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Almost Sure Convergence for Self-Normalized Products of Sums of Partial Sums of ρ^--Mixing Sequences

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Abstract. Let $X, X_1, X_2, \ldots$ be a stationary sequence of ρ^--mixing positive random variables. A universal result in the area of almost sure central limit theorems for the self-normalized products of sums of partial sums $(\prod_{j=1}^{k}(T_j/(j(j+1)\mu/2)))^{\mu/(\beta V_k)}$ is established, where: $T_j = \sum_{i=1}^{j} S_i, S_i = \sum_{k=1}^{i} X_k, V_k = \sqrt{\sum_{i=1}^{k} X_i^2}, \mu = \mathbb{E}X, \beta > 0$. Our results generalize and improve those on almost sure central limit theorems obtained by previous authors from the independent case to ρ^--mixing sequences and from partial sums case to self-normalized products of sums of partial sums.

1. Introduction

Starting with Brosamler [6] and Schatte [22] established the almost sure central limit theorem (ASCLT) for partial sums S_n/σ_n of independent random variables. Several authors investigated ASCLT for partial sums S_n/σ_n of random variables in the last two decades. Some improved and generalized ASCLT results for partial sums were obtained by Brosamler [6], Schatte [22], Lacey and Philipp [16], Ibragimov and Lifshits [14], Berkes and Csáki [4], Hörmann [11], Miao [18], Zang [33] and Wu [27]. If σ_n is replaced by an estimate from the given data, usually denoted by $V_n =: \sqrt{\sum_{i=1}^{n} X_i^2}$, V_n is called a self-normalizer of partial sums. A class of self-normalized random variables has been proposed and studied in Peligrad and Shao [20], Peña et al. [19] and references therein. The limit theorems for the self-normalized sums S_n/V_n have been developed significantly in the past decade. We refer the reader to: Bentkus and Gótze [3] for the Berry-Esseen bound, Giné et al. [10] for the asymptotic normality, Hu et al. [12] for the Cramér type moderate deviations, Csörgo et al. [9] for the Donsker's theorem, Huang and Pang [13], Zhang and Yang [37], Wu [28], Wu and Jiang [31] and Wu [32] for the almost sure central limit theorems.

The study of the sums of partial sums was initiated by Resnick [21] and [2] who obtained the central limit theorem (CLT) for sums of records. As we know, the sum of exponential records is the sum of partial sums of exponential random variables. So it is necessary to study the sum of partial sums. [38] obtained the ASCLT for products of sums of partial sums. Furthermore, [29] proved the ASCLT for the self-normalized

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products of sums of partial sums that reads as follows: Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. positive random variables in the domain of attraction of the normal law with mean $\mu > 0$. Then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left(\left(\prod_{j=1}^{k}\left(\frac{T_j}{j(j+1)\mu/2}\right)\right)^{\mu/V_k} \leq x\right) = F(x) \quad \text{a.s.} \quad x \in \mathbb{R},$$

where sums of partial sums $T_j =: \sum_{i=1}^{j} S_i$, $d_k =: k^{-1}\exp(\ln^{\alpha} k)$, $D_n =: \sum_{k=1}^{n} d_k$, $0 \leq \alpha < 1/2$, I denotes indicator function, F is the distribution function of the random variable $\exp(\sqrt{10/3}\mathcal{N})$, and $\mathcal{N}$ stands for the standard normal random variable.

Following introduced the related concept of ρ^--mixing. Let $\sigma(S)$ be the σ-field generated by $\{X_k; k \in S \subset \mathbb{N}\}$. Let C be a class of functions which are increasing for every variable (or decreasing for every variable).

Random variables $X_1, X_2, \ldots, X_n, n \geq 2$, are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f_1(X_i; i \in A_1), f_2(X_j; j \in A_2)) \leq 0,$$

where $f_1, f_2 \in C$ such that this covariance exists. A sequence of random variables $\{X_n; n \geq 1\}$ is said to be NA if its every finite subfamily is NA.

A sequence of random variables $\{X_n; n \geq 1\}$ is called ρ^*-mixing if

$$\rho^*(n) =: \sup\{\rho(S,T); S, T \subset \mathbb{N}, \mathrm{dist}(S,T) \geq n\} \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$\rho(S,T) =: \sup\left\{\frac{|\mathbb{E}(f - \mathbb{E}f)(g - \mathbb{E}g)|}{\|f - \mathbb{E}f\|_2 \|g - \mathbb{E}g\|_2}; f \in \mathcal{L}_2(\sigma(S)), g \in \mathcal{L}_2(\sigma(T))\right\}, \quad \|X\|_p =: (\mathbb{E}|X|^p)^{1/p},$$

and

$$\mathrm{dist}(S,T) =: \min\{|j-k|; j \in S, k \in T\}.$$

A sequence of random variables $\{X_n; n \geq 1\}$ is called ρ^--mixing if

$$\rho^-(n) =: \sup\{\rho^-(S,T); S, T \subset \mathbb{N}, \mathrm{dist}(S,T) \geq n\} \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$\rho^-(S,T) =: 0 \vee \sup\left\{\frac{\mathrm{Cov}(f(X_i, i \in S), g(X_j, j \in T))}{\sqrt{\mathrm{Var}(f(X_i, i \in S))\mathrm{Var}(g(X_j, j \in T))}}; f, g \in C\right\},$$

and

$$a \vee b =: \max(a,b).$$

The concept of negative association was introduced by Alam and Saxena [1] and Joag-Dev and Proschan [15]. The concept of ρ^--mixing was introduced by Zhang and Wang [34]. Obviously, ρ^--mixing random variables include NA and ρ^*-mixing random variables. Because of the wide applications of ρ^--mixing random variables in multivariate statistical analysis and reliability theory, the limit behaviors of ρ^--mixing random variables have received extensive attention recently. One can refer to: Zhang and Wang [34] for fundamental properties, Zhang [35, 36] for central limit theorem (CLT), Cai [7] for the moment inequalities and convergence rates in the strong laws, Wang and Lu [26] for the inequalities of maximum of partial sums and weak convergence, and Tan et al. [25] for the ASCLT.

Many results concerning the limit theory for the self-normalized random sequences and for the ρ^--mixing random sequences have been obtained, respectively. However, since the denominator of the self-normalized random sequences contains random variables, the study of limit theory for the self-normalized random sequences of ρ^- random variables is very difficult, and so far, there are very few research results in this field. Thus, this is a challenging, difficult and meaningful research topic.

The purpose of this article is based on the Wu [29], to establish the ASCLT for the self-normalized products of sums of partial sums $(\prod_{j=1}^{k}(T_j/(j(j+1)\mu/2)))^{\mu/(\beta V_k)}$ of ρ^--mixing random variables, where $T_j=\sum_{i=1}^{j}S_i, S_i=\sum_{k=1}^{i}X_k, V_k=\sqrt{\sum_{i=1}^{k}X_i^2}, \mu=\mathbb{E}X, \beta>0$. We will show that the ASCLT holds under a fairly general growth condition.

In the following, $a_n \sim b_n$ denotes $\lim_{n\to\infty} a_n/b_n = 1$, and the symbol c stands for a generic positive constant which may differ from one place to another. We assume that $\{X, X_n; n\geq 1\}$ is a stationary sequence of ρ^--mixing positive random variables with $\mathbb{E}X=\mu>0$.

For every $1\leq i\leq k\leq n$, define:

$$S_k =: \sum_{i=1}^{k} X_i, \quad T_k =: \sum_{i=1}^{k} S_i, \quad V_k^2 =: \sum_{i=1}^{k}(X_i-\mu)^2,$$
$$\bar{X}_{i,k} =: -\sqrt{k}I(X_i-\mu<-\sqrt{k}) + (X_i-\mu)I(|X_i-\mu|\leq\sqrt{k}) + \sqrt{k}I(X_i-\mu>\sqrt{k}),$$
$$\bar{S}_{i,k} =: \sum_{j=1}^{i} c_{j,k}\bar{X}_{j,k}, \quad \text{where} \quad c_{j,k} =: 2\sum_{l=j}^{k}\frac{l+1-j}{l(l+1)},$$

and

$$\sigma_k^2 =: \mathrm{Var}\bar{S}_{k,k}, \quad \delta_k^2 =: \mathbb{E}\bar{X}_{1,k}^2.$$

Our theorem is formulated in a general setting.

