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Golden Riemannian Structures On the Tangent Bundle with g-Natural Metrics

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Abstract. Starting from the g-natural Riemannian metric G on the tangent bundle TM of a Riemannian manifold (M, g), we construct a family of the Golden Riemannian structures φ on the tangent bundle (TM, G). Then we investigate the integrability of such Golden Riemannian structures on the tangent bundle TM and show that there is a direct correlation between the locally decomposable property of $(\mathrm{TM}, \varphi, G)$ and the locally flatness of manifold (M, g).

1. Introduction

S. Sasaki in his original paper ([11]) introduced the concept of lifted metric on the tangent bundle TM of a Riemannian manifold (M, g), as a Riemannian metric G_S which depends on the base metric g. The Sasaki metric G_S motivated many mathematicians to study and to develop various types of lifted metrics on the tangent bundle TM of (M, g). The notion of g-natural metrics on the tangent bundle of a Riemannian manifold (M, g) first constructed in [4], as the most general type of lifted metrics on the tangent bundle TM. Abbassi et al., investigated some properties of g-natural metrics on the tangent bundle and the unit tangent sphere bundle of Riemannian manifolds ([1], [2], [3], [5]).

In [8], the authors introduced the notion of Golden Riemannian structures on a Riemannian manifold (M, g). The name of the Golden structure φ refers to the Golden Ratio $\Phi \approx 1.618\ldots$, which was first employed by Phidias $(490 - 430$ BC) and was first defined by Euclid. The structure φ on the Riemannian manifold (M, g) is called a Golden Riemannian structure, if the polynomial $X^2 - X - 1$ is the minimal polynomial for φ satisfying $\varphi^2 - \varphi - 1 = 0$. Gezer et al., have investigated some properties of the Golden structures on a Riemannian manifold (M, g) and published valuable papers in this context (see for example [9]).

The aim of this paper is to construct a family of Golden Riemannian structures on the tangent bundle TM of a Riemannian manifold (M, g) equipped with the g-natural Riemannian metrics G. Also, we determine some requirements for such Golden structures to be integrable on the tangent bundle (TM, G) of M.

The work is organized in the following way. In Section 2, we study the concept of g-natural metrics on the tangent bundle TM and also, the notion and some properties of the Golden Riemannian structures are introduced in this section. In Section 3, we construct and introduce a family of the Golden Riemannian structures φ on the tangent bundle TM equipped with the g-natural metrics G. Section 4 includes some statements on the integrability of the Golden Riemannian structures φ defined in the preceding section.

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2. g-natural metrics and Golden structures on the tangent bundle

This section contains some essential information on g-natural metrics and Golden Riemannian structures on the tangent bundle TM of a Riemannian manifold (M, g).

2.1. g-Natural metrics on the tangent bundle

Let (M, g) be an n-dimensional Riemannian manifold, and we denote by ∇ its Levi-Civita connection. If $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical spaces concerning ∇, then the tangent space $TM_{(x,u)}$ of the tangent bundle TM at a point (x, u) splits as

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}.$$

For a vector $X \in M_x$, the *horizontal lift* of X to $(x, u) \in TM$, is a unique vector $X^h \in \mathcal{H}_{(x,u)}$ such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. Also, the *vertical lift* of a vector $X \in M_x$ is defined by a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(\mathrm{d}f) = Xf$, for all functions f on M. Remark that 1-forms $\mathrm{d}f$ on M are considered as functions on TM (i.e., $(\mathrm{d}f)(x, u) = uf$). Both maps $X \to X^h$ and $X \to X^v$ are isomorphisms between the vector spaces M_x and $\mathcal{H}_{(x,u)}$ and between M_x and $\mathcal{V}_{(x,u)}$ respectively. Each tangent vector $Z \in (TM)_{(x,u)}$ can be written in the form $Z = X^h + Y^v$, where $X, Y \in M_x$, are uniquely determined vectors. Moreover, the vector field $u^h_{(x,u)} = u^i(\frac{\partial}{\partial x^i})^h_{(x,u)}$ for any point $x \in M$ and $u \in TM_x$, uniquely defines the geodesic flow vector field on TM with respect to the local coordinates $\{\frac{\partial}{\partial x^i}\}$ on (M, g). Now we see how to define the g-natural metric G on the tangent bundle TM of (M, g).

