l!} t^r \cos\left(t\sqrt{\tau} + r\frac{\pi}{2}\right) \left(\frac{\frac{1}{2}\tau^{\frac{1}{2}-1}}{1!}\right)^{b_1} \left(\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\tau^{\frac{1}{2}-2}}{2!}\right)^{b_2} \cdots \\ &\times \left(\frac{\frac{1}{2}\left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-l+1\right)\tau^{\frac{1}{2}-l}}{l!}\right)^{b_l}, \end{aligned}$$

where $l \in \mathbb{N}_0$ and the sum is taken over all distinct non-negative integral solutions b_1, b_2, \dots, b_l , satisfying the following conditions:

$$b_1 + 2b_2 + \cdots + lb_l = l \quad \text{and} \quad b_1 + b_2 + \cdots + b_l = r.$$

Thus

$$\left| \frac{d^l}{d\tau^l} \cos(t\sqrt{\tau}) \right| \leq l! \sum_{r=0}^l C_r t^r \tau^{-(2l-r)/2}, \tag{24}$$

where C_r are positive constants.

Similarly, we obtain

$$\left| \frac{d^{m-l}}{d\tau^{m-l}} \cosh(\pi\sqrt{\tau}) \right| \leq (m-l)! e^{\pi\sqrt{\tau}} \sum_{s=0}^{m-l} C_s \tau^{-(2(m-l)-s)/2}, \tag{25}$$

where $C_s > 0$ are constants. From (24) and (25), we have

$$\left| \frac{d^m}{d\tau^m} P_{i\sqrt{\tau} - \frac{1}{2}}(2\sqrt{x}) \right| \leq e^{\pi\sqrt{\tau}} \frac{\sqrt{2}}{\pi} \sum_{l=0}^m m! \sum_{s=0}^{m-l} C_s \tau^{-(2(m-l)-s)/2} \sum_{r=0}^l C_r \tau^{-(2l-r)/2} \int_0^\infty \frac{t^r}{\sqrt{(1 + \cosh t)}} dt$$

the last integral converges for every $r \in \mathbb{N}_0$. Hence we get inequality (23). \square

Theorem 2.3. If $f \in L^p(I; x^{-\frac{1}{2}} dx)$, $1 \leq p < \infty$, then $\mathcal{T}_x f \in L^p(I; x^{-\frac{1}{2}} dx)$ such that

$$\|\mathcal{T}_x f\|_{L^p(I; x^{-\frac{1}{2}} dx)} \leq \|f\|_{L^p(I; x^{-\frac{1}{2}} dx)}.$$

Proof. From (14) and using Hölder’s inequality, we have

$$|(\mathcal{T}_x f)(y)| \leq \left(\int_{\frac{1}{4}}^\infty D(x, y, z) |f(z)|^p \frac{dz}{\sqrt{z}} \right)^{\frac{1}{p}} \left(\int_{\frac{1}{4}}^\infty D(x, y, z) \frac{dz}{\sqrt{z}} \right)^{\frac{1}{q}}.$$

Using (13), we get

$$\int_{\frac{1}{4}}^\infty |(\mathcal{T}_x f)(y)|^p \frac{dy}{\sqrt{y}} \leq \int_{\frac{1}{4}}^\infty |f(z)|^p \frac{dz}{\sqrt{z}} \int_{\frac{1}{4}}^\infty D(x, y, z) \frac{dy}{\sqrt{y}}.$$

Again using (13), we have

$$\|(\mathcal{T}_x f)\|_{L^p(I; x^{-\frac{1}{2}} dx)} \leq \|f\|_{L^p(I; x^{-\frac{1}{2}} dx)}.$$

Hence we obtained the Theorem. \square

Theorem 2.4. If $f, g \in L^p(I; x^{-\frac{1}{2}} dx)$, $1 \leq p < \infty$ then $f * g \in L^p(I; x^{-\frac{1}{2}} dx)$ such that

$$\|f * g\|_{L^p(I; x^{-\frac{1}{2}} dx)} \leq \|f\|_{L^p(I; x^{-\frac{1}{2}} dx)} \|g\|_{L^1(I; x^{-\frac{1}{2}} dx)}. \tag{26}$$