Theorem 1.1. *Let $\{X, X_n; n\geq 1\}$ be a stationary sequence of ρ^--mixing positive random variables with $\mathbb{E}X=\mu>0$ satisfying*

$$\sum_{k=1}^{\infty}\rho^-(k)<\infty, \tag{1}$$

$$\mathbb{E}(X^2h(X))<\infty, \quad \mathbb{P}(X\geq\mu)>0, \quad \mathbb{P}(X<\mu)>0, \tag{2}$$

where $h>0$ is a increasing slowly varying function at infinity satisfying $\int_1^{\infty}\frac{1}{xh(x)}<\infty$,

$$\sum_{k=2}^{\infty}|\mathrm{Cov}(X_1,X_k)|<\infty, \quad \mathrm{Var}X_1+2\sum_{k=2}^{\infty}\mathrm{Cov}(X_1,X_k)>0, \tag{3}$$

and

$$\sigma_n^2 \sim \frac{10n\beta^2\delta_n^2}{3} =: B_n^2 \quad \text{for some} \quad \beta>0. \tag{4}$$

Set

$$d_k=\frac{L(k)}{k}, \quad D_n=\sum_{k=1}^{n}d_k, \tag{5}$$

where $L(\cdot)>0$ is a slowly varying function at infinity and there exist constants $c>0$ and $\theta>0$ such that

$$\max_{1\leq k\leq n}L(k)\leq c\frac{D_n}{(\ln D_n)^{1+\theta}}. \tag{6}$$

Then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\left(\prod_{j=1}^{k}\left(\frac{T_j}{j(j+1)\mu/2}\right)\right)^{\mu/(\beta V_k)}\le x\right)=F(x)\quad \text{a.s.}\quad \text{for any}\quad x\in\mathbb{R}, \tag{7}$$

here and in the sequel, F is the distribution function of the random variable $\exp(\sqrt{10/3}\mathcal{N})$*, and* $\mathcal{N}$ *is a standard normal random variable.*

By the terminology of summation procedures (see e.g. Chandrasekharan and Minakshisundaram [8], p.35), Theorem 1.1 remains valid if we replace the weight sequence $\{d_k; k\ge 1\}$ by any $\{d_k^*; k\ge 1\}$ such that $0\le d_k^*\le d_k$, $\sum_{k=1}^{\infty}d_k^*=\infty$.

Suppose that $\{X, X_n; n\ge 1\}$ is a sequence of NA random variables, then $\rho^-(k)=0$ for any $k\ge 1$, further, by the following proof of Theorem 1.1, the condition $\mathbb{E}(X^2h(X))<\infty$ can be reduced to the condition $\mathbb{E}X^2<\infty$. Therefore, we have the following Corollary.

Corollary 1.2. *Let* $\{X, X_n; n\ge 1\}$ *be a stationary sequence of NA positive random variables with* $\mathbb{E}X=\mu>0$ *satisfying conditions (3)-(6), and* $0<\mathbb{E}(X-\mu)^2I(X-\mu\ge 0)<\infty, 0<\mathbb{E}(X-\mu)^2I(X-\mu<0)<\infty$*. Then (7) holds.*

Remark 1.3. *If* $\{X, X_n; n\ge 1\}$ *is a sequence of i.i.d. random variables, then by the Lemma 3.1 (iii) in Appendix and* $\mathbb{E}\bar{X}_{1,n}\to 0$ *as* $n\to\infty$*,*

$$\sigma_n=:\mathrm{Var}\left(\sum_{j=1}^{n}c_{j,n}\bar{X}_{j,n}\right)=\sum_{j=1}^{n}c_{j,n}^2\mathrm{Var}\bar{X}_{1,n}\sim\frac{10n}{3}\mathrm{Var}\bar{X}_{1,n}\sim\frac{10n\delta_n^2}{3}.$$

Hence, (4) holds and $\beta=1$*.*

Remark 1.4. *Let* $L(k)=\exp(\ln^{\gamma}k), 0\le\gamma<1/2$*. Then from (13) in Wu [30], we get*

$$\max_{1\le k\le n}L(k)=\exp(\ln^{\gamma}n)\le c\frac{D_n}{(\ln D_n)^{1+\theta}},$$

where, $\theta=1/\gamma-2>0$*. Hence, condition (6) holds for* $L(k)=\exp(\ln^{\gamma}k), 0\le\gamma<1/2$*. Therefore, Theorem 1.1 generalizes theorem 1.1 in Wu [29].*

2. Proofs

We will point out that it is of great difficulties and challenges to extend the sequence of random variables from independent be extended to ρ^--mixing for self-normalized random sequences and, to overcome the difficulties and challenges we need the following two Lemmas. The moment inequality of Lemma 2.1 is obtained by Wang and Lu [26] and it is a basic tool for studying the limit theory of the partial sums of ρ^--mixed random variables. Lemma 2.2 plays a key role in proving Theorem 1.1. The proof of Lemma 2.2 is very difficult and tedious, so the proof of Lemma 2.2 is given in Appendix. In the appendix, in order to prove Lemma 2.2, Lemmas 3.1 to 3.4 are required.

Lemma 2.1. ([26]) *Let* $\{X_i; i\ge 1\}$ *be a sequence of* ρ^-*-mixing random variables with zero means and such that* $\mathbb{E}|X_i|^p<\infty$, $i=1,2,\ldots$ *and* $p\ge 2$*. Then for* $S_n=\sum_{i=1}^{n}X_i$*,*

$$\mathbb{E}\left(\max_{1\le j\le n}|S_j|^p\right)\le c_p\left(\sum_{i=1}^{n}\mathbb{E}|X_i|^p+\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{p/2}\right),$$

where $c_p>0$ *only depends on p.*

Lemma 2.2. *Suppose that the assumptions of Theorem* 1.1 *hold. Then:*

$$\frac{\bar{S}_{n,n}-\mathbb{E}\bar{S}_{n,n}}{B_n}\xrightarrow{d}\mathcal{N},\quad \text{as}\quad n\to\infty, \tag{8}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\le x\right\}=\Phi(x)\quad \text{a.s.}\quad \text{for}\quad \text{any}\quad x\in\mathbb{R}, \tag{9}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\left(I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\sqrt{k})\right)-\mathbb{E}I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\sqrt{k})\right)\right)=0\quad \text{a.s.}, \tag{10}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\left(f\left(\frac{\bar{V}_{k,l}^2}{k\delta_{k,l}^2}\right)-\mathbb{E}f\left(\frac{\bar{V}_{k,l}^2}{k\delta_{k,l}^2}\right)\right)=0\quad \text{a.s.},\quad l=1,2 \tag{11}$$

where B_k , d_k and D_n are defined by (4)-(6), *respectively,* $\Phi(x)$ *is the standard normal distribution function, and f is a bounded function with bounded continuous derivatives.*

Proof of Theorem 1.1. Let $Z_j=T_j/(j(j+1)\mu/2)$; then (7) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\frac{\sqrt{3}\mu}{\sqrt{10}\beta V_k}\sum_{i=1}^{k}\ln Z_i\le x\right)=\Phi(x)\quad \text{a.s.}\quad \text{for}\quad \text{any}\quad x\in\mathbb{R}, \tag{12}$$

where $\Phi(x)$ is the standard normal distribution function.

Let q be a real number $q\in(4/3,2)$. By condition (2) and (3), using the Marcinkiewicz-Zygmund strong law of large numbers for ρ^--mixing sequences (see Lemma 2.7 in Tan et al. [25]), we have

$$S_k-\mu k=o(k^{1/q})\quad \text{a.s.}\quad k\to\infty.$$

Thus,

$$|Z_i-1|=\frac{\left|\sum_{j=1}^{i}S_j-i(i+1)\mu/2\right|}{i(i+1)\mu/2}\le\frac{\sum_{j=1}^{i}|S_j-\mu j|}{i(i+1)\mu/2}\le\frac{\sum_{j=1}^{i}j^{1/q}}{i(i+1)\mu/2}\le c\frac{i^{1/q+1}}{i^2}=ci^{1/q-1}\to 0\quad \text{a.s.}$$

Hence let $a_k=:\sqrt{10(1\pm\varepsilon)k/3}\beta\delta_k$ for any given $0<\varepsilon<1$, by $|\ln(1+x)-x|=O(x^2)$ for $|x|<1/2$, and $\delta_k^2\to\mathbb{E}(X-\mu)^2>0$ as $k\to\infty$,

$$\begin{aligned}\left|\frac{1}{a_k}\sum_{i=1}^{k}\ln Z_i-\frac{1}{a_k}\sum_{i=1}^{k}(Z_i-1)\right| &\le c\frac{1}{\sqrt{k}}\sum_{i=1}^{k}(Z_i-1)^2\le\frac{c}{\sqrt{k}}\sum_{i=1}^{k}i^{2(1/q-1)}\\ &\le c\frac{1}{k^{3/2-2/q}}\to 0\quad \text{a.s.}\quad k\to\infty,\end{aligned}$$

from $3/2-2/q>0$.

