Filomat 33:8 (2019), 2529–2542
https://doi.org/10.2298/FIL1908529S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/filomat`

A Competition Model with Herd Behaviour and Allee Effect

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Abstract. In this work we have studied the deterministic behaviours of a competition model with herd behaviour and Allee effect. The uniform boundedness of the system has been studied. Criteria for local stability at equilibrium points are derived. The effect of discrete time-delay on the model is investigated. We have carried out numerical simulations to validate the analytical findings. The biological implications of our analytical and numerical findings are discussed.

1. Introduction

In many situations, two or more species live in proximity and share the same basic resources (such as food, water, habitat, or territory). As these resources are not unlimited, therefore it is quite obvious that these species might have to fight for these. *Competition* among organisms or species can be defined as an interaction in which the fitness of one is diminished by the presence of the other [10]. Competition among individuals of the same species is called *intraspecific competition*. On the other hand, *interspecific competition* is the competition between individuals of different species. The so called *Competitive exclusion principle* (or *Gause's law of competitive exclusion* [26]) states that stronger (or best suited) species will always dominate the weaker (or less suited) leading to either the extinction of the weaker or an evolutionary or behavioral shift toward a different ecological niche. But there are many evidences where this principle fails, the best known example being the *paradox of plankton* [31].

So far as the growth of a single-species population is concerned, it has long been recognised that the famous *logistic growth function* is a logical choice. The function is introduced in 1838 by the Belgian mathematician Pierre Francois Verhulst [57] and later it is rediscovered in 1920 by American biologists Reymon Pearl and Lowell Reed [43]. If $X(T)$ denotes the population density at time T, then the logistic growth equation is given by

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right), \tag{1}$$

where r is the intrinsic per capita growth rate and K is the carrying capacity of the environment. The logic behind this is very simple. As the resources (e.g., space, food, essential nutrients) are limited, every

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2010 *Mathematics Subject Classification*. Primary 92B05; Secondary 92D25, 92D30
Keywords. Competition model; Herd; Allee effect; Stability; Time-delay.
Received: 08 May 2017; Revised: 20 August 2018; Accepted: 03 November 2018
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population grows into a saturated phase from which it cannot grow further; the ecological habitat of the population can carry just so much of it and no more. This suggests that the per capita growth rate is a decreasing function of the size of the population, and reaches zero as the population achieved a size K (in the saturated phase). Further, any population reaching a size that is above this value will experience a negative growth rate. The term $-rX^2/K$ may also be regarded as the loss due to intraspecific competition. Although logistic growth function became extremely popular, but, in real life situations, researchers found many evidences where the populations show a reverse trend in low population density [14, 19, 22, 23, 41, 44]. This phenomenon of positive density dependence of population growth at low densities is known as the *Allee effect* [23, 51].

The phenomenon of *Allee effect* is named after the US Behavioral scientist Warder Clyde Allee (although Allee never used the term 'Allee effect'). Allee described this concept in three of his papers [4–6]. Actually, the term 'Allee effect' was introduced by Odum [42]. Since the late eighties of the 20th century, the concept gained importance but there were necessity of clear-cut definitions and clarification of concepts. The necessity was fulfilled when three reviews by Stephens et al. [51], Courchamp et al. [22], Stephens and Sutherland [50]. There are many reasons for Allee effect, such as difficulty in mate finding, reduced antipredator vigilance, problem of environmental conditioning, reduced defense against predators, and many others (for thorough reviews, see references [14, 23]).

The Allee effect can be divided into two main types, depending on how strong the per capita growth rate is depleted at low population densities. These two types are called the strong Allee effect [39, 46, 47, 52, 53, 56, 59, 60] or critical depensation [17, 18, 34], and the weak Allee effect [49, 51, 58] or noncritical depensation [17, 18, 34]. Usually, the Allee effect is modelled by a growth equation of the form

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right)(X - m)\,, \tag{2}$$

where $X(T)$ denotes the population density at time T, r is the intrinsic per capita growth rate, and K is the carrying capacity of the environment. Here $0 < m << K$. When $m > 0$ and the population size is below the threshold level m, then the population growth rate decreases [9, 20, 25, 32], and the population goes to extinction. In this case, the equation describes the *strong Allee effect* [53, 56, 59, 60]. On the contrary, the description of *weak Allee effect* is also available (see references [27, 49, 59]). In this paper, we are concerned with strong Allee effect. The above growth is often said to have a *multiplicative Allee effect*. There is another mathematical form of the growth function featuring the *additive Allee effect*. In this paper, we are not interested in additive Allee effect (interested readers might see the works of Aguirre et al. [1, 2]). A comparison of the logistic growth function of (1) and the function representing Allee effect in equation (2) can be found in [36].