Let (M, g) be a Riemannian manifold and G be the g-natural metric on TM. Then there are six smooth functions $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that for every $u, X, Y \in M_x$, we have

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(v^2)g(X, Y) + (\beta_1 + \beta_3)(v^2)g(X, u)g(Y, u), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = \alpha_2(v^2)g(X, Y) + \beta_2(v^2)g(X, u)g(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(v^2)g(X, Y) + \beta_1(v^2)g(X, u)g(Y, u), \end{cases} \tag{1}$$

where $v^2 = g(u, u)$ (for more details, see [5]). It can be checked that the g-natural metric G is Riemannian if and only if ([7])

$$\alpha_1(t) > 0, \quad \psi_1(t) > 0, \quad \alpha(t) > 0, \quad \psi(t) > 0,$$

for all $t \in \mathbb{R}^+$, where

$$\psi_i(t) = \alpha_i(t) + t\beta_i(t), \quad \alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t), \quad \psi(t) = \psi_1(t)(\psi_1 + \psi_3)(t) - \psi_2^2(t).$$

Lemma 2.1 ([6]). *Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection and* R *be the Riemann curvature tensor of ∇. The Lie bracket on the tangent bundle* TM *of M satisfies the following*

i. $[X^v, Y^v] = 0$,

ii. $[X^h, Y^v] = (\nabla_X Y)^v$,

iii. $[X^h, Y^h] = [X, Y]^h - (R(X, Y)u)^v$,

for all X, Y on M at any point (p, u) in TM.

2.2. Golden Riemannian structures on the tangent bundle

Definition 2.2. *A Golden Riemannian structure (as defined in [8]) on the n-dimensional Riemannian manifold (M, g) is a $(1, 1)$-tensor field φ and a Riemannian metric g which satisfy the following relations*

$$\varphi^2 = \varphi + I, \tag{2}$$

$$g(\varphi X, \varphi Y) = g(\varphi X, Y) + g(X, Y), \tag{3}$$

for all vector fields X and Y on M.

The Riemannian metric g satisfying (3) is called φ-compatible and triple (M, φ, g) is called a Golden Riemannian manifold. The minimal polynomial for the Golden Riemannian structure φ on M satisfying $\varphi^2 - \varphi - I = 0$ is $X^2 - X - 1$. Remark that if $\varphi = aI$ then $X - a$ is the minimal polynomial for φ (see [9]).

If the Nijenhuis tensor N_φ of φ vanishes, then the Golden Riemannian structure φ is integrable ([8]). A Golden Riemannian manifold (M, φ, g) with an integrable Golden structure φ is called locally Golden Riemannian manifold. In [9], the authors have introduced the notion of locally decomposable Golden Riemannian manifolds and they have proven that a necessary and sufficient condition for (M, φ, g) to be a locally decomposable Golden Riemannian manifold is that $\phi_\varphi g = 0$, where ϕ_φ is the Tachibana operator defined by

$$\phi_\varphi : \mathfrak{T}^0_s(M) \to \mathfrak{T}^0_{s+1}(M),$$

applied to the pure tensor field t of type $(0, s)$ with respect to φ by

$$\begin{aligned}(\phi_\varphi t)(X, Y_1, \ldots, Y_s) =& (\varphi X)\, t(Y_1, \ldots, Y_s) - Xt(\varphi Y_1, \ldots, Y_s) \\ &+ \sum_{\lambda=1}^{s} t(Y_1, \ldots, (L_{Y_\lambda}\varphi)X, \ldots, Y_s),\end{aligned} \tag{4}$$

for any $X, Y_1, \ldots, Y_s \in \mathfrak{T}^1_0(M)$. In the latter equality L_Y stands for the Lie derivation concerning Y and $\mathfrak{T}^r_s(M)$ denotes the module of all tensor fields of type (r, s) on M over $F(M)$, where $F(M)$ is the algebra of C^∞-functions on M (see [9]). Notice that a tensor field t of type (r, s) is called a pure tensor field concerning φ if