Therefore, for any $\delta>0$ and almost every event ω, there exists $k_0=k_0(\omega,\delta,x)$ such that for $k>k_0$,

$$\left\{\frac{\mu}{a_k}\sum_{i=1}^{k}(Z_i-1)\le x-\delta\right\}\subseteq\left\{\frac{\mu}{a_k}\sum_{i=1}^{k}\ln Z_i\le x\right\}\subseteq\left\{\frac{\mu}{a_k}\sum_{i=1}^{k}(Z_i-1)\le x+\delta\right\}. \tag{13}$$

By (2.30) of [29], under the condition $|X_j - \mu| \le \sqrt{k}, 1 \le j \le k$, we have

$$\mu \sum_{i=1}^{k} (Z_i - 1) = \bar{S}_{k,k}. \tag{14}$$

Thus, by (13) and (14) for any given $0 < \varepsilon < 1, \delta > 0$, we have for $x \ge 0$ and $k > k_0$,

$$\begin{aligned}
\left\{\frac{\sqrt{3}\mu}{\sqrt{10}\beta V_k}\sum_{i=1}^{k}\ln Z_i \le x\right\} &\subseteq \left\{\frac{\sqrt{3}\mu}{\sqrt{10}\beta V_k}\sum_{i=1}^{k}\ln Z_i \le x, \forall 1 \le i \le k, |X_i - \mu| \le \sqrt{k}, \bar{V}_k^2 \le (1+\varepsilon)k\delta_k^2\right\} \\
&\quad \bigcup \left\{\bar{V}_k^2 > (1+\varepsilon)k\delta_k^2\right\} \bigcup \left\{\exists 1 \le i \le k, |X_i - \mu| > \sqrt{k}\right\} \\
&\subseteq \left\{\frac{\sqrt{3}\mu}{\beta\delta_k\sqrt{10(1+\varepsilon)k}}\sum_{i=1}^{k}(Z_i - 1) \le x + \delta, \forall 1 \le i \le k, |X_i - \mu| \le \sqrt{k}\right\} \\
&\quad \bigcup \left\{\bar{V}_k^2 > (1+\varepsilon)k\delta_k^2\right\} \bigcup \left\{\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right\} \\
&\subseteq \left\{\frac{\sqrt{3}\bar{S}_{k,k}}{\beta\delta_k\sqrt{10(1+\varepsilon)k}} \le x + \delta\right\} \bigcup \left\{\bar{V}_k^2 > (1+\varepsilon)k\delta_k^2\right\} \bigcup \left\{\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right\},
\end{aligned}$$

where $\bar{V}_k^2 =: \sum_{j=1}^{k} \bar{X}_{j,k}^2$. Hence, combine (4)

$$I\left(\frac{\sqrt{3}\mu}{\sqrt{10}\beta V_k}\sum_{i=1}^{k}\ln Z_i \le x\right) \le I\left(\frac{\bar{S}_{k,k}}{\sqrt{(1+\varepsilon)}B_k} \le x + \delta\right) + I\left(\bar{V}_k^2 > (1+\varepsilon)k\delta_k^2\right) + I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right), \quad \text{for } x \ge 0.$$

Similarly, we have for any given $0 < \varepsilon < 1$ and $x < 0$,

$$I\left(\frac{\sqrt{3}\mu}{\sqrt{10}\beta V_k}\sum_{i=1}^{k}\ln Z_i \le x\right) \le I\left(\frac{\bar{S}_{k,k}}{\sqrt{(1-\varepsilon)}B_k} \le x + \delta\right) + I\left(\bar{V}_k^2 < (1-\varepsilon)k\delta_k^2\right) + I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right).$$

Furthermore, we get

$$I\left(\frac{\sqrt{3}\mu}{\sqrt{10}\beta V_k}\sum_{i=1}^{k}\ln Z_i \le x\right) \ge I\left(\frac{\bar{S}_{k,k}}{\sqrt{(1-\varepsilon)}B_k} \le x - \delta\right) - I\left(\bar{V}_k^2 < (1-\varepsilon)k\delta_k^2\right) - I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right), \quad \text{for } x \ge 0,$$

$$I\left(\frac{\sqrt{3}\mu}{\sqrt{10}\beta V_k}\sum_{i=1}^{k}\ln Z_i \le x\right) \ge I\left(\frac{\bar{S}_{k,k}}{\sqrt{(1+\varepsilon)}B_k} \le x - \delta\right) - I\left(\bar{V}_k^2 > (1+\varepsilon)k\delta_k^2\right) - I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right), \quad \text{for } x < 0.$$

Hence, in order to establish (12), it suffices to prove

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\frac{\bar{S}_{k,k}}{B_k} \le x\right) = \Phi(x) \ \text{ a.s. } \ \text{for any } x \in \mathbb{R}, \tag{15}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right) = 0 \ \text{ a.s.}, \tag{16}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I(\bar{V}_k^2 > (1+\varepsilon)k\delta_k^2) = 0 \ \text{ a.s. } \ \text{for any } 0 < \varepsilon < 1, \tag{17}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I(\bar{V}_k^2 < (1-\varepsilon)k\delta_k^2) = 0 \ \text{ a.s. for any } \ 0<\varepsilon<1. \tag{18}$$

Next, we prove (15)-(18) with Lemma 2.2, first we prove (15). By $\mathbb{E}(X_i-\mu)=0$ and $\mathbb{E}(X-\mu)^2<\infty$, similar to (2.37) of [29], we have $|\mathbb{E}\bar{S}_{k,k}| = o(\sqrt{k})$, as $k\to\infty$. This, and the fact that $B_k = O(\sqrt{k})$, when $k\to\infty$, implies for any $x\in\mathbb{R}$ and $\alpha>0$

$$I\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\le x-\alpha\right)\le I\left(\frac{\bar{S}_{k,k}}{B_k}\le x\right)\le I\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\le x+\alpha\right).$$

Thus, by (9) in Lemma 2.2, we get as $n\to\infty$,

$$\begin{aligned}
\Phi(x-\alpha) &\leftarrow \frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\le x-\alpha\right)\\
&\le \frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\frac{\bar{S}_{k,k}}{B_k}\le x\right)\\
&\le \frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\le x+\alpha\right)\\
&\to \Phi(x+\alpha)\ \text{ a.s.}
\end{aligned}$$

Letting $\alpha\to 0$ in the above formula, by the continuity of Φ, we obtain that (15) holds.

Now, we prove (16). Note that $\mathbb{E}(X-\mu)^2<\infty$ implies $k\mathbb{P}(|X-\mu|>\sqrt{k})\to 0$ as $k\to\infty$. Thus, by (10) in Lemma 2.2 and the Toeplitz lemma,

$$\begin{aligned}
0 &\le \frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\sqrt{k})\right)\\
&\sim \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E}\left(I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\sqrt{k})\right)\right)\\
&= \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{P}\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\sqrt{k})\right)\\
&\le \frac{1}{D_n}\sum_{k=1}^{n} d_k k\mathbb{P}(|X-\mu|>\sqrt{k})\\
&\to 0 \ \text{ a.s. as } \ n\to\infty.
\end{aligned}$$

That is, (16) holds.

Finally, we prove (17) and (18). If $\{X_i; i\ge 1\}$ is a sequence of ρ^--mixing random variables, and $\{f_i; i\ge 1\}$ is a sequence of increasing (or decreasing) functions, then from Property P2 in [35], $\{f_i(X_i); i\ge 1\}$ is also a sequence of ρ^--mixing random variables. And so for each fixed n, $\{\bar{X}_{i,n}; 1\le i\le n\}$ is also a sequence of ρ^--mixing random variables from $\bar{X}_{i,n}$ being increasing on X_i. However, $\bar{X}_{i,n}^2$ is not monotonous about $\bar{X}_{i,n}$, so we consider $\bar{X}_{i,n}^2 I(\bar{X}_{i,n}\ge 0)$ and $\bar{X}_{i,n}^2 I(\bar{X}_{i,n}<0)$ respectively. For each fixed n, $\{\bar{X}_{i,n}^2 I(\bar{X}_{i,n}\ge 0); 1\le i\le n\}$ and $\{\bar{X}_{i,n}^2 I(\bar{X}_{i,n}<0); 1\le i\le n\}$ are also two sequences of ρ^--mixing random variables from $\bar{X}_{i,n}^2 I(\bar{X}_{i,n}\ge 0)$ and $\bar{X}_{i,n}^2 I(\bar{X}_{i,n}<0)$ being increasing and decreasing on $\bar{X}_{i,n}$ respectively. Let

$$\bar{V}_{k,1}^2 =: \sum_{j=1}^{k}\bar{X}_{j,k}^2 I(\bar{X}_{j,k}\ge 0),\quad \bar{V}_{k,2}^2 =: \sum_{j=1}^{k}\bar{X}_{j,k}^2 I(\bar{X}_{j,k}<0),\ \delta_{k,1}^2 =: \mathbb{E}\bar{X}_{1,k}^2 I(\bar{X}_{1,k}\ge 0),\quad \delta_{k,2}^2 =: \mathbb{E}\bar{X}_{1,k}^2 I(\bar{X}_{1,k}<0).$$