Proof. From (15) and using Hölder’s inequality, we have

$$|(f * g)(x)| \leq \left(\int_{\frac{1}{4}}^{\infty} \int_{\frac{1}{4}}^{\infty} D(x, y, z) |f(y)|^p |g(z)| \frac{dy}{\sqrt{y}} \frac{dz}{\sqrt{z}} \right)^{\frac{1}{p}} \left(\int_{\frac{1}{4}}^{\infty} \int_{\frac{1}{4}}^{\infty} D(x, y, z) |g(z)| \frac{dy}{\sqrt{y}} \frac{dz}{\sqrt{z}} \right)^{\frac{1}{q}}.$$

Using (13), we have

$$|(f * g)(x)|^p \leq \left(\int_{\frac{1}{4}}^{\infty} \int_{\frac{1}{4}}^{\infty} D(x, y, z) |f(y)|^p |g(z)| \frac{dy}{\sqrt{y}} \frac{dz}{\sqrt{z}} \right) \left(\int_{\frac{1}{4}}^{\infty} |g(z)| \frac{dz}{\sqrt{z}} \right)^{\frac{p}{q}}.$$

Again using (13), we obtain

$$\left(\int_{\frac{1}{4}}^{\infty} |(f * g)(x)|^p \frac{dx}{\sqrt{x}} \right)^{\frac{1}{p}} \leq \left(\int_{\frac{1}{4}}^{\infty} |f(y)|^p \frac{dy}{\sqrt{y}} \right)^{\frac{1}{p}} \left(\int_{\frac{1}{4}}^{\infty} |g(z)| \frac{dz}{\sqrt{z}} \right).$$

Hence completes the desired result. \square

Generalized Minkowski inequaltity: The genralized Minkowski inequality for suitable function h is defined as

$$\left[\int_a^b \left| \int_c^d h(x, y) dy \right|^p dx \right]^{\frac{1}{p}} \leq \int_c^d \left[\int_a^b |h(x, y)|^p dx \right]^{\frac{1}{p}} dy, \quad 1 \leq p < \infty. \tag{27}$$

Theorem 2.5. The MFC-transform is a bounded linear operator from $L^1(I; x^{-\frac{1}{2}} dx)$ into $L^q(\mathbb{R}_+; e^{-\alpha\tau} d\tau)$, $1 \leq q < \infty$, $\alpha > 0$.

Proof. Let $f \in L^1(I; x^{-\frac{1}{2}} dx)$. From (4), we have

$$\begin{aligned} \|\mathfrak{M}_{\mathbb{C}} f\|_{L^q(\mathbb{R}_+; e^{-\alpha\tau} d\tau)} &= \left(\int_0^{\infty} |\mathfrak{M}_{\mathbb{C}} f|^q e^{-\alpha\tau} d\tau \right)^{\frac{1}{q}} \\ &= \left(\int_0^{\infty} \left| \int_{\frac{1}{4}}^{\infty} f(x) P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) \frac{dx}{\sqrt{x}} \right|^q e^{-\alpha\tau} d\tau \right)^{\frac{1}{q}}. \end{aligned}$$

Using generalized Minkowski inequality (27) and (22), we get

$$\|\mathfrak{M}_{\mathbb{C}} f\|_{L^q(\mathbb{R}_+; e^{-\alpha\tau} d\tau)} \leq C \frac{1}{2} \int_{\frac{1}{4}}^{\infty} |f(x)| \frac{dx}{\sqrt{x}} \left(\int_0^{\infty} e^{-\alpha\tau} d\tau \right)^{\frac{1}{q}},$$

for $\alpha > 0$ the integral converges. Thus

$$\|\mathfrak{M}_{\mathbb{C}} f\|_{L^q(\mathbb{R}_+; e^{-\alpha\tau} d\tau)} \leq C' \|f\|_{L^p(I; x^{-\frac{1}{2}} dx)},$$

where $C' > 0$ is a constant. Hence proved. \square

3. Test function spaces

Definition 3.1. An infinitely differentiable complex valued function $\varphi(x)$ for $x \in I$ is said to be in the space \mathcal{F}_α , such that