$$\begin{aligned}
t(\varphi X_1, X_2, \ldots, X_s; \overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{r}{\xi}) &= t(X_1, \varphi X_2, \ldots, X_s; \overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{r}{\xi}),\\
&\;\;\vdots\\
t(\varphi X_1, X_2, \ldots, X_s; \overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{r}{\xi}) &= t(X_1, X_2, \ldots, \varphi X_s; \overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{r}{\xi}),\\
t(\varphi X_1, X_2, \ldots, X_s; \overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{r}{\xi}) &= t(X_1, X_2, \ldots, X_s; \varphi_\star \overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{r}{\xi}),\\
&\;\;\vdots\\
t(\varphi X_1, X_2, \ldots, X_s; \overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{r}{\xi}) &= t(X_1, X_2, \ldots, X_s; \overset{1}{\xi}, \overset{2}{\xi}, \ldots, \varphi_\star \overset{r}{\xi}),
\end{aligned}$$

for any $X_1, X_2, \ldots, X_s \in \mathfrak{T}^1_0(M)$ and $\overset{1}{\xi}, \overset{2}{\xi}, \ldots, \overset{r}{\xi} \in \mathfrak{T}^0_1(M)$, where $\varphi_\star$ is the adjoint operator of φ defined by

$$(\varphi_\star \xi)(X) = \xi(\varphi X),\; X \in \mathfrak{T}^1_0(M),\; \xi \in \mathfrak{T}^0_1(M).$$

We now present an example of the Golden Riemannian Structures from [8] as follows .

Example 2.3 ([8]). *Let $(\mathbb{R}^2, g_{Euc})$ be the 2-dimensional Euclidean manifold. Defining the distributions R and S by*

$$R = \operatorname{span}\left\{x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right\}, \qquad S = \operatorname{span}\left\{\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right\},$$

it can be verified that R and S are orthogonal complementary distributions concerning g_{Euc}. These distributions are associated to the structure

$$\varphi\Big(\frac{\partial}{\partial x}\Big) = \frac{\Phi x^2 + (1-\Phi)}{1+x^2}\frac{\partial}{\partial x} + \frac{\sqrt{5}x}{1+x^2}\frac{\partial}{\partial y}, \qquad \varphi\Big(\frac{\partial}{\partial y}\Big) = \frac{\sqrt{5}x}{1+x^2}\frac{\partial}{\partial x} + \frac{(1-\Phi)x^2 + \Phi}{x^2+1}\frac{\partial}{\partial y},$$

where $\Phi = \frac{1+\sqrt{5}}{2}$ *is the Golden Ratio. It is easy to see that*

$$\varphi^2(\frac{\partial}{\partial x}) = \varphi(\frac{\partial}{\partial x}) + I(\frac{\partial}{\partial x}) = \frac{x^2\sqrt{5}+3x^2-\sqrt{5}+3}{2x^2+2}\frac{\partial}{\partial x} + \frac{\sqrt{5}x}{x^2+1}\frac{\partial}{\partial y},$$
$$\varphi^2(\frac{\partial}{\partial y}) = \varphi(\frac{\partial}{\partial y}) + I(\frac{\partial}{\partial y}) = \frac{\sqrt{5}x}{x^2+1}\frac{\partial}{\partial x} + \frac{-x^2\sqrt{5}+3x^2+\sqrt{5}+3}{2x^2+2}\frac{\partial}{\partial y},$$

and also,

$$g_{Euc}(\varphi(\frac{\partial}{\partial x}),\varphi(\frac{\partial}{\partial x})) = g_{Euc}(\varphi(\frac{\partial}{\partial x}),\frac{\partial}{\partial x}) + g_{Euc}(\frac{\partial}{\partial x},\frac{\partial}{\partial x}) = \frac{x^2\sqrt{5}+3x^2-\sqrt{5}+3}{2x^2+2},$$
$$g_{Euc}(\varphi(\frac{\partial}{\partial x}),\varphi(\frac{\partial}{\partial y})) = g_{Euc}(\varphi(\frac{\partial}{\partial x}),\frac{\partial}{\partial y}) + g_{Euc}(\frac{\partial}{\partial x},\frac{\partial}{\partial y}) = \frac{\sqrt{5}x}{x^2+1},$$
$$g_{Euc}(\varphi(\frac{\partial}{\partial y}),\varphi(\frac{\partial}{\partial y})) = g_{Euc}(\varphi(\frac{\partial}{\partial y}),\frac{\partial}{\partial y}) + g_{Euc}(\frac{\partial}{\partial y},\frac{\partial}{\partial y}) = \frac{-x^2\sqrt{5}+3x^2+\sqrt{5}+3}{2x^2+2}.$$

Hence, the conditions (2) and (3) are satisfied by φ *and so,* φ *is a Golden Riemannian structure. Moreover, a straightforward computation yields that* $N_\varphi(\frac{\partial}{\partial x},\frac{\partial}{\partial y}) = 0$. *Therefore, the Golden Riemannian Structure* φ *is integrable.*

3. Golden Riemannian Structures on the tangent bundle with g-natural metrics

In this section we see how to define a Golden Riemannian structure φ on the tangent bundle TM, having g-natural metrics G as φ-compatible metrics.