Obviously,

$$\delta_k^2 = \delta_{k,1}^2 + \delta_{k,2}^2, \quad \bar{V}_k^2 = \bar{V}_{k,1}^2 + \bar{V}_{k,2}^2, \quad \mathbb{E}\bar{V}_k^2 = k\delta_k^2 = k\delta_{k,1}^2 + k\delta_{k,2}^2.$$

It follows that

$$\begin{aligned} I(\bar{V}_k^2 > (1+\varepsilon)k\delta_k^2) &= I(\bar{V}_k^2 - \mathbb{E}\bar{V}_k^2 > \varepsilon k\delta_k^2) \le I(\bar{V}_{k,1}^2 - \mathbb{E}\bar{V}_{k,1}^2 > \varepsilon k\delta_k^2/2) + I(\bar{V}_{k,2}^2 - \mathbb{E}\bar{V}_{k,2}^2 > \varepsilon k\delta_k^2/2) \\ &\le I(\bar{V}_{k,1}^2 > (1+\varepsilon/2)k\delta_{k,1}^2) + I(\bar{V}_{k,2}^2 > (1+\varepsilon/2)k\delta_{k,2}^2). \end{aligned}$$

Therefore, by the arbitrariness of $\varepsilon > 0$, in order to prove (17), it suffices to show that,

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I(\bar{V}_{k,l}^2 > (1+\varepsilon)k\delta_{k,l}^2) = 0 \text{ a.s. for } l = 1, 2. \tag{19}$$

Note that for each fixed n, $\{\bar{X}_{i,n}^2 I(\bar{X}_{i,n} \ge 0) - \mathbb{E}\bar{X}_{i,n}^2 I(\bar{X}_{i,n} \ge 0); 1 \le i \le n\}$ is a sequence of ρ^--mixing random variables with mean zero. From Lemma 2.1, the Markov inequality, the c_r inequality, $\mathbb{E}\bar{V}_{k,1}^2 = k\delta_{k,1}^2$, $\delta_{k,1}^2 \to \mathbb{E}(X-\mu)^2 I(X-\mu \ge 0)$ as $k \to \infty$, and condition $\mathbb{P}(X \ge \mu) > 0$ in (2) implies $\mathbb{E}(X-\mu)^2 I(X-\mu \ge 0) > 0$, we get

$$\begin{aligned} \mathbb{P}\left(\bar{V}_{k,1}^2 > (1+\varepsilon/2)k\delta_{k,1}^2\right) &= \mathbb{P}\left(\bar{V}_{k,1}^2 - \mathbb{E}\bar{V}_{k,1}^2 > \varepsilon k\delta_{k,1}^2/2\right) \le c\frac{\mathbb{E}(\bar{V}_{k,1}^2 - \mathbb{E}\bar{V}_{k,1}^2)^2}{k^2} \\ &= ck^{-2}\mathbb{E}\left(\sum_{i=1}^{k}\left(\bar{X}_{i,k}^2 I(\bar{X}_{i,k} \ge 0) - \mathbb{E}\bar{X}_{i,k}^2 I(\bar{X}_{i,k} \ge 0)\right)\right)^2 \\ &\le ck^{-2}\sum_{i=1}^{k}\mathbb{E}\left(\bar{X}_{i,k}^2 I(\bar{X}_{i,k} \ge 0) - \mathbb{E}\bar{X}_{i,k}^2 I(\bar{X}_{i,k} \ge 0)\right)^2 \\ &\le ck^{-1}\mathbb{E}\bar{X}_{1,k}^4 I(\bar{X}_{1,k} \ge 0) \\ &\le ck^{-1}\left(\mathbb{E}(X-\mu)^4 I(0 \le X-\mu \le \sqrt{k}) + k^2 P(|X-\mu| > \sqrt{k})\right). \end{aligned} \tag{20}$$

Since $\mathbb{E}(X-\mu)^2 < \infty$ implies $x^2 P(|X-\mu| > x) = o(1)$, as $x \to \infty$, we have $kP(|X-\mu| > \sqrt{k}) \to 0$. Hence

$$\begin{aligned} \mathbb{E}(X-\mu)^4 I(0 \le X-\mu \le \sqrt{k}) &= \int_0^\infty \mathbb{P}\left(|X-\mu| I(0 \le X-\mu \le \sqrt{k}) > t\right) 4t^3 \mathrm{d}t \\ &\le c\int_0^{\sqrt{k}} \mathbb{P}(|X-\mu| > t)t^3 \mathrm{d}t = \int_0^{\sqrt{k}} o(1)t\mathrm{d}t \\ &= o(1)k. \end{aligned}$$

From this, and (20) yields,

$$\mathbb{P}\left(\bar{V}_{k,1}^2 > (1+\varepsilon/2)k\delta_{k,1}^2\right) \to 0, \quad \text{as } k \to \infty.$$

For a given $\varepsilon > 0$, let f denote a bounded function with bounded continuous derivatives, such that

$$I(x > 1+\varepsilon) \le f(x) \le I(x > 1+\varepsilon/2).$$

Therefore, it follows from (11) in Lemma 2.2 and the Toeplitz lemma that,

$$\begin{aligned} 0 &\le \frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\bar{V}_{k,1}^2 > (1+\varepsilon)k\delta_{k,1}^2\right) \le \frac{1}{D_n}\sum_{k=1}^{n} d_k f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) \\ &\sim \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E} f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) \le \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E} I\left(\bar{V}_{k,1}^2 > (1+\varepsilon/2)k\delta_{k,1}^2\right) \\ &= \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{P}(\bar{V}_{k,1}^2 > (1+\varepsilon/2)k\delta_{k,1}^2) \\ &\to 0 \quad \text{a.s.} \quad \text{as } n\to\infty \end{aligned}$$

Hence, (19) holds for $l = 1$. Using similar methods to those used in the proof of (19) for $l = 1$, we can prove that (19) holds for $l = 2$. Consequently, (17) holds. Moreover, applying identical methods to those used in the proof of (17), we can prove (18).

This completes the proof of Theorem 1.1.

The idea of proving Theorem 1.1 is to transform almost sure central limit theorem (ASCLT) for self-normalized products of sums of partial sums into ASCLT for self-normalized partial sums. Then the ASCLT for self-normalized partial sums is transformed into the ASCLT for partial sums and the ASCLT for three tail sequences, that is, the proof (12) is converted into the proof (15)-(18). Finally, the four ASCLT are proved by Lemma 2.2. In addition, it is important to point out the following two points. First, in order to ensure that the sequence of truncated random variables is still ρ^--mixed, the truncated function must be monotone and cannot be truncated as a sequence of independent random variables. Second, the proof of (17) needs to be translated into a proof of (19). Because $\bar{V}_k^2 =: \sum_{j=1}^k \bar{X}_{j,k}^2$, and $\bar{X}_{j,k}^2$ is not monotone about $\bar{X}_{j,k}^2$ and cannot guarantee that $\{\bar{X}_{j,k}^2; 1 \le j \le k\}$ is still a ρ^--mixed sequence. Therefore, the moment inequality of Lemma 2.1 cannot be used for $\{\bar{X}_{j,k}^2; 1 \le j \le k\}$. The conversion of $\bar{V}_k^2$ to $\bar{V}_{k,1}^2$ and $\bar{V}_{k,2}^2$ is precisely because $\bar{X}_{j,k}^2 I(\bar{X}_{j,k} \ge 0)$ and $\bar{X}_{j,k}^2 I(\bar{X}_{j,k} < 0)$ in $\bar{V}_{k,1}^2 =: \sum_{j=1}^k \bar{X}_{j,k}^2 I(\bar{X}_{j,k} \ge 0)$ and $\bar{V}_{k,2}^2 =: \sum_{j=1}^k \bar{X}_{j,k}^2 I(\bar{X}_{j,k} < 0)$ are monotone with respect to $\bar{X}_{j,k}^2$, so $\{\bar{X}_{j,k}^2 I(\bar{X}_{j,k} \ge 0); 1 \le j \le k\}$ and $\{\bar{X}_{j,k}^2 I(\bar{X}_{j,k} < 0); 1 \le j \le k\}$ are still ρ^--mixed sequences for which the moment inequality of Lemma 2.1 can be used. For the sequence of independent random variables, we can prove (17) directly, which is the difference between ρ^--mixed sequence and independent sequence.