A *herd* or *pack* is a social grouping of different animals of the same species. When a species shows herd behaviour, the individuals of the species show a collective social behavior, and each individual chooses a behaviour that corresponds to that of the majority of other members (for example, all moving in the same direction at a given time). There are several reasons for herd behaviour (see [11–13, 36–38, 48] and references therein).

Now, when a population lives forming groups, then all members of a group do not interact at a time. There are many reasons for this herd behaviour, such as searching for food resources, defending the predators, etc. Usually the species, which exhibits this social behaviour, allows the weakest individuals to occupy the interior of the herd, leaving the healthier and stronger animals around it. As a consequence, it is necessary to search for suitable function to describe this social behaviour. Only a few works have so far tried to enlighten this area. These works demonstrated an ingenious idea that suitable powers of the state variables can account for the social behaviour of the populations. For example, to explore the consequence of forming spatial group of fixed shape by predators, Cosner et al. [21] introduced the idea that the square root of the predator variable is to be used in the function describing the encounter rate in two-dimensional systems. Unfortunately, such an idea has not been used by the researchers for about a decade. The work of Chattopadhyay et al. [16] may be regarded as a strong recognition of this concept. Then came the most innovative work of Ajraldi et al. [3], which gave such modelling a new dimension. Their idea is very interesting. If X is the density of a population that gathers in herds and the herd occupies an area A, then the number of individuals staying at outermost positions in the herd is proportional to the length of the perimeter of the patch where the herd is located. Clearly, its length is proportional to $\sqrt{A}$. Since X is distributed over a two-dimensional domain, $\sqrt{X}$ would therefore count the individuals at the edge of the patch.

In the Lotka-Volterra competition model (which was later studied empirically by Gause [26]), the competition between two species is depicted and this model becomes extremely popular. In this model, it is assumed that each of the competing species follows logistic growth dynamics in the absence of the other. The description of this model with clear interpretation of different terms can be found in the classical book of Edelstein-Keshet [24]. Recently, Ajraldi et al. [3] have derived and analyzed a competition model, where one of the species shows a herd behaviour. Here we have considered a competition model where one species shows herd behaviour, whereas the other species is subject to strong Allee effect.

In recent times, it is well understood that many of the processes, both natural and manmade, in biology, medicine, et cetera, involve time-delays or time-lags. Time-delays occur so often, in almost every situation, that to ignore them is to ignore reality. Kuang [33] mentioned that animals must take time to digest their food before further activities and responses take place and hence any model of species dynamics without delays is an approximation at best. Now it is beyond doubt that in an improved analysis, the effect of time-delay due to the time required in going from egg stage to the adult stage, gestation period, et cetera, has to be taken into account. Detailed arguments on importance and usefulness of time-delays in realistic models may be found in the classical books of Gopalsamy [28], Kuang [33] and MacDonald [35].

Hutchinson [30] pointed out that the logistic equation would be inappropriate for the description of population growth in the case where there is a time-delay in some of the processes involved. The delayed logistic equation or Hutchinson's equation is one of the first examples of a delay differential equation that has been thoroughly examined (see for example [7, 28, 33, 35], and the references within).

In this paper, we have considered a two-species competition model. The first species is vulnerable to Allee effect and displays a herd behaviour. On the other hand, the second species has a logistic growth with time-delay in it. The paper is structured as follows. In section 2, the basic deterministic model (without time-delay) is introduced. In section 3, positivity and boundedness of the model are discussed. The equilibria and their stability are studied in section 4. The effect of time-delay in the logistic growth of the second species is analyzed in section 5. In section 6, the analytical findings of sections 4 and 5 are verified through computer simulation. Section 7 contains the general discussions of the paper and biological implications of our mathematical findings.

2. The basic mathematical model

The model we analyze in this paper is composed of two competing species, whose population biomass at time t are denoted by x and y. Before we introduce the model and dip into the depth of the rigorous analysis, we would like to present a brief sketch of the construction of the model which may indicate the biological relevance of it.

1. We assume that the first species (with biomass x) follows a logistic growth in absence of the other species.

2. It is assumed that the second species is subject to a strong Allee effect.

3. Naturally, we assume that the presence of each population leads to the depression of its competitor's growth rate.

4. We also assume that the first species shows a herd behaviour. Then its members, who are at the edge of the herd, interact mostly with the members of the second species, which lives independently. As it is a two-dimensional system, following the main idea of Ajraldi et al.[3], Melchionda et al. [40], we notice that $\sqrt{x}$ will count the number of individuals of the first species at the outer edge of the herd. Hence the loss due to interaction of two competing species will be proportional to $\sqrt{x}y$.