Theorem 3.1. *Let G be a g-natural Riemannian metric on the tangent bundle* TM *described by (1). The necessary and sufficient condition for the structure tensor* φ *defined by*

$$\begin{cases} \varphi(X^h) = pX^h + qX^v, \\ \varphi(X^v) = rX^h + sX^v, \end{cases} \tag{5}$$

to be a Golden Riemannian structure on the tangent bundle (TM, G) *is that the relations*

$$\begin{cases} q = -\dfrac{p^2-p-1}{r}, \qquad s = 1-p, \qquad \alpha_2(v^2) = \beta_2(v^2) = 0, \\ \alpha_3(v^2) = -\dfrac{\alpha_1(v^2)(p^2+r^2-p-1)}{r^2}, \qquad \beta_3(v^2) = -\dfrac{\beta_1(v^2)(p^2+r^2-p-1)}{r^2}, \end{cases} \tag{6}$$

for non-zero real constants p, q, r, s *and all vector fields* $X \in \mathfrak{T}^1_0(M)$ *hold.*

Proof. The structure φ defined by (5) is a Golden Riemannian structure on the tangent bundle TM with g-natural metric G as a φ-compatible metric, if and only if both conditions (2) and (3) hold. Substituting (5) and (1) into (2) and (3) conclude that these conditions hold if and only if the following system of equations satisfying

$$\begin{cases} (\alpha_1+\alpha_3)(v^2)p^2 + \Big(2q\alpha_2(v^2) - \alpha_1(v^2) - \alpha_3(v^2)\Big)p + q^2\alpha_1(v^2) - \alpha_1(v^2) - \alpha_3(v^2) = 0, \\ (\beta_1+\beta_3)(v^2)p^2 + \Big(2q\beta_2(v^2) - \beta_1(v^2) - \beta_3(v^2)\Big)p + q^2\beta_1(v^2) - \beta_1(v^2) - \beta_3(v^2) = 0, \\ pr(\alpha_1+\alpha_3)(v^2) + (p(s-1)+qr-1)\alpha_2(v^2) + q(s-1)\alpha_1(v^2) = 0, \\ pr(\beta_1+\beta_3)(v^2) + (p(s-1)+qr-1)\beta_2(v^2) + q(s-1)\beta_1(v^2) = 0, \\ r(q\alpha_2(v^2) + (p-1)(\alpha_1+\alpha_3)(v^2)) + (ps-s-1)\alpha_2(v^2) + qs\alpha_1(v^2) = 0, \\ r(q\beta_2(v^2) + (p-1)(\beta_1+\beta_3)(v^2)) + (ps-s-1)\beta_2(v^2) + qs\beta_1(v^2) = 0, \\ (r^2+s^2-s-1)\alpha_1(v^2) + r^2\alpha_3(v^2) + (2s\alpha_2(v^2) - \alpha_2(v^2))r = 0, \\ (r^2+s^2-s-1)\beta_1(v^2) + r^2\beta_3(v^2) + (2s\beta_2(v^2) - \beta_2(v^2))r = 0, \\ p^2+qr-p-1 = 0, \qquad q(p+s-1) = 0, \qquad r(p+s-1) = 0, \qquad qr+s^2-s-1 = 0. \end{cases} \tag{7}$$

Standard calculations show that this system of equations satisfies if and only if (6) holds, which proves the truthfulness of the assertion. □