3. Appendix

As it has been mentioned, we give the proof of Lemma 2.2 in this part of our paper. In order to prove Lemma 2.2, the following four Lemmas are required. Lemma 3.1 can be directly verified, Lemma 3.2 is due to Zhang [35] and it is mainly used to prove the (8) of Lemma 2.2. Lemma 3.3 is due to Zhang [36] and it is mainly used to estimate the covariance of functions of random variables. Lemma 3.4 is of our authorship and it is a powerful tool to prove almost sure central limit theorem. In this paper, Lemma 3.4 is mainly used to prove (9)-(11) of Lemma 2.2.

Lemma 3.1. (i) $c_{i,n} \le 2b_{i,n}$, *where* $b_{i,n} =: \sum\limits_{j=i}^{n} \frac{1}{j}$;

(ii) $\sum\limits_{i=1}^{n} b_{i,n}^2 = 2n - b_{1,n} \sim 2n$;

(iii) $\sum\limits_{i=1}^{n} c_{i,n}^2 = \frac{10n}{3} - 4b_{1,n} + \frac{10n}{3(n+1)} \sim \frac{10n}{3}$.

Lemma 3.2. ([36]) *Let $\{X_{ni}; 1 \le i \le n, n \ge 1\}$ be an array random variables with zero means and $\mathbb{E}X_{ni}^2 < \infty$ for each $i = 1, 2, \ldots, n$. Assume that for fixed n, $\{X_{ni}; 1 \le i \le n\}$ is a sequence of ρ^--mixing random variables. Let*

$\{a_{ni}; 1 \le i \le n, n \ge 1\}$ be an array of real numbers with $a_{ni} = \pm 1$ for each $i = 1, 2, \ldots, n$. Denote $A_n^2 =: \mathrm{Var}\left(\sum_{i=1}^{n} a_{ni} X_{ni}\right)$ and suppose that

$$\sup_{n\ge 1} \frac{1}{A_n^2} \sum_{i=1}^{n} \mathbb{E}X_{ni}^2 < \infty,$$

$$\limsup_{n\to\infty} \frac{1}{A_n^2} \sum_{1\le i,j\le n, |i-j|\ge k} \left(\mathrm{Cov}(X_{ni}, X_{nj})\right)^- \to 0, \text{ as } k \to \infty,$$

where $a^- =: \max(-a, 0)$, and the following Lindeberg condition is satisfied:

$$\frac{1}{A_n^2} \sum_{i=1}^{n} \mathbb{E}X_{ni}^2 I\{|X_{ni}| \ge \varepsilon A_n\} \to 0 \text{ as } n \to \infty \text{ for every } \varepsilon > 0.$$

Then

$$\frac{1}{A_n} \sum_{i=1}^{n} a_{ni} X_{ni} \xrightarrow{d} \mathcal{N}, \text{ as } n \to \infty,$$

where $\xrightarrow{d}$ denotes the convergence in distribution.

Lemma 3.3. ([35]) *Suppose that $f(x)$ and $g(x)$ are real, bounded and absolutely continuous functions on $\mathbb{R}$ with $|f'(x)| \le c_1$ and $|g'(x)| \le c_2$. Then for any random variables X and Y,*

$$|\mathrm{Cov}(f(X), g(Y))| \le c_1 c_2 \left\{|\mathrm{Cov}(X, Y)| + 8\rho^-(X, Y)\|X\|_{2,1}\|Y\|_{2,1}\right\},$$

where $\|X\|_{2,1} =: \int_0^\infty \mathbb{P}^{1/2}(|X| > x)\mathrm{d}x$.

Lemma 3.4. *Let $\{\xi, \xi_n; n \ge 1\}$ be a sequence of uniformly bounded random variables. If there exist constants $c > 0$ and $\delta > 0$ such that*

$$\mathbb{E}(\xi_k \xi_j) \le c\left(\frac{k}{j}\right)^{\delta} + c\rho^-(k), \text{ for } 1 \le 2k < j, \tag{21}$$

and $\sum_{k=1}^{\infty} \frac{\rho^-(k)}{k} < \infty$, then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \xi_k = 0 \quad \text{a.s.}, \tag{22}$$

where d_k and D_n are defined by (5) and (6).

Proof. From the proof of Theorem 1 in Wu [27], in order to prove (22), it suffices to prove that there exists a constant $\lambda > 0$ such that

$$\mathbb{E}\left(\sum_{k=1}^{n} d_k \xi_k\right)^2 \le c\frac{D_n^2}{(\ln D_n)^{1+\lambda}}. \tag{23}$$

Note that

$$\begin{aligned}\mathbb{E}\left(\sum_{k=1}^{n} d_k \xi_k\right)^2 &\le 2\sum_{1\le k\le j\le n, 2k\ge j} d_k d_j \mathbb{E}(\xi_k \xi_j) + 2\sum_{1\le k\le j\le n, 2k<j} d_k d_j \mathbb{E}(\xi_k \xi_j) \\ &=: T_1 + T_2. \end{aligned} \tag{24}$$

By $\mathbb{E}(\xi_k\xi_j) \le c$ for any $k, j \ge 1$, and (6)

$$T_1 \le c\sum_{k=1}^{n}\sum_{j=k}^{\min(2k,n)} d_k d_j \le \max_{1\le k\le n} L(k)\sum_{k=1}^{n} d_k \sum_{j=k}^{2k}\frac{1}{j} \le c\frac{D_n^2}{(\ln D_n)^{1+\theta}}. \tag{25}$$

Using the property of slowly varying function: $\sum_{j=k}^{\infty} L(j)/j^{1+\delta} \le ck^{-\delta}L(k)$, (21) and condition $\sum_{k=1}^{\infty}(\rho^-(k))/k < \infty$,

$$\begin{aligned} T_2 &\le c\sum_{1\le k\le j\le n, 2k<j} d_k d_j\left(\left(\frac{k}{j}\right)^{\delta} + \rho^-(k)\right) \\ &\le c\sum_{k=1}^{n}\frac{L(k)}{k^{1-\delta}}\sum_{j=k}^{n}\frac{L(j)}{j^{1+\delta}} + \max_{1\le k\le n} L(k)\sum_{j=1}^{n} d_j\sum_{k=1}^{n}\frac{\rho^-(k)}{k} \\ &\le c\max_{1\le k\le n} L(k)\sum_{k=1}^{n} d_k \le c\frac{D_n^2}{(\ln D_n)^{1+\theta}}. \end{aligned}$$

This, combining with (24) and (25) implies that (23) holds.

Proof of Lemma 2.2. Firstly, we prove (8). For fixed n, $\{c_{i,n}\bar{X}_{i,n}; 1 \le i \le n\}$ is a sequence of ρ^--mixing random variables. Let $a_{in} \equiv 1$ in Lemma 3.2, using Lemma 3.2 for $\{c_{i,n}(\bar{X}_{i,n} - \mathbb{E}\bar{X}_{i,n}); 1 \le i \le n\}$, thus, by (4): $\sigma_n \sim B_n$ as $n \to \infty$, in order to prove (8), it suffices to show that

$$\sup_{n\ge1}\frac{1}{\sigma_n^2}\sum_{i=1}^{n} c_{i,n}^2\mathbb{E}(\bar{X}_{i,n} - \mathbb{E}\bar{X}_{i,n})^2 < \infty, \tag{26}$$

$$\limsup_{n\to\infty}\frac{1}{\sigma_n^2}\sum_{1\le i,j\le n, |i-j|\ge k}\left(\mathrm{Cov}(c_{i,n}\bar{X}_{i,n}, c_{j,n}\bar{X}_{j,n})\right)^- \to 0, \ \text{ as } \ k \to \infty, \tag{27}$$

and for every $\varepsilon > 0$,

$$\frac{1}{\sigma_n^2}\sum_{i=1}^{n}\mathbb{E}c_{i,n}^2\bar{X}_{i,n}^2 I\{|c_{i,n}\bar{X}_{i,n}| \ge \varepsilon\sigma_n\} \to 0, \ \text{ as } \ n \to \infty. \tag{28}$$

By $\delta_n^2 =: \mathbb{E}\bar{X}_{i,n}^2 = \mathbb{E}\bar{X}_{1,n}^2 \to \mathbb{E}(X_1 - \mu)^2 > 0$ as $n \to \infty$, conditions (2) and (4): $\sigma_n^2 \sim cn$, and Lemma 3.1 (iii),

$$\sup_{n\ge1}\frac{1}{\sigma_n^2}\sum_{i=1}^{n} c_{i,n}^2\mathbb{E}(\bar{X}_{i,n} - \mathbb{E}\bar{X}_{i,n})^2 \le \sup_{n\ge1}\frac{cn}{\sigma_n^2} < \infty.$$

That is, (26) holds.