$$\gamma_{\alpha,k}(\varphi) = \sup_{x \in I} |\lambda_\alpha^-(x) x^{-\frac{1}{2}} A_x^k \varphi(x)| < \infty,$$

where $\alpha > 0, k \in \mathbb{N}_0, A_x$ is the differential operator defined as (6) and $\lambda_\alpha^-(x)$ is the continuous function given by

$$\lambda_\alpha^-(x) = \begin{cases} e^{-\frac{\alpha}{4x-1}}, & x \in (\frac{1}{4}, \frac{1}{2}] \\ e^{-\alpha(4x-1)}, & x \in [\frac{1}{2}, \infty). \end{cases}$$

Definition 3.2. An infinitely differentiable complex valued function $\psi(x)$ for $x \in I$ is said to be in the space \mathcal{G}_α , such that

$$\Gamma_{\alpha,k}(\psi) = \sup_{x \in I} |\lambda_\alpha^+(x) x^{-\frac{1}{2}} A_x^k \psi(x)| < \infty,$$

where $\alpha > 0, k \in \mathbb{N}_0, A_x$ is the differential operator defined as (6) and $\lambda_\alpha^+(x)$ is the continuous function given by

$$\lambda_\alpha^+(x) = \begin{cases} e^{\frac{\alpha}{4x-1}}, & x \in (\frac{1}{4}, \frac{1}{2}] \\ e^{\alpha(4x-1)}, & x \in [\frac{1}{2}, \infty). \end{cases}$$

For every $\varphi \in \mathcal{G}_\alpha$, we have

$$\begin{aligned} \gamma_{\alpha,k}(\varphi) &= \sup_{x \in I} |\lambda_\alpha^-(x) x^{-\frac{1}{2}} A_x^k \varphi(x)| \\ &= \sup_{x \in I} |(\lambda_\alpha^-(x))^2 \lambda_\alpha^+(x) x^{-\frac{1}{2}} A_x^k \varphi(x)| \\ &\leq C \Gamma_{\alpha,k}(\varphi) < \infty. \end{aligned}$$

where $C > 0$ is a constant.

Moreover, $P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) \notin \mathcal{G}_\alpha$. Hence \mathcal{G}_α is proper subset of \mathcal{F}_α and the topology of \mathcal{G}_α is stronger than that induced on it by \mathcal{F}_α . So, $\mathcal{F}'_\alpha \subset \mathcal{G}'_\alpha$.

- Remark 3.3.** (i) The differential operator A_x is continuous linear mapping from \mathcal{G}_α onto itself.
- (ii) The differential operator A_x is continuous linear mapping from \mathcal{F}_α onto itself.
- (iii) The differential operator A_x is continuous linear mapping from \mathcal{G}_α into \mathcal{F}_α .

Theorem 3.4. The MFC-transform is continuous linear mapping from \mathcal{G}_α into \mathcal{F}_α .

Proof. Consider $\varphi \in \mathcal{G}_\alpha$, and using (4) and (8), we have

$$A_\tau^k(\mathfrak{M}_\mathbb{C}\varphi)(\tau) = \sum_{m=1}^{2k} q_m^k(\tau) \int_{\frac{1}{4}}^\infty D_\tau^m P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) \varphi(x) \frac{dx}{\sqrt{x}}.$$

Using (23) and Definition 3.1, we have

$$\begin{aligned} \gamma_{\alpha,k}(\mathfrak{M}_\mathbb{C}\varphi) &\leq \Gamma_{\alpha,0}(\varphi) \sup_{\tau \in I} \left| \tau^{-\frac{1}{2}} \lambda_\alpha^-(\tau) e^{\pi\sqrt{\tau}} \sum_{m=1}^{2k} m! q_m^{2k}(\tau) \sum_{l=0}^m \sum_{s=0}^{m-l} C_s \tau^{-(2(m-l)-s)/2} \sum_{r=0}^l C'_r \tau^{-(2(l-r)/2)} \right| \\ &\quad \times \int_{\frac{1}{4}}^\infty \frac{1}{x^{-\frac{1}{2}} \lambda_\alpha^+(x) \sqrt{x}} dx \end{aligned}$$