These considerations motivate us to introduce the following prey-predator system under the framework of the following set of nonlinear ordinary differential equations:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - a\sqrt{x}y, \quad x(0)>0, \\ \frac{dy}{dt} &= sy\left(1-\frac{y}{c}\right)(y-m) - b\sqrt{x}y, \quad y(0)>0. \end{aligned} \tag{3}$$

Here r is the intrinsic growth rate of the first species, k is its carrying capacity. The parameter s is the intrinsic growth rate of the second species, c is its carring capacity, m indicates the Allee threshold for the first species (since the Allee effect is strong, we have $0 < m << c$). The term $a\sqrt{x}y$ in the first equation can be thought of as the contribution made by the second species to a decline in the growth rate of the first species. Similarly, $b\sqrt{x}y$ is the decline in the growth rate of the second species (caused by individuals of the first species).

3. Positivity and boundedness

In this section, we discuss the positivity and boundedness of the solutions of the system (3). The first theorem is on positivity of solutions.

Theorem 3.1. *All solutions of the system* (3) *that start in* $\mathbb{R}^2_+$ *remain positive forever.*

The proof is simple and therefore it is omitted. The following theorem ensures the boundedness of the system (3).

Theorem 3.2. *All solutions of the system* (3) *that start in* $\mathbb{R}^2_+$ *are uniformly bounded.*

Proof. Let $(x(t), y(t))$ be any solution of the system (3). From the first equation of (3), we obtain

$$\frac{dx}{dt} \le rx\left(1-\frac{x}{k}\right),$$

which implies that

$$\limsup_{t\to\infty} x(t) \le k.$$

Next we prove that $\limsup_{t\to\infty} y(t) \le c$. We consider two possible cases separately.

Case-I. Let $y(0) \le c$. We claim that $y(t) \le c$ for all $t \ge 0$.
If possible, assume that our claim is not true. Then it is possible to find two positive real numbers t' and t'' such that $y(t') = c$ and $y(t) > c$ for all $t \in (t', t'')$.
Now, for all $t \in (t', t'')$, we have from the second equation of(3)

$$y(t) = y(0)\exp\left(\int_0^t \phi(x(s), y(s))ds\right),$$

where $\phi(x(t), y(t)) = s\left(1-\frac{y(t)}{c}\right)(y(t)-m) - b\sqrt{x(t)}$.
This implies that

$$\begin{aligned} y(t) &= y(0)\left[\exp\left(\int_0^{t'} \phi(x(s), y(s))ds\right)\right]\left[\exp\left(\int_{t'}^{t} \phi(x(s), y(s))ds\right)\right] \\ &= y(t')\exp\left(\int_{t'}^{t} \phi(x(s), y(s))ds\right), \quad \text{for all } t \in (t', t''). \end{aligned}$$

Since $m < c$, we have $\phi(x(t), y(t)) < 0$ for all $t \in (t', t'')$. Consequently, we have

$$y(t) < y(t'), \text{ where } y(t') = c.$$

This is contrary to the assumption that $y(t) > c$ for all $t \in (t', t'')$. Thus our claim is true.

Case-II. Let $y(0) > c$. We claim that $\limsup_{t\to\infty} y(t) \leq c$.
If possible, assume that this claim is false. Then $y(t) > c$ for all $t > 0$. So $\phi(x(t), y(t)) < 0$ (where ϕ has the same expression as in Case-I); and consequently, we have from the second equation of (3) that

$$y(t) = y(0)\exp\left(\int_0^t \phi(x(s), y(s))ds\right) < y(0).$$

Also from the second equation of (3), we obtain

$$\frac{dy}{dt} < s(y(0) - m)y\left(1 - \frac{y}{c}\right), \text{ where } y(0) - m > 0.$$

This implies that $\limsup_{t\to\infty} y(t) \leq c$, which is contradictory to our assumption. Therefore our claim is true.

The above two cases can be combined together as

$$\limsup_{t\to\infty} y(t) \leq c.$$

Hence the theorem follows. □

4. Equilibria and their stability

In this section, we find the equilibrium points of the system (3) and study their stability. The eqilibrium points and the conditions of their existence is given in the following lemma.

Lemma 4.1. *The trivial equilibrium $E_0(0,0)$ of the system* (3) *always exists. There are three boundary equilibrium points $E_1(k,0)$, $E_2(0,c)$ and $E_3(0,m)$, each of which also exists unconditionally. The first component x^* of the interior or coexistence equilibrium $E^*(x^*, y^*)$ exists if and only if the equation*

$$\frac{sr^2x}{a^2c}\left(1 - \frac{x}{k}\right)^2 - \frac{rs\sqrt{x}}{a}\left(1 + \frac{m}{c}\right)\left(1 - \frac{x}{k}\right) + b\sqrt{x} + sm = 0 \tag{4}$$

has a positive root. When this condition is satisfied, then x^* is a positive root of (4). *If further, $x^* < k$, then y^* exists and is given by*

$$y^* = \frac{r\sqrt{x^*}}{a}\left(1 - \frac{x^*}{k}\right).$$

4.1. Behaviour near the boundary equilibria E_0, E_2, E_3

System (3) cannot be linearized at $E_0 = (0,0)$, $E_2 = (0,c)$ and $E_3 = (0,m)$, so local stability of E_0, E_2 and E_3 cannot be studied in the usual way. However, we provide some logical arguments to understand the behaviours of the system (3) near the above equilibrium points.