There are some well-known examples of Riemannian metrics on the tangent bundle TM which are special cases of Riemannian g-natural metrics. For example, the Sasaki metric G_S is obtained from (1) for

$$\alpha_1(t) = 1, \qquad \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0,$$

and the Cheeger-Gromoll metric G_{CG} is obtained when

$$\alpha_2(t) = \beta_2(t) = 0, \qquad \alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}, \qquad \alpha_3(t) = \frac{t}{1+t},$$

and metrics of Cheeger-Gromoll type G_{ml} are obtained for

$$\alpha_2(t) = \beta_2(t) = 0, \qquad \alpha_1(t) = \frac{1}{(1+t)^m}, \qquad \alpha_3(t) = 1 - \alpha_1(t), \qquad \beta_1(t) = -\beta_3(t) = \frac{l}{(1+t)^m},$$

and the Kaluza-Klein metric G_{KK} is obtained when

$$\alpha_2(t) = \beta_2(t) = (\beta_1 + \beta_3)(t) = 0,$$

and the class of metrics of Kaluza-Klein type is defined by

$$\alpha_2(t) = \beta_2(t) = 0.$$

In the following theorems, we investigate Theorem 3.1 for such g-natural Riemannian metrics.

Theorem 3.2. *Let (M, g) be a Riemannian manifold and let* TM *be its tangent bundle equipped to the Sasaki metric G_S. The structure φ on* TM *defined by*

$$\begin{cases} \varphi(X^h) = cX^h + \sqrt{-c^2+c+1}X^v, \\ \varphi(X^v) = \sqrt{-c^2+c+1}X^h + (1-c)X^v, \end{cases} \tag{8}$$

for all non-zero real constants c with $-c^2 + c + 1 > 0$, and all vector fields $X \in \mathfrak{I}^1_0(M)$ is a Golden Riemannian structure and the triple $(\mathrm{TM}, \varphi, G_S)$ is a Golden Riemannian manifold.

Proof. Taking into account Theorem 3.1 and substituting $\alpha_1(t) = 1$ and $\alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0$ into (7), after some direct calculations, it deduces that (7) holds if and only if

$$p = c, \qquad q = r = \sqrt{-c^2+c+1}, \qquad s = 1 - c,$$

for all non-zero real constants c with $-c^2 + c + 1 > 0$, which completes the proof. □

Corollary 3.3. *If we put $c = \frac{1}{2}$ into the proof of Theorem 8, it concludes that the structure φ defined on (TM, G_S) by*

$$\begin{cases} \varphi(X^h) = \frac{1}{2}(X^h + \sqrt{5}X^v), \\ \varphi(X^v) = \frac{1}{2}(\sqrt{5}X^h + X^v), \end{cases} \tag{9}$$

is a Golden Riemannian structure. Notice that the number $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618$, which is a solution of $x^2 - x - 1 = 0$, is called the Golden Ratio. (Gezer et al., have presented such a Golden Riemannian structure and have studied its integrability in [9]).

Example 3.4. *Let $(\mathbb{R}^2, g_{Euc})$ be the 2-dimensional Euclidean manifold. Given the local coordinate system (x^1, x^2), the tangent vectors $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}$ define a local coordinate basis of the tangent space to $\mathbb{R}^2$ at each point of its domain. Also, the components of the Euclidean metric g_{Euc} are*

$$g_{ij} = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

for $i, j \in \{1, 2\}$. It is easy to see that the Christoffel symbols Γ^k_{ij} on $(\mathbb{R}^2, g_{Euc})$ vanish, i.e., $\Gamma^k_{ij} = 0$, for $i, j, k \in \{1, 2\}$. Let $T\mathbb{R}^2$ be the tangent bundle over $\mathbb{R}^2$ and $(\bar{x}^1, \bar{x}^2, y^1, y^2)$ be the local coordinate system on it, where $\bar{x}^i = x^i \circ \pi$, for $i \in \{1, 2\}$. Taking into account the vanishing of the Christoffel symbols Γ^k_{ij}, we have $\frac{\delta}{\delta \bar{x}^i} = \frac{\partial}{\partial \bar{x}^i}$ and so, the horizontal space $\mathcal{H}$ is spanned by $\left\{\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^2}\right\}$. Also, the vertical space $\mathcal{V}$ is spanned by the vector fields $\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right\}$. The Sasaki metric G_S corresponded to $(\mathbb{R}^2, g_{Euc})$ on $T\mathbb{R}^2$ is of the following form.