Now using Definition 3.2, then for $\alpha > 0$ and sufficiently large constant $M' > 0$, we have

$$\gamma_{\alpha,k}(\mathfrak{M}_\mathbb{C}\varphi) \leq M' \Gamma_{\alpha,0}(\varphi) < \infty.$$

Hence proved. \square

4. Pseudo-differential operators

The theory of pseudo-differential operators (p.d.o.) have been first developed in 1960 to treat the problems of partial differential equations. The theory of p.d.o. already discussed by using the theory of integral transforms like Fourier transforms, Hankel transforms, Fourier-Jacobi transforms, etc. in the work of [4, 6, 13, 15, 17, 20]. Motivated by them here we define p.d.o. in terms of the Mehler-Fock-Clifford transform.

Let a partial differential operator $P(x, A_x)$ on I is

$$P(x, A_x) = \sum_{r=0}^m a_r(x) A_x^r, \tag{28}$$

where $a_r(x)$ are functions defined on I and A_x is the differential operator as (6). If we replace A_x by monomial $(-1)(\tau + \frac{1}{4})$ in (28), we obtain

$$P(x, \tau) = \sum_{r=0}^m a_r(x) (-1)^r \left(\tau + \frac{1}{4}\right)^r. \tag{29}$$

Now from (28), we have

$$P(x, A_x)f(x) = \sum_{r=0}^m a_r(x) \left[\mathfrak{M}_{\mathbb{C}}^{-1} \mathfrak{M}_{\mathbb{C}}(A_x^r f) \right](x).$$

Using (19), we have

$$P(x, A_x)f(x) = \sum_{r=0}^m a_r(x) \left[\mathfrak{M}_{\mathbb{C}}^{-1} \left((-1)^r \left(\tau + \frac{1}{4}\right)^r \mathfrak{M}_{\mathbb{C}} f \right) \right](x).$$

From (5) and (29), we get

$$P(x, A_x)f(x) = \frac{1}{2} \int_0^\infty \tanh(\pi \sqrt{\tau}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) P(x, \tau) (\mathfrak{M}_{\mathbb{C}} f)(\tau) d\tau. \tag{30}$$

If we replace $P(x, \tau)$ by more general symbol $\sigma(x, \tau)$ in (30), which is no longer polynomial in τ only, the operator so obtained is called pseudo-differential operator associated with MFC-transform.

Definition 4.1. The symbol class \mathbb{S}^m is the collection of infinitely differentiable complex valued function $\sigma(x, \tau)$ for $(x, \tau) \in (I \times \mathbb{R}_+)$. The function $\sigma(x, \tau) \in \mathbb{S}^m$ iff for $\mu, \nu, l \in \mathbb{N}_0$, and $m \in \mathbb{R}_+$ there exists a constant $C = C_{\mu, \nu, l} > 0$ such that

$$(1+x)^l |D_x^\nu D_\tau^\mu \sigma(x, \tau)| \leq C e^{-m\tau}. \tag{31}$$

Definition 4.2. For the symbol $\sigma(x, \tau) \in \mathbb{S}^m$, the pseudo-differential operator associated with MFC-transform is defined as

$$(\mathcal{P}_\sigma \varphi)(x) = \frac{1}{2} \int_0^\infty \tanh(\pi \sqrt{\tau}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) \sigma(x, \tau) (\mathfrak{M}_{\mathbb{C}} \varphi)(\tau) d\tau. \tag{32}$$

Theorem 4.3. The pseudo-differential operator is the continuous linear mapping from \mathcal{G}_α into \mathcal{F}_α .