If the first populations disappears, the system reduces to one equation. In this circumstance, the surviving population y follows the equation $\frac{dy}{dt} = sy\left(1 - \frac{y}{c}\right)(y - m)$ having three equilibrium points $0, m, c$. It is well known that the equilibrium points 0 and c are asymptotically stable, and m is unstable. Therefore, if $y(0) < m$, then y will approach 0; otherwise it will grow toward its own carrying capacity c. Thus $E_2(0,c)$ is locally asymptotically stable, and $E_3(0,m)$ is unstable.

To study the behaviour near $E_0(0,0)$, it is reasonable to assume that $x << \min\{1,k\}$ and $y << \min 1, m$. Then $x < \sqrt{x}$ and $y < \sqrt{y}$, and x, y are so small that, following the approach of Melchionda et al. [40], we have

$$\frac{dx}{dt} \approx r\sqrt{x} > 0, \quad \frac{dy}{dt} \approx s\sqrt{y} > 0.$$

This indicates that the origin is unstable.

4.2. Stability of E_1 and E^*

The Jacobian matrix of system (3) at an equilibrium point (x, y) (when $x \neq 0$) is

$$J(x,y) = \begin{bmatrix} r - \frac{2r}{k}x - \frac{ay}{2\sqrt{x}} & -a\sqrt{x} \\ -\frac{by}{2\sqrt{x}} & (y-m)(s - \frac{2sy}{c}) + sy(1 - \frac{y}{c}) - b\sqrt{x} \end{bmatrix}.$$

Theorem 4.2. *The equilibrium point $E_1(k, 0)$ of system is stable if and only if $b\sqrt{k} + ms > 0$.*

Proof. The Jcobian matrix at $E_1 = (k, 0)$ is

$$J(E_1) = \begin{bmatrix} -r & -a\sqrt{k} \\ 0 & -b\sqrt{k} - ms \end{bmatrix}.$$

Clearly, its eigenvalues are $\lambda_1 = -r$ and $\lambda_2 = -b\sqrt{k} - ms$.
Since $\lambda_1 < 0$, E_1 is stable or unstable according as $\lambda_2 <$ or > 0.
Hence the theorem follows. □

The Jacobian matrix at $E^*(x^*, y^*)$ is given by

$$J(E^*) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where

$$\begin{aligned} a_{11} &= r - \frac{2rx^*}{k} - \frac{ay^*}{2\sqrt{x^*}} \\ a_{12} &= -a\sqrt{x^*} \\ a_{21} &= -\frac{by^*}{2\sqrt{x^*}} \\ a_{22} &= (y^* - m)(s - \frac{2sy^*}{c}) + sy^*(1 - \frac{y^*}{c}) - b\sqrt{x^*}. \end{aligned}$$

The characteristic equation of $J(E^*)$ is

$$\lambda^2 + P\lambda + Q = 0,$$

where $P = -\text{tr}\, J(E^*) = -(a_{11} + a_{22})$ and $Q = \det J(E^*) = a_{11}a_{22} - a_{12}a_{21}$.
Then we have the following theorem guaranteeing the stability of E^*.

Theorem 4.3. *If the equilibrium point $E^*(x^*, y^*)$ exists with $P > 0$ and $Q > 0$, then E^* is locally asymptotically stable.*

Proof. It is easy to notice that, if the conditions of the theorem are satisfied, then all the eigenvalues of $J(E^*)$ have negative real parts. Hence the theorem follows. □

5. Effect of discrete time-delay

It is mentioned that time-delay is a vital factor in biological systems. In particular, delay in logistic growth function is extremely important [7, 28, 30, 33, 35, 45]. As a starting point of this section, we consider the following generalization of the model (3) involving discrete time delay:

$$\begin{aligned} \frac{dx}{dt} &= rx\left[1 - \frac{x(t-\tau)}{k}\right] - a\sqrt{x}y, \\ \frac{dy}{dt} &= sy\left(1 - \frac{y}{c}\right)(y - m) - b\sqrt{x}y. \end{aligned} \tag{5}$$

Initial conditions: $x(0) > 0$, $y(0) > 0$ with $x(\psi) \geq 0, \forall \psi \in [-\tau, 0)$.

Here it is assumed that the net per capita rate of change might depend on the state of the system τ time units in the past [7].

The system (5) has the same equilibria as in the previous case. The main purpose of this section is to study the stability behaviour of $E^*(x^*, y^*)$ in presence of delay ($\tau \neq 0$).