In order to estimate $\|X_i\|_{2,1}$, we first prove that for any r.v. $X > 0$ and a increasing slowly varying function at infinity h,

$$\mathbb{E}(X^2h(X)) < \infty \Longleftrightarrow \int_1^{\infty} xh(x)\mathbb{P}(X > x)\mathrm{d}x < \infty. \tag{29}$$

Let $f(x) = x^2h(x), x \ge 0$, and f^{-1} be its inverse function. By Karamata's representation in Seneta [23], we have $h(x) \sim c\exp\left(\int_1^x \frac{b(u)}{u}\mathrm{d}u\right)$, where $\lim_{x\to\infty} b(x) = 0$. This implies that $f'(x) \sim 2xh(x) + xh(x)b(x) \sim 2xh(x)$. Therefore,

$$\begin{aligned} \mathbb{E}(X^2h(X)) &\sim \int_1^{\infty}\mathbb{P}(X^2h(X) > x)\mathrm{d}x = \int_1^{\infty}\mathbb{P}(X > f^{-1}(x))\mathrm{d}x = \int_1^{\infty}\mathbb{P}(X > y)f'(y)\mathrm{d}y \qquad (\text{let } \ y = f^{-1}(x)) \\ &\sim 2\int_1^{\infty} yh(y)\mathbb{P}(X > y)\mathrm{d}y. \end{aligned}$$

This implies that (29) holds.

From (2), (29) and Cauchy-Scharz inequality, for any $i \geq 1$,

$$\begin{aligned} \|X_i\|_{2,1} &\leq 1 + \int_1^\infty \mathbb{P}^{1/2}(X > x)\mathrm{d}x = 1 + \int_1^\infty \sqrt{xh(x)\mathbb{P}(X > x)}\frac{1}{\sqrt{xh(x)}}\mathrm{d}x \\ &\leq 1 + \sqrt{\int_1^\infty xh(x)\mathbb{P}(X > x)\mathrm{d}x}\sqrt{\int_1^\infty \frac{1}{xh(x)}\mathrm{d}x} \\ &\leq c. \end{aligned} \tag{30}$$

By Lemma 3.1 (iii), (4), the stationarity assumption on the $\{X_i\}$, and Lemma 3.3 is applied with: $f(x) =: \sqrt{k}I(x-\mu < -\sqrt{k}) + (x-\mu)I(|x-\mu| \leq \sqrt{k}) + \sqrt{k}I(x-\mu > \sqrt{k})$, $g(y) =: \sqrt{j}I(y-\mu < -\sqrt{j}) + (y-\mu)I(|y-\mu| \leq \sqrt{j}) + \sqrt{j}I(y-\mu > \sqrt{j})$, we get

$$\begin{aligned} 0 &\leq \frac{1}{\sigma_n^2}\sum_{1\leq i,j\leq n,|i-j|\geq k}\left(\mathrm{Cov}(c_{i,n}\bar{X}_{i,n}, c_{j,n}\bar{X}_{j,n})\right)^- \\ &\leq \frac{c}{\sigma_n^2}\sum_{1\leq i,j\leq n,|i-j|\geq k} c_{i,n}c_{j,n}\left(|\mathrm{Cov}(X_i, X_j)| + \rho^-(|i-j|)\|X_i\|_{2,1}\|X_j\|_{2,1}\right) \\ &\leq \frac{c}{n}\sum_{1\leq i,j\leq n,j-i\geq k} c_{i,n}^2\left(|\mathrm{Cov}(X_1, X_{j-i+1})| + \rho^-(j-i)\right) \\ &\leq \frac{c}{n}\sum_{i=1}^n c_{i,n}^2\sum_{m\geq k}(|\mathrm{Cov}(X_1, X_m)| + \rho^-(m)) \\ &\leq c\sum_{m\geq k}(|\mathrm{Cov}(X_1, X_m)| + \rho^-(m)). \end{aligned}$$

This implies that (27) holds from (1) and (3).

By Lemma 3.1 (i), $|\bar{X}_{i,n}| \leq |X_i|$ and $\mathbb{E}\bar{X}_{i,n}^2 \leq \mathbb{E}X^2 < \infty$, for any $1 \leq i \leq n, n \geq 1$,

$$\mathbb{E}\bar{X}_{i,n}^2 I(|c_{i,n}\bar{X}_{i,n}| > \varepsilon\sigma_n) \leq \mathbb{E}X^2 I\left(|X| > \varepsilon\sigma_n/c_{1,n} \geq c\sqrt{n}/\ln n\right) \to 0, \quad \text{as} \quad n \to \infty.$$

Hence, by the Toeplitz lemma and Lemma 3.1 (iii), Lindeberg condition (28)

$$\frac{1}{\sigma_n^2}\sum_{i=1}^n c_{i,n}^2\mathbb{E}\bar{X}_{i,n}^2 I(|c_{i,n}\bar{X}_{i,n}^2| > \varepsilon\sigma_n) \to 0, \quad \text{as} \quad n \to \infty$$

holds.

Now, we prove (9). (8) implies that for any function $g \in \mathcal{A}$, where $\mathcal{A}$ denotes the class of bounded function with bounded continuous derivatives,

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k\mathbb{E}g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{B_k}\right) = \mathbb{E}g(\mathcal{N}).$$

On the other hand, it follows from Theorem 7.1 of Billingsley [5] and Section 2 of Peligrad and Shao [20] that (11) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k g\left(\frac{\bar{S}_{k,k} - \mathbb{E}\bar{S}_{k,k}}{B_k}\right) = \mathbb{E}g(\mathcal{N}) \quad \text{a.s.}$$

Hence, in order to prove (9), it suffices to show that

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\left(g\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\right)-\mathbb{E}g\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\right)\right)=0\quad\text{a.s.},\tag{31}$$

for any $g\in\mathcal{A}$.

Let for $k\geq 1$,

$$\xi_k=g\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\right)-\mathbb{E}g\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\right).$$

Observe that, for any $1\leq 2k<j$, we get,

$$\begin{aligned}|\mathbb{E}\xi_k\xi_j| &= \left|\mathrm{Cov}\left(g\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\right),g\left(\frac{\bar{S}_{j,j}-\mathbb{E}\bar{S}_{j,j}}{B_j}\right)\right)\right|\\ &\leq \left|\mathrm{Cov}\left(g\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\right),g\left(\frac{\bar{S}_{j,j}-\mathbb{E}\bar{S}_{j,j}}{B_j}\right)-g\left(\frac{\sum_{i=2k+1}^{j}c_{i,j}(\bar{X}_{i,j}-\mathbb{E}\bar{X}_{i,j})}{B_j}\right)\right)\right|\\ &\quad+\left|\mathrm{Cov}\left(g\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{B_k}\right),g\left(\frac{\sum_{i=2k+1}^{j}c_{i,j}(\bar{X}_{i,j}-\mathbb{E}\bar{X}_{i,j})}{B_j}\right)\right)\right|\\ &=: I_1+I_2.\end{aligned}\tag{32}$$

Clearly, since g is a bounded Lipschitz function, there exists a constant $c>0$ such that $|g(x)|\leq c$, and $|g(x)-g(y)|\leq c|x-y|$, for any $x,y\in\mathbb{R}$. For fixed j, as $\{c_{i,j}\bar{X}_{i,j};1\leq i\leq j\}$ is a sequence of ρ^--mixing random variables, as well as Lemma 3.1 (i) (ii), Lemma 2.1, $\ln x\leq\beta^{-1}x^{\beta},\beta>0,x\geq 1$ and condition $\delta_n^2\to\mathbb{E}(X-\mu)^2$, $\mathbb{E}\bar{X}_{i,j}^2\leq\mathbb{E}(X-\mu)^2$, $0<\mathbb{E}(X-\mu)^2<\infty$ and (4): $B_j\sim c\sqrt{j}$, we obtain that

$$\begin{aligned}I_1 &\leq c\frac{\mathbb{E}\left|\sum_{i=1}^{2k}c_{i,j}(\bar{X}_{i,j}-\mathbb{E}\bar{X}_{i,j})\right|}{\sqrt{j}}\leq c\frac{\sqrt{\mathbb{E}\left(\sum_{i=1}^{2k}c_{i,j}(\bar{X}_{i,j}-\mathbb{E}\bar{X}_{i,j})\right)^2}}{\sqrt{j}}\\ &\leq c\frac{\sqrt{\sum_{i=1}^{2k}b_{i,j}^2\mathbb{E}\bar{X}_{i,j}^2}}{\sqrt{j}}\leq c\frac{\sqrt{\sum_{i=1}^{2k}(b_{i,k}+b_{k+1,j})^2}}{\sqrt{j}}\\ &\leq c\frac{\sqrt{\sum_{i=1}^{2k}b_{i,2k}^2+\sum_{i=1}^{2k}b_{k+1,j}^2}}{\sqrt{j}}\leq c\frac{\sqrt{k+k\ln^2(j/k)}}{\sqrt{j}}\\ &\leq c\left(\frac{k}{j}\right)^{1/4}.\end{aligned}\tag{33}$$