$$\begin{cases} G_S(\frac{\partial}{\partial \bar{x}^i}, \frac{\partial}{\partial \bar{x}^j}) = g_{Euc}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \\ G_S(\frac{\partial}{\partial \bar{x}^i}, \frac{\partial}{\partial y^j}) = 0, \\ G_S(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = g_{Euc}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \end{cases}$$

for $i, j, \in \{1, 2\}$. Using (8), we put

$$\begin{cases} \varphi(\frac{\partial}{\partial \bar{x}^i}) = c\frac{\partial}{\partial \bar{x}^i} + \sqrt{-c^2 + c + 1}\frac{\partial}{\partial y^i}, \\ \varphi(\frac{\partial}{\partial y^i}) = \sqrt{-c^2 + c + 1}\frac{\partial}{\partial \bar{x}^i} + (1 - c)\frac{\partial}{\partial y^i}, \end{cases} \tag{10}$$

for all non-zero real constants c with $-c^2 + c + 1 > 0$, and we show that the conditions (2) and (3) are satisfied for $(T\mathbb{R}^2, \varphi, G_S)$ and so, φ is a Golden Riemannian structure on $(T\mathbb{R}^2, G_S)$. For $i \in \{1, 2\}$, using (10) we have

$$\begin{aligned} \varphi^2(\frac{\partial}{\partial \bar{x}^i}) &= \varphi(c\frac{\partial}{\partial \bar{x}^i} + \sqrt{-c^2 + c + 1}\frac{\partial}{\partial y^i}) = c\varphi(\frac{\partial}{\partial \bar{x}^i}) + \sqrt{-c^2 + c + 1}\varphi(\frac{\partial}{\partial y^i}) \\ &= c(c\frac{\partial}{\partial \bar{x}^i} + \sqrt{-c^2 + c + 1}\frac{\partial}{\partial y^i}) + \sqrt{-c^2 + c + 1}(\sqrt{-c^2 + c + 1}\frac{\partial}{\partial \bar{x}^i} + (1 - c)\frac{\partial}{\partial y^i}) \\ &= (1 + c)\frac{\partial}{\partial \bar{x}^i} + \sqrt{-c^2 + c + 1}\frac{\partial}{\partial y^i}, \end{aligned} \tag{11}$$

and

$$\varphi(\frac{\partial}{\partial \bar{x}^i}) + I(\frac{\partial}{\partial \bar{x}^i}) = c\frac{\partial}{\partial \bar{x}^i} + \sqrt{-c^2 + c + 1}\frac{\partial}{\partial y^i} + \frac{\partial}{\partial \bar{x}^i} = (1 + c)\frac{\partial}{\partial \bar{x}^i} + \sqrt{-c^2 + c + 1}\frac{\partial}{\partial y^i}. \tag{12}$$

Employing (11) and (12) it deduces that $\varphi^2(\frac{\partial}{\partial \bar{x}^i}) = \varphi(\frac{\partial}{\partial \bar{x}^i}) + I(\frac{\partial}{\partial \bar{x}^i})$. Also, for $i \in \{1, 2\}$ we obtain

$$\begin{aligned} \varphi^2(\frac{\partial}{\partial y^i}) &= \varphi(\sqrt{-c^2 + c + 1}\frac{\partial}{\partial \bar{x}^i} + (1 - c)\frac{\partial}{\partial y^i}) = \sqrt{-c^2 + c + 1}\varphi(\frac{\partial}{\partial \bar{x}^i}) + (1 - c)\varphi(\frac{\partial}{\partial y^i}) \\ &= \sqrt{-c^2 + c + 1}(c\frac{\partial}{\partial \bar{x}^i} + \sqrt{-c^2 + c + 1}\frac{\partial}{\partial y^i}) + (1 - c)(\sqrt{-c^2 + c + 1}\frac{\partial}{\partial \bar{x}^i} + (1 - c)\frac{\partial}{\partial y^i}) \\ &= \sqrt{-c^2 + c + 1}\frac{\partial}{\partial \bar{x}^i} + (-c^2 + c + 1 + (1 - c)^2)\frac{\partial}{\partial y^i} = \sqrt{-c^2 + c + 1}\frac{\partial}{\partial \bar{x}^i} + (2 - c)\frac{\partial}{\partial y^i}, \end{aligned} \tag{13}$$

and

$$\varphi(\frac{\partial}{\partial y^i}) + I(\frac{\partial}{\partial y^i}) = (\sqrt{-c^2 + c + 1}\frac{\partial}{\partial \bar{x}^i} + (1 - c)\frac{\partial}{\partial y^i}) + \frac{\partial}{\partial y^i} = \sqrt{-c^2 + c + 1}\frac{\partial}{\partial \bar{x}^i} + (2 - c)\frac{\partial}{\partial y^i}. \tag{14}$$