We linearize the system (5) by using the following transformations:

$$x = x^* + x_1, \quad y = y^* + y_1.$$

Then the linear system is given by

$$\frac{du}{dt} = Au(t) + Bu(t-\tau) \tag{6}$$

where $u(t) = [x_1\ y_1]^T$, $A = (a_{ij})_{2\times 2}$, $B = (b_{ij})_{2\times 2}$,
and $a_{11} = 0$, $a_{12} = 0$, $a_{21} = -\frac{by^*}{(2\sqrt{x^*})}$, $a_{22} = (y^* - m)(s - \frac{2sy^*}{c}) + sy^*(1 - \frac{y^*}{c}) - b\sqrt{x^*}$,
$b_{11} = -\frac{rx^*}{k}$, $b_{12} = 0$, $b_{21} = 0$, $b_{22} = 0$.
We look for solution of the model (6) of the form $u(t) = \rho e^{\lambda t}$, $0 \neq \rho \in \mathbb{R}^2$. This leads to the following characteristic equation:

$$\lambda^2 + a_1\lambda + (a_2\lambda + a_3)e^{-\lambda\tau} = 0, \tag{7}$$

where $a_1 = -a_{22}$, $a_2 = -b_{11}$, $a_3 = b_{11}a_{22}$.
It is well known that the signs of the real parts of the solutions of (7) characterize the stability behaviour of E^*. Therefore, substituting $\lambda = \xi + i\eta$ in (7), we obtain real and imaginary parts, respectively, as

$$\xi^2 - \eta^2 + a_1\xi + \left[\{a_2\xi + a_3\}\cos\eta\tau + a_2\eta\sin\eta\tau\right]e^{-\xi\tau} = 0, \tag{8}$$

and

$$2\xi\eta + a_1\eta + \left[a_2\eta\cos\eta\tau - \{a_2\xi + a_3\}\sin\eta\tau\right]e^{-\xi\tau} = 0. \tag{9}$$

A necessary condition for a stability change of E^* is that the characteristic equation (7) has purely imaginary solutions. Hence to obtain the stability criterion, we set $\xi = 0$ in (8) and (9) to obtain

$$\eta^2 = a_3\cos\eta\tau + a_2\eta\sin\eta\tau, \tag{10}$$

and

$$a_1\eta = a_3\sin\eta\tau - a_2\cos\eta\tau. \tag{11}$$

Eliminating τ by squaring and adding (10) and (11), we get the equation for determining η as

$$\eta^4 + d_1\eta^2 + d_2 = 0, \tag{12}$$

where $d_1 = a_1^2 - a_2^2$, $d_2 = -a_3^2$.
Substituting $\eta^2 = \sigma$ in (12), we get a quadratic equation given by

$$\sigma^2 + d_1\sigma + d_2 = 0. \tag{13}$$

The quadratic (13) always has one and only one positive real root. Let σ_0 be the unique positive root of (13). So there exist $\sigma_0 = \eta_0^2$ that satisfy equation (12). In the following theorem, we have given a criterion for switching the stability behaviour of E^*.

Theorem 5.1. *Let $E^*(x^*, y^*)$ exists with $P > 0$ and $Q > 0$. Then E^* is locally asymptotically stable for $0 \leq \tau < \tau^*$, and unstable for $\tau > \tau^*$ and the system exhibits a hopf bifurcation near E^* for $\tau = \tau^*$, provided*

$$f(\eta_0) = \frac{1}{\eta_0}(a_1a_3\eta_0 \sin\eta_0\tau - a_1a_2\eta_0^2 \cos\eta_0\tau + 2a_2\eta_0^3 \sin\eta_0\tau + 2a_3\eta_0^2 \cos\eta_0\tau - a_2^2\eta_0^2) > 0,$$

where

$$\tau^* = \frac{1}{\eta_0} \arcsin\left[\frac{\eta_0^3 a_2 + a_1a_3\eta_0}{a_3^2 + a_2^2\eta_0^2}\right].$$

Proof First we note that η_0 is a solution of (12). Solving (10) for $\cos\tau\eta_0$ and substituting in (11), we find that for $\tau = \tau^*$, the characteristic equation (7) has purely imaginary roots, $\pm i\eta_0$. Again it may be noted that if $\pm\eta_0$ is a solution of (10) and (11), then η_0^2 is a solution of (12). The theorem will be proved if we can show that $\left[\frac{d\xi}{d\tau}\right]_{\tau=\tau^*} > 0$. To show this, we differentiate (10) and (11) with respect to τ and then set $\xi = 0$ to obtain

$$C(\eta)\frac{d\xi}{d\tau} + D(\eta)\frac{d\eta}{d\tau} = G(\eta) \tag{14}$$

and

$$-D(\eta)\frac{d\xi}{d\tau} + C(\eta)\frac{d\eta}{d\tau} = H(\eta), \tag{15}$$

where

$$\begin{aligned} C(\eta) &= a_1 + a_2\cos\eta\tau - a_3\tau\cos\eta\tau - a_2\eta\tau\sin\eta\tau \\ D(\eta) &= -2\eta - a_3\tau\sin\eta\tau + a_2\sin\eta\tau + a_2\eta\tau\cos\eta\tau \\ G(\eta) &= a_3\eta\sin\eta\tau - a_2\eta^2\cos\eta\tau \\ H(\eta) &= a_2\eta^2\sin\eta\tau + a_3\eta\cos\eta\tau. \end{aligned}$$