Note that g is a bounded function with bounded continuous derivatives, so, from Lemma 3.3,

$$\begin{aligned}I_2 &\leq c\left|\mathrm{Cov}\left(\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{\sqrt{k}},\frac{\sum_{i=2k+1}^{j}c_{i,j}(\bar{X}_{i,j}-\mathbb{E}\bar{X}_{i,j})}{\sqrt{j}}\right)\right|\\ &\quad+8\rho^-(k)\left\|\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{\sqrt{k}}\right\|_{2,1}\left\|\frac{\sum_{i=2k+1}^{j}c_{i,j}(\bar{X}_{i,j}-\mathbb{E}\bar{X}_{i,j})}{\sqrt{j}}\right\|_{2,1}\\ &=: I_{21}+I_{22}.\end{aligned}\tag{34}$$

Thus, From (1), (3), (30), the stationarity of $\{X_i\}$, Lemma 3.1 (i), (iii), and Lemma 3.3, we have

$$\begin{aligned}
I_{21} &\le \frac{c}{\sqrt{kj}}\sum_{l=1}^{k} c_{l,k}\sum_{i=2k+1}^{j} c_{i,j}\left|\mathrm{Cov}(\bar{X}_{l,k},\bar{X}_{i,j})\right| \\
&\le \frac{c}{\sqrt{kj}}\sum_{l=1}^{k} c_{l,k}\sum_{i=2k+1}^{j} c_{i,j}\{|\mathrm{Cov}(X_l,X_i)|+\rho^-(i-l)\|X_i\|_{2,1}\|X_l\|_{2,1}\} \\
&\le \frac{c}{\sqrt{kj}}\sum_{l=1}^{k} c_{l,k}\sum_{m=2k-l+1}^{j-l} c_{m+l-1,j}\{|\mathrm{Cov}(X_1,X_{m+1})|+\rho^-(m)\} \\
&\le \frac{c}{\sqrt{kj}}\sum_{l=1}^{k} c_{l,k}\sum_{m=k}^{j} c_{k,j}\{|\mathrm{Cov}(X_1,X_{m+1})|+\rho^-(m)\} \\
&\le \frac{c}{\sqrt{kj}}\left(\sum_{l=1}^{k} c_{l,k}^2\right)^{1/2}\left(\sum_{l=1}^{k} 1^2\right)^{1/2}\ln\frac{j}{k}\sum_{m=1}^{\infty}\{|\mathrm{Cov}(X_1,X_{m+1})|+\rho^-(m)\} \\
&\le c\left(\frac{k}{j}\right)^{1/4}.
\end{aligned} \tag{35}$$

On the other hand, by following inequality (cf. Ledoux and Talagrand [17], p. 251)

$$\|X\|_{2,1}\le \frac{r}{r-2}\|X\|_r \quad (r>2). \tag{36}$$

Since $\int_1^\infty \frac{\mathrm{d}t}{th(t)}<\infty$ and h is increasing. By Cauchy criterion, for $\varepsilon=1$, there is a constant $M>0$ such that

$$1>\int_{\sqrt{x}}^{x}\frac{\mathrm{d}t}{th(t)}\ge\frac{1}{h(x)}\int_{\sqrt{x}}^{x}\frac{\mathrm{d}t}{t}=\frac{\ln x}{2h(x)} \quad \text{for all } x>M.$$

Hence, $\mathbb{E}(X^2\ln X)\le c\mathbb{E}(X^2h(X))<\infty$. Combining with (36), Lemma 3.1 (iii) and Lemma 2.1, for $2<r<3$, we have

$$\begin{aligned}
\left\|\frac{\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}}{\sqrt{k}}\right\|_{2,1} &\le c\frac{(\mathbb{E}|\bar{S}_{k,k}-\mathbb{E}\bar{S}_{k,k}|^r)^{1/r}}{\sqrt{k}} \\
&\le ck^{-1/2}\left(\sum_{i=1}^{k} c_{i,k}^r\mathbb{E}|\bar{X}_{i,k}|^r+\left(\sum_{i=1}^{k} c_{i,k}^2\mathbb{E}\bar{X}_{i,k}^2\right)^{r/2}\right)^{1/r} \\
&\le ck^{-1/2}\left(\ln^{r-2}k\sum_{i=1}^{k} c_{i,k}^2\mathbb{E}(\bar{X}_{1,k}^2\ln|\bar{X}_{1,k}|)\frac{k^{(r-2)/2}}{\ln k}+k^{r/2}\right)^{1/r} \\
&\le ck^{-1/2}\left(k^{r/2}\ln^{r-3}k+k^{r/2}\right)^{1/r} \\
&\le c,
\end{aligned}$$

and

$$\begin{aligned}
\left\|\frac{\sum_{i=2k+1}^{j} c_{i,j}(\bar{X}_{i,j}-\mathbb{E}\bar{X}_{i,j})}{\sqrt{j}}\right\|_{2,1} &\le cj^{-1/2}\left(\mathbb{E}\left(\sum_{i=2k+1}^{j} c_{i,j}(\bar{X}_{i,j}-\mathbb{E}\bar{X}_{i,j})\right)^r\right)^{1/r} \\
&\le cj^{-1/2}\left(\sum_{i=2k+1}^{j} c_{i,j}^r\mathbb{E}|\bar{X}_{i,j}|^r+\left(\sum_{i=2k+1}^{j} c_{i,j}^2\mathbb{E}\bar{X}_{i,j}^2\right)^{r/2}\right)^{1/r}
\end{aligned}$$

$$\begin{aligned}&\leq c j^{-1/2}\left(\ln^{r-2} j \sum_{i=1}^{j} c_{i,j}^2 \mathbb{E}(\bar{X}_{i,j}^2 \ln|\bar{X}_{i,j}|)\frac{j^{(r-2)/2}}{\ln j} + j^{r/2}\right)^{1/r}\\ &\leq c.\end{aligned}$$

Hence, $I_{22} \leq c\rho^-(k)$, this combining with (32)-(35), we get $|\mathbb{E}\xi_k\xi_j| \leq c((k/j)^{1/4} + \rho^-(k))$ for $1 \leq 2k < j$. Hence, by Lemma 3.4, (31) holds.

Next, we prove (10). Let

$$Z_k = I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right) - \mathbb{E}I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right) \quad \text{for any } k \geq 1.$$

For $1 \leq 2k < j$,

$$\begin{aligned}\mathbb{E}(Z_kZ_j) &= \operatorname{Cov}\left(I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right), I\left(\bigcup_{i=1}^{j}(|X_i - \mu| > \sqrt{j})\right)\right)\\ &= \operatorname{Cov}\left(I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right), I\left(\bigcup_{i=1}^{j}(|X_i - \mu| > \sqrt{j})\right) - I\left(\bigcup_{i=2k+1}^{j}(|X_i - \mu| > \sqrt{j})\right)\right)\\ &\quad + \operatorname{Cov}\left(I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right), I\left(\bigcup_{i=2k+1}^{j}(|X_i - \mu| > \sqrt{j})\right)\right)\\ &=: I_3 + I_4. \end{aligned} \tag{37}$$

It is known that $I(A \cup B) - I(B) \leq I(A)$ for any sets A and B, we get

$$\begin{aligned} I_3 &\leq \mathbb{E}\left|I\left(\bigcup_{i=1}^{j}(|X_i - \mu| > \sqrt{j})\right) - I\left(\bigcup_{i=2k+1}^{j}(|X_i - \mu| > \sqrt{j})\right)\right|\\ &\leq \mathbb{E}I\left(\bigcup_{i=1}^{2k}(|X_i - \mu| > \sqrt{j})\right) \leq ck\mathbb{P}(|X - \mu| > \sqrt{j})\\ &\leq c\frac{k}{j}. \end{aligned} \tag{38}$$

From the definition of $\rho^-(k)$, we have

$$\begin{aligned} I_4 &\leq \rho^-(k)\sqrt{\operatorname{Var}\left(I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right)\right)\operatorname{Var}\left(I\left(\bigcup_{i=2k+1}^{j}(|X_i - \mu| > \sqrt{j})\right)\right)}\\ &\leq \rho^-(k)\sqrt{\mathbb{E}\left(I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \sqrt{k})\right)\right)\mathbb{E}\left(I\left(\bigcup_{i=2k+1}^{j}(|X_i - \mu| > \sqrt{j})\right)\right)}\\ &\leq \rho^-(k)\sqrt{\sum_{i=1}^{k}\mathbb{P}(|X_i - \mu| > \sqrt{k})\sum_{i=2k+1}^{j}\mathbb{P}(|X_i - \mu| > \sqrt{j})}\\ &\leq \rho^-(k)\sqrt{k\frac{\mathbb{E}(X - \mu)^2}{k}j\frac{\mathbb{E}(X - \mu)^2}{j}}\\ &\leq c\rho^-(k). \end{aligned}$$

This implies $\mathbb{E}(Z_kZ_j) \le c(k/j + \rho^-(k))$ for $1 \le 2k < j$ from (37) and (38). Hence, by Lemma 3.4, (10) holds.