Proof. From (8) and (32), we have

$$\begin{aligned}
 A_x^k(\mathcal{P}_\sigma\varphi)(x) &= \frac{1}{2} \int_0^\infty \tanh(\pi \sqrt{\tau}) \sum_{j=1}^{2k} q_j^{2k}(x) D_x^j [P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x})\sigma(x, \tau)] (\mathfrak{M}_\mathbb{C}\varphi)(\tau) d\tau \\
 &= \frac{1}{2} \int_0^\infty \tanh(\pi \sqrt{\tau}) \sum_{j=1}^{2k} q_j^{2k}(x) \sum_{r=0}^j \binom{j}{r} D_x^r P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{\tau}) D_x^{j-r} \sigma(x, \tau) (\mathfrak{M}_\mathbb{C}\varphi)(\tau) d\tau. \\
 &= \frac{1}{2} \int_0^\infty \tanh(\pi \sqrt{\tau}) \sum_{j=1}^{2k} q_j^{2k}(x) \sum_{r=0}^j \binom{j}{r} D_x^r P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{\tau}) D_x^{j-r} \sigma(x, \tau) \left(\frac{5}{4} + \tau\right)^{-n} \\
 &\quad \times \sum_{s=0}^n \binom{n}{s} (-1)^s \left[(-1)^s \left(\tau + \frac{1}{4}\right)^s\right] (\mathfrak{M}_\mathbb{C}\varphi)(\tau) d\tau. \tag{33}
 \end{aligned}$$

Now

$$\begin{aligned}
 (-1)^s \left(\tau + \frac{1}{4}\right)^s (\mathfrak{M}_\mathbb{C}\varphi)(\tau) &= \int_{\frac{1}{4}}^\infty (-1)^s \left(\tau + \frac{1}{4}\right)^s P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y}) \varphi(y) \frac{dy}{\sqrt{y}} \\
 &= \int_{\frac{1}{4}}^\infty (A_y^s P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y})) \varphi(y) \frac{dy}{\sqrt{y}} \\
 &= \int_{\frac{1}{4}}^\infty P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y}) (A_y^s)^s (y^{-1/2} \varphi(y)) dy.
 \end{aligned}$$

Using (17) and (22), we have

$$\left|(-1)^s \left(\tau + \frac{1}{4}\right)^s (\mathfrak{M}_\mathbb{C}\varphi)(\tau)\right| \leq C \Gamma_{\alpha,s}(\varphi) \int_{\frac{1}{4}}^\infty \frac{1}{\lambda_\alpha^+(y)} dy,$$

where $C > 0$ is a constant. Invoking Definition 3.2, the last integral converges. Therefore there exists $C' > 0$, such that

$$\left|(-1)^s \left(\tau + \frac{1}{4}\right)^s (\mathfrak{M}_\mathbb{C}\varphi)(\tau)\right| \leq C' \Gamma_{\alpha,s}(\varphi). \tag{34}$$

Now from (20), (31), (33) and (34), we have

$$\begin{aligned}
 \gamma_{\alpha,k}(\mathcal{P}_\sigma\varphi) &\leq \sup_{x \in I} \left| x^{-\frac{1}{2}} \lambda_\alpha^-(x) C (1+x)^{-l} \sum_{j=1}^{2k} 2^j q_j^{2k}(x) \right| \sum_{s=0}^n \binom{n}{s} C' \Gamma_{\alpha,s}(\varphi) \\
 &\quad \times M \int_0^\infty e^{-m\tau} \cosh(\pi \sqrt{\tau}) \left(\frac{5}{4} + \tau\right)^{-n} d\tau.
 \end{aligned}$$

Thus integral converges for any $m > 0$ and supremum exists finitely. Hence

$$\gamma_{\alpha,k}(\mathcal{P}_\sigma\varphi) \leq C'' \Gamma_{\alpha,s}(\varphi),$$

where $C'' > 0$ is a constant. \square

An integral representation of p.d.o.:

If we consider a function $g_x(y)$ associated with the symbol $\sigma(x, \tau)$ as

$$g_x(y) = \frac{1}{2} \int_0^\infty \tanh(\pi \sqrt{\tau}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{x}) P_{i\sqrt{\tau-\frac{1}{2}}}(2\sqrt{y}) \sigma(x, \tau) d\tau, \tag{35}$$

then from (32), (4) and using Fubini’s theorem, the pseudo-differential operator can be defined as

$$(\mathcal{P}_\sigma\varphi)(x) = \int_{\frac{1}{4}}^\infty \left(\frac{1}{2} \int_0^\infty \tanh(\pi\sqrt{\tau}) P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{y}) \sigma(x, \tau) d\tau \right) \varphi(y) \frac{dy}{\sqrt{y}}.$$

Now using (35), the integral is reduced as

$$(\mathcal{P}_\sigma\varphi)(x) = \int_{\frac{1}{4}}^\infty g_x(y) \varphi(y) y^{-\frac{1}{2}} dy. \tag{36}$$

This is another integral representation of the p.d.o.