Solving (14) and (15) with $\tau = \tau^*$ and $\eta = \eta_0$, we get

$$\left[\frac{d\xi}{d\tau}\right]_{\tau=\tau^*} = \frac{\eta_0^2 f(\eta_0)}{C^2(\eta_0) + D^2(\eta_0)},$$

which is positive under the condition of the theorem ($f(\eta_0) > 0$).
Thus the theorem is established. □

6. Numerical simulation

Analytical studies can never be complete without numerical verification of the results. In this section we present computer simulation of different solutions of the system (3) using MATLAB.

First we choose the parameters of the system as $a = 0.05,\ b = 3,\ c = 0.02,\ r = 8,\ s = 1,\ k = 3.5,\ m = 0.2$ and $x(0) = 1,\ y(0) = 0.5$. Then condition ($b\sqrt{k} + ms = 5.81 > 0$) of Theorem 4.2 is satisfied, and as such $E_1(k, 0)$ is locally asymptotically stable. The behaviour of x and y with time is depicted in Figure 1. Clearly x approaches $k = 3.5$ and y approaches 0 in finite time.

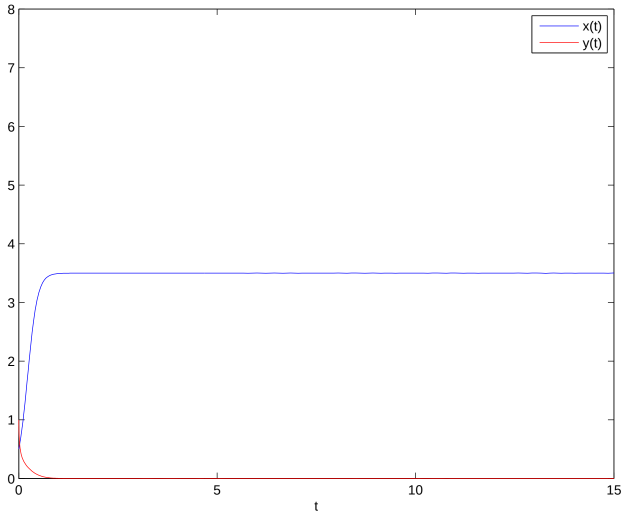

Figure 1. *Behaviour of* x *and* y *with* t *for the system* (3) *when* $a = 0.05,\ b = 3,\ c = 0.02,\ r = 8,\ s = 1,\ k = 3.5,\ m = 0.2$ *and* $x(0) = 1,\ y(0) = 0.5$.

Now we choose the parameters of the system as $a = 0.05,\ b = 3,\ c = 0.02,\ r = 10,\ s = 1,\ k = 4,\ m = 0.75$. Then conditions of Theorem 4.3 are satisfied, and as such $E^*(4, 0.5)$ is locally asymptotically stable. The corresponding phase portrait with different initial choices $[(x(0), y(0)) = (1, 2), (2, 3), (3, 4)]$ is shown in Figure 2. The behaviour of x and y with time is shown in Figure 3 when $x(0) = 1$ and $y(0) = 1$.

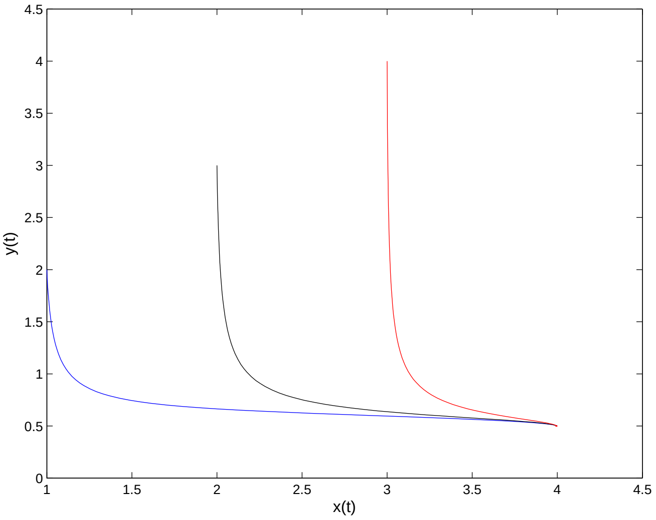

Figure 2. *Phase portrait of the system* (3) *for different values of* $x(0)$ *and* $y(0)$ *when* $a = 0.05,\ b = 3,\ c = 0.02,\ r = 10,\ s = 1,\ k = 4,\ m = 0.75$.