By means of (13) and (14) we get $\varphi^2(\frac{\partial}{\partial y^i}) = \varphi(\frac{\partial}{\partial y^i}) + I(\frac{\partial}{\partial y^i})$. Therefore, the condition (2) is satisfied. Now, we investigate the condition (3) for $(T\mathbb{R}^2, \varphi, G_S)$. We have

$$\begin{aligned} G_S(\varphi(\frac{\partial}{\partial \bar{x}^1}), \varphi(\frac{\partial}{\partial \bar{x}^1})) &= G_S((c\frac{\partial}{\partial \bar{x}^1} + \sqrt{-c^2+c+1}\frac{\partial}{\partial y^1}), (c\frac{\partial}{\partial \bar{x}^1} + \sqrt{-c^2+c+1}\frac{\partial}{\partial y^1})) \\ &= c^2 G_S(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^1}) + (-c^2+c+1)G_S(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^1}) \\ &= c^2 g_{Euc}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) + (-c^2+c+1)g_{Euc}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) = 1+c, \end{aligned} \tag{15}$$

and

$$\begin{aligned} G_S(\varphi(\frac{\partial}{\partial \bar{x}^1}), \frac{\partial}{\partial \bar{x}^1}) + G_S(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^1}) &= G_S(c\frac{\partial}{\partial \bar{x}^i} + \sqrt{-c^2+c+1}\frac{\partial}{\partial y^i}, \frac{\partial}{\partial \bar{x}^1}) + G_S(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^1}) \\ &= cG_S(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^1}) + G_S(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^1}) = (1+c)G_S(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^1}) \\ &= (1+c)g_{Euc}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) = 1+c. \end{aligned} \tag{16}$$

So, (15) and (16) imply that

$$G_S(\varphi(\frac{\partial}{\partial \bar{x}^1}), \varphi(\frac{\partial}{\partial \bar{x}^1})) = G_S(\varphi(\frac{\partial}{\partial \bar{x}^1}), \frac{\partial}{\partial \bar{x}^1}) + G_S(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^1}) = 1+c. \tag{17}$$

Similarly, it can be checked that

$$G_S(\varphi(\frac{\partial}{\partial \bar{x}^2}), \varphi(\frac{\partial}{\partial \bar{x}^2})) = G_S(\varphi(\frac{\partial}{\partial \bar{x}^2}), \frac{\partial}{\partial \bar{x}^2}) + G_S(\frac{\partial}{\partial \bar{x}^2}, \frac{\partial}{\partial \bar{x}^2}) = 1+c, \tag{18}$$

$$G_S(\varphi(\frac{\partial}{\partial \bar{x}^1}), \varphi(\frac{\partial}{\partial \bar{x}^2})) = G_S(\varphi(\frac{\partial}{\partial \bar{x}^1}), \frac{\partial}{\partial \bar{x}^2}) + G_S(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^2}) = 0, \tag{19}$$

$$G_S(\varphi(\frac{\partial}{\partial y^1}), \varphi(\frac{\partial}{\partial y^1})) = G_S(\varphi(\frac{\partial}{\partial y^1}), \frac{\partial}{\partial y^1}) + G_S(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^1}) = 2-c, \tag{20}$$

$$G_S(\varphi(\frac{\partial}{\partial y^2}), \varphi(\frac{\partial}{\partial y^2})) = G_S(\varphi(\frac{\partial}{\partial y^2}), \frac{\partial}{\partial y^2}) + G_S(\frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^2}) = 2-c, \tag{21}$$

$$G_S(\varphi(\frac{\partial}{\partial y^1}), \varphi(\frac{\partial}{\partial y^2})) = G_S(\varphi(\frac{\partial}{\partial y^1}), \frac{\partial}{\partial y^2}) + G_S(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}) = 0. \tag{22}$$

Therefore, using (17), (18), (19), (20), (21) and (22) it deduces that the condition (3) is satisfied for $(T\mathbb{R}^2, \varphi, G_S)$ and so, φ is a Golden Riemannian structure and the triple $(T\mathbb{R}^2, \varphi, G_S)$ is a Golden Riemannian manifold.

Now, we check Theorem 3.1 for the Cheeger-Gromoll metric G_{CG} as follows.