Finally, we prove (11). Let

$$\eta_k = f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) - \mathbb{E}f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right) \quad \text{for} \quad \text{any} \quad k \ge 1.$$

Since f is a bounded function with bounded continuous derivatives, so, from Lemma 3.3, $\delta_{j,1}^2 \to \mathbb{E}(X-\mu)^2 I(X-\mu \ge 0)$, $0 < \mathbb{E}(X-\mu)^2 I(X-\mu \ge 0) < \infty$, and $\sum_{m=2}^{\infty} |\mathrm{Cov}(X_1, X_m)| < \infty$, we have, for $1 \le 2k < j$,

$$\begin{aligned}
|\mathbb{E}\eta_k\eta_j| &= \left|\mathrm{Cov}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\bar{V}_{j,1}^2}{j\delta_{j,1}^2}\right)\right)\right| \\
&= \left|\mathrm{Cov}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\bar{V}_{j,1}^2}{j\delta_{j,1}^2}\right) - f\left(\frac{\sum_{i=2k+1}^{j} \bar{X}_{i,j}^2 I(\bar{X}_{i,j} \ge 0)}{j\delta_{j,1}^2}\right)\right)\right| \\
&\quad + \left|\mathrm{Cov}\left(f\left(\frac{\bar{V}_{k,1}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\sum_{i=2k+1}^{j} \bar{X}_{i,j}^2 I(\bar{X}_{i,j} \ge 0)}{j\delta_{j,1}^2}\right)\right)\right| \\
&\le c\frac{\mathbb{E}\left(\sum_{i=1}^{2k} \bar{X}_{i,j}^2 I(\bar{X}_{i,j} \ge 0)\right)}{j} + c\left|\mathrm{Cov}\left(\frac{\bar{V}_{k,1}^2}{k}, \frac{\sum_{i=2k+1}^{j} \bar{X}_{i,j}^2 I(\bar{X}_{i,j} \ge 0)}{j}\right)\right| \\
&\quad + 8\rho^-(k)\left\|\frac{\bar{V}_{k,1}^2}{k}\right\|_{2,1}\left\|\frac{\sum_{i=2k+1}^{j} \bar{X}_{i,j}^2 I(\bar{X}_{i,j} \ge 0)}{j}\right\|_{2,1} \\
&=: I_5 + I_6 + I_7. \qquad (39)
\end{aligned}$$

Obviously, $I_5 \le ck/j$, following estimates I_6. From (30), (1), (3), and Lemma 3.3 is applied with: $f(x) =: (x-\mu)^2 I(0 \le x-\mu \le \sqrt{k}) + kI(x-\mu > \sqrt{k})$, $g(y) =: (y-\mu)^2 I(0 \le y-\mu \le \sqrt{j}) + jI(y-\mu > \sqrt{j})$, the stationarity assumption on the $\{X_i\}$, it follows that

$$\begin{aligned}
I_6 &\le \frac{c}{kj}\sum_{l=1}^{k}\sum_{i=2k+1}^{j}\left|\mathrm{Cov}\left(\bar{X}_{l,k}^2 I(\bar{X}_{l,k} \ge 0), \bar{X}_{i,j}^2 I(\bar{X}_{i,j} \ge 0)\right)\right| \\
&\le \frac{c}{\sqrt{kj}}\sum_{l=1}^{k}\sum_{i=2k+1}^{j}\left(|\mathrm{Cov}(X_l, X_i)| + \rho^-(i-l)\|X_l\|_{2,1}\|X_i\|_{2,1}\right) \\
&\le \frac{c}{\sqrt{kj}}\sum_{l=1}^{k}\sum_{m=2k-l+1}^{j-l}\left(|\mathrm{Cov}(X_1, X_{m+1})| + \rho^-(m)\right) \\
&\le \frac{c\sqrt{k}}{\sqrt{j}}\sum_{m=1}^{\infty}\left(|\mathrm{Cov}(X_1, X_{m+1})| + \rho^-(m)\right) \\
&\le c\left(\frac{k}{j}\right)^{1/2}. \qquad (40)
\end{aligned}$$

By the c_r inequality and Lemma 2.1,

$$\begin{aligned}
\mathbb{E}\bar{V}_{k,1}^{2r} &= \mathbb{E}\left(\sum_{l=1}^{k}\bar{X}_{l,k}^2 I(\bar{X}_{l,k}\geq 0)\right)^r \\
&\leq c\mathbb{E}\left(\sum_{l=1}^{k}\left(\bar{X}_{l,k}^2 I(\bar{X}_{l,k}\geq 0)-\mathbb{E}\bar{X}_{l,k}^2 I(\bar{X}_{l,k}\geq 0)\right)\right)^r+\left(\sum_{l=1}^{k}\mathbb{E}\bar{X}_{l,k}^2 I(\bar{X}_{l,k}\geq 0)\right)^r \\
&\leq c\sum_{l=1}^{k}\mathbb{E}\bar{X}_{l,k}^{2r} I(\bar{X}_{l,k}\geq 0)+\left(\sum_{l=1}^{k}\mathbb{E}\bar{X}_{l,k}^4\right)^{r/2}+k^r \\
&\leq ck^{(2r-2)/2}\sum_{l=1}^{k}\mathbb{E}\bar{X}_{l,k}^2+\left(k\sum_{l=1}^{k}\mathbb{E}\bar{X}_{l,k}^2\right)^{r/2}+k^r \\
&\leq ck^r.
\end{aligned}$$

Thus, let $r > 2$, by (36)

$$\left\|\frac{\bar{V}_{k,1}^2}{k}\right\|_{2,1} \leq ck^{-1}\left(\mathbb{E}\bar{V}_{k,1}^{2r}\right)^{1/r} \leq c, \tag{41}$$

and

$$\begin{aligned}
\left\|\frac{\sum_{i=2k+1}^{j}\bar{X}_{i,j}^2 I(\bar{X}_{i,j}\geq 0)}{j}\right\|_{2,1} &\leq cj^{-1}\left(\mathbb{E}\left(\sum_{i=2k+1}^{j}\bar{X}_{i,j}^2 I(\bar{X}_{i,j}\geq 0)\right)^r\right)^{1/r} \\
&\leq cj^{-1}\left(\mathbb{E}\left(\sum_{i=2k+1}^{j}\left(\bar{X}_{i,j}^2 I(\bar{X}_{i,j}\geq 0)-\mathbb{E}\bar{X}_{i,j}^2 I(\bar{X}_{i,j}\geq 0)\right)\right)^r+\left(\sum_{i=2k+1}^{j}\mathbb{E}\bar{X}_{i,j}^2 I(\bar{X}_{i,j}\geq 0)\right)^r\right)^{1/r} \\
&\leq cj^{-1}\left(\sum_{i=2k+1}^{j}\mathbb{E}|\bar{X}_{i,j}|^{2r}+\left(\sum_{i=2k+1}^{j}\mathbb{E}\bar{X}_{i,j}^4\right)^{r/2}+j^r\right)^{1/r} \\
&\leq c.
\end{aligned}$$

Thus,combining this with (39)-(41), we have $|\mathbb{E}\eta_k\eta_j| \leq c((k/j)^{1/2}+\rho^-(k))$ for $1 \leq 2k < j$. Hence, by Lemma 3.4, (11) holds for $l = 1$. Using similar methods to those used in the proof of (11) for $l = 1$, we can prove that (11) holds for $l = 2$. Consequently (11) holds. This completes the proof of Lemma 2.2.

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