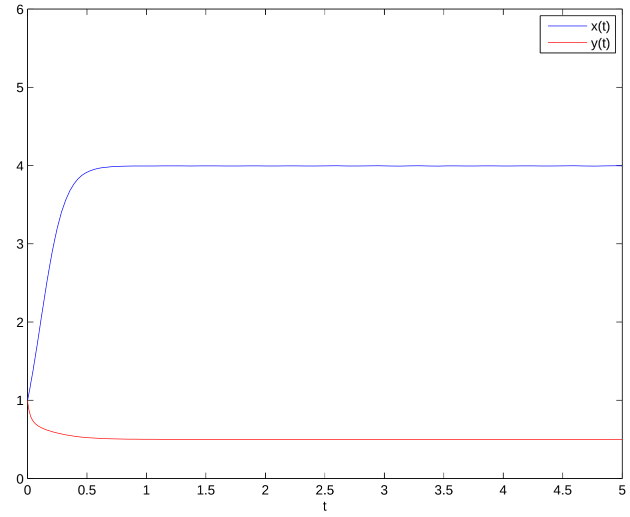

Figure 3. *Behaviour of x and y with t for the system* (3) *when the parameter values are same as in Figure 2 and* $x(0) = 1,\ y(0) = 1$.

For the above choices of parameters as in Figure 2, using Theorem 5.1, we see that $\tau^* = 0.1566$. Therefore, by Theorem 5.1, $E^*(4, 0.5)$ is locally asymptotically stable for $0 \leq \tau < \tau^*$, and Hopf bifurcation occurs at $\tau = \tau^*$. We verify that, for $\tau = 0.15 < \tau^*$, E^* is locally asymptotically stable. The corresponding phase portrait for different initial choices $[(x(0), y(0)) = (0.1, 0.2), (0.3, 0.4), (0.4, 0.5)]$ is shown in Figure 4. The stable behaviour of x and y with time is presented in Figure 5, when $x(0) = 0.2,\ y(0) = 0.3$.

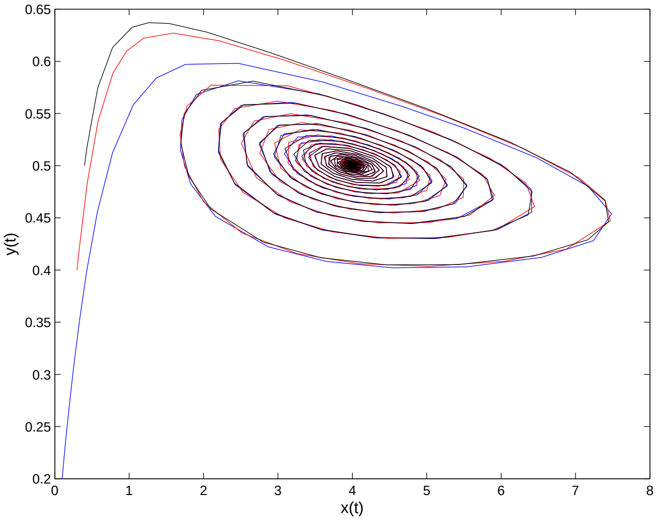

Figure 4. *Phase portrait of the system* (5) *for different values of* $x(0)$ *and* $y(0)$ *when* $a = 0.05,\ b = 3,\ c = 0.02,\ r = 10,\ s = 1,\ k = 4,\ m = 0.75$ *and* $\tau = 0.15$.

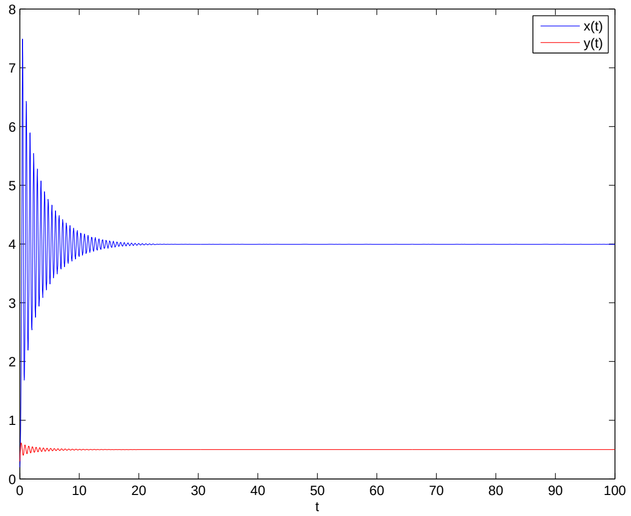

Figure 5. *Stable behaviour of x and y with t for the system* (5) *when the parameter values are same as in Figure 4 and* $x(0) = 0.2,\ y(0) = 0.3$.