Theorem 4.4. *If $g_x(y)$ is defined as (35), then*

$$|g_x(y)| \leq C' (1+x)^{-l}, \tag{37}$$

where $C' > 0$ is a constant.

Proof. From (35), (22) and (31), we have

$$|g_x(y)| \leq C \frac{1}{2} \int_0^\infty (1+x)^{-l} e^{-m\tau} d\tau,$$

where $C > 0$ is constant. Clearly the integral is convergent for $m > 0$, therefore

$$|g_x(y)| \leq C' (1+x)^{-l},$$

where $C' > 0$ is a constant. \square

Now we obtain an estimate of p.d.o. defined as (32). From (36) and (37), we have

$$\begin{aligned} |(\mathcal{P}_\sigma\varphi)(x)| &\leq C'(1+x)^{-l} \int_{\frac{1}{4}}^\infty |\varphi(y)| y^{-\frac{1}{2}} dy \\ &\leq C'(1+x)^{-l} \|\varphi\|_{L^1(I; x^{-\frac{1}{2}} dx)}, \end{aligned}$$

where $C' > 0$ is certain constant and $l \in \mathbb{N}_0$.

$$\|(\mathcal{P}_\sigma\varphi)\|_{L^\infty(I)} \leq C'' \|\varphi\|_{L^1(I; x^{-\frac{1}{2}} dx)},$$

where $C'' > 0$ is a constant.

Special Case: If we consider symbol $\sigma(x, \tau)$ in such a way that it can be represented explicitly as $\sigma(x, \tau) = V(x)W(\tau)$, provided $V(x) \neq 0$, then p.d.o. defined as (32) is represented as

$$(\mathcal{P}_\sigma\varphi)(x) = \frac{1}{2} \int_0^\infty P_{i\sqrt{\tau}-\frac{1}{2}}(2\sqrt{x}) \tanh(\pi\sqrt{\tau}) V(x) W(\tau) (\mathfrak{M}_\mathbb{C}\varphi)(\tau) d\tau. \tag{38}$$

Now by application of inverse MFC-transform, (38) reduces to

$$\left(\mathfrak{M}_\mathbb{C} \left[\frac{\mathcal{P}_\sigma\varphi}{V} \right] \right) (\tau) = W(\tau) (\mathfrak{M}_\mathbb{C}\varphi)(\tau). \tag{39}$$

Further, if we suppose $W(\tau) = (\mathfrak{M}_\mathbb{C}\psi)(\tau)$ in (39) then from (16), we have

$$\left(\mathfrak{M}_\mathbb{C} \left[\frac{\mathcal{P}_\sigma\varphi}{V} \right] \right) (\tau) = \mathfrak{M}_\mathbb{C}(\varphi * \psi)(\tau).$$

Invoking inverse MFC-transform, we obtain

$$(\mathcal{P}_\sigma\varphi)(x) = V(x)(\varphi * \psi)(x).$$

Furthermore, by using Hölder's inequality and (26), we have

$$\begin{aligned} \|\mathcal{P}_\sigma \varphi\|_{L^1(I; x^{-\frac{1}{2}} dx)} &\leq \|\psi * \varphi\|_{L^p(I; x^{-\frac{1}{2}} dx)} \|V\|_{L^q(I; x^{-\frac{1}{2}} dx)} \\ &\leq \|\psi\|_{L^1(I; x^{-\frac{1}{2}} dx)} \|\varphi\|_{L^p(I; x^{-\frac{1}{2}} dx)} \|V\|_{L^q(I; x^{-\frac{1}{2}} dx)}, \end{aligned}$$

where $p, q > 1$ and $1/p + 1/q = 1$.

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