On the other hand, if we take $\tau = 0.16 > \tau^*$ keeping other parameters fixed, then E^* becomes unstable. The corresponding phase portrait for different initial choices $[(x(0), y(0)) = (0.1, 0.2), (0.3, 0.4), (0.4, 0.5)]$ is shown in Figure 6, which shows that there is a limit cycle that grows out of E^*. The oscillations of x and y with time is presented in Figure 7, when $x(0) = 0.2,\ y(0) = 0.3$.

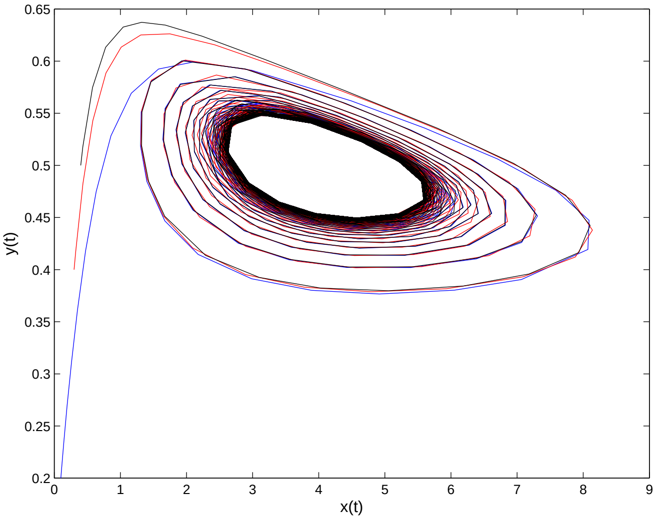

Figure 6. *Phase portrait of the system* (5) *for different values of* $x(0)$ *and* $y(0)$ *when the parameter values are same as in Figure 4 except* $\tau = 0.16 > \tau^*$.

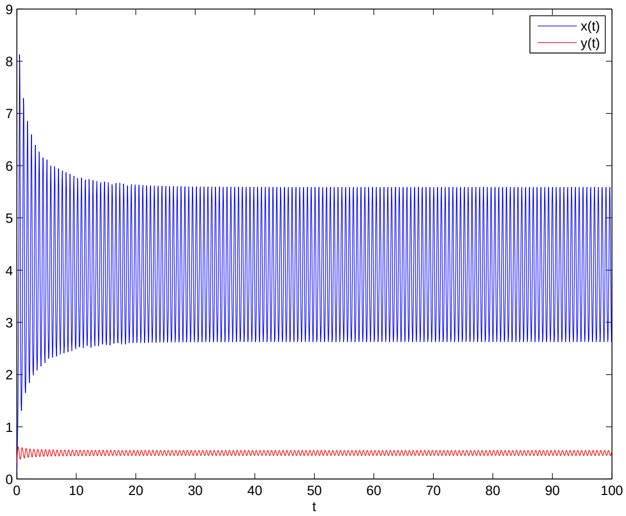

Figure 7. *Oscillations of x and y with t for the system* (5) *when the parameter values are same as in Figure 6 and* $x(0) = 0.2,\ y(0) = 0.3$.

7. Concluding remarks

Many populations in nature live in herd for numerous reasons. When two or more species share a common resource, there might be competition among them. Further, a population might be susceptible to Allee effect due to many reasons. There should be no denying that, studies of interplay between these phenomena would be very exciting and challenging. In this paper, we have constructed a two-species competition model (3), where one species shows a herd behaviour and the other is subject to strong Allee effect.

It is shown (in Theorem 3.1 and Theorem 3.2) that the solutions of the system (3) remains non-negative forever, and they are uniformly bounded. These, in turn, imply that the system is biologically well-behaved. Analysis of stability of the equilibrium points is presented. The effect of discrete time-delay in the growth term of the first species is studied. Our results are illustrated through computer simulation using MATLAB.

One of the main goals of mathematical modeling is to observe formation of different patterns, which are commonly exhibited in nature, through mathematical models [54, 55]. Again, on the issue of population control, we usually get very useful suggestions from mathematical modeling of the concerned system. Very often, modelers identify the factors which might leave a stabilizing or destabilizing influence on the underlying system [8, 15]. In our model, we have observed some patterns like spiral pattern, cyclic pattern, etc. Also it is shown that the time-delay has the capability to control the dynamics of the system.

Nowadays, preservation of ecological balance in nature is an issue which the entire globe is concerned with. Stability analysis of the interior equilibrium might provide the conditions for ecological balance in nature. From our analysis, it is evident that if conditions of Theorem 4.3 are satisfied, and τ lies below τ^*, then *competitive exclusion* can be avoided and ecological balance can be preserved.

Acknowledgment: We are grateful to the anonymous referees and Prof. Miljana Jovanovic, Editor for their careful reading, valuable comments and helpful suggestions which have helped us to improve the presentation of this work significantly.

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