Theorem 3.5. *There is not any Golden Riemannian structure φ of the form (5) on the tangent bundle* TM *with the Cheeger-Gromoll metric G_{CS}.*

Proof. Substituting $\alpha_2(t) = \beta_2(t) = 0$, and $\alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}$ and $\alpha_3(t) = \frac{t}{1+t}$ into (7), it deduces that this system of equations has no solution for p, q, r and s, which gives the assertion. □

Remark 3.6. *In [10], the authors presented a structurer of the form (9) on the tangent bundle* TM *with the Cheeger-Gromoll metric G_{CG} and claimed that it introduces a Golden Riemannian structure. It is easy to check that such a structure does not satisfy (3) and hence, it dose not define a Golden Riemannian structure on* (TM, G_{CG}).

Also, as the other immediate consequence of Theorem 3.1, we have the following statement.

Theorem 3.7. *Let (M, g) be a Riemannian manifold and let* TM *be its tangent bundle equipped to the Cheeger-Gromoll type metric G_{ml}. The structure φ defined by*

$$\begin{cases} \varphi(X^h) = (1-c_1)X^h + c_2X^v, \\ \varphi(X^v) = -\dfrac{c_1^2 - c_1 - 1}{c_2}X^h + c_1X^v, \end{cases} \tag{23}$$

for all non-zero real constants c_1 and c_2 when $c_1^2 - c_1 - 1 > 0$, is a Golden Riemannian structure on (TM, G_{ml}) *if and only if $l = 0$ and $m = \dfrac{\ln(\frac{c_2^2}{c_1^2-c_1-1})}{\ln(1+(g(u,u))^2)} = m_\lambda$.*

Proof. Let φ be a structure of the form (5) on the tangent bundle (TM, G_{ml}). Substituting $\alpha_2(t) = \beta_2(t) = 0$, and $\alpha_1(t) = \frac{1}{(1+t)^m}$, and $\alpha_3(t) = 1 - \alpha_1(t)$ and $\beta_1(t) = -\beta_3(t) = \frac{l}{(1+t)^m}$ into (7) and some calculations, it deduces that this system of equations has one solution of the following form

$$s = c_1, \quad q = c_2, \quad p = 1 - c_1, \quad r = -\frac{c_1^2 - c_1 - 1}{c_2}, \quad l = 0, \quad m = \frac{\ln(\frac{c_2^2}{c_1^2-c_1-1})}{\ln(1+(g(u,u))^2)},$$

for all non-zero real constants c_1 and c_2 when $c_1^2 - c_1 - 1 > 0$. Therefore, if $l = 0$ and $m = m_\lambda$, then the structure φ defined by (23) is a Golden Riemannian structure on the tangent bundle (TM, G_{ml}). □

Taking into account Theorem 3.7, we establish immediately the truthfulness of the following

Corollary 3.8. *If we substitute $c_1 = \frac{1}{2}$ and $c_2 = \frac{\sqrt{5}}{2}$ into (23), it concludes that $m = \ln(1) = 0$ and consequently, the structure φ defined by*

$$\begin{cases} \varphi(X^h) = \dfrac{1}{2}X^h + \dfrac{\sqrt{5}}{2}X^v, \\ \varphi(X^v) = \dfrac{\sqrt{5}}{2}X^h + \dfrac{1}{2}X^v, \end{cases} \tag{24}$$

for all vector fields $X \in \mathfrak{X}_0^1$, is a Golden Riemannian structure on the tangent bundle (TM, G_{00}). *Notice that the Cheeger-Gromoll metric type G_{ml} for $m = l = 0$ is the Sasaki metric G_S and hence, Corollary 3.3 is proven again.*

Corollary 3.9. *According to Theorem 3.7, for $l \neq 0$, the tangent bundle* (TM, G_{ml}) *has no Golden Riemannian structure of the form (5). Notice that the Cheeger-Gromoll metric G_{CG} is obtained from the Cheeger-Gromoll type metric G_{ml} for $m = l = 1$ and hence, the tangent bundle* (TM, G_{CG}) *has no Golden Riemannian structure of the form (5), which proves the truthfulness of Theorem 3.5 again.*

Now we check Theorem 3.1 for the Kaluza-Klein metric G_{KK} as follows.

Theorem 3.10. *Let (M, g) be a Riemannian manifold and let* TM *be its tangent bundle equipped to the Kaluza-Klein metric G_{KK}. The structure φ on* TM *defined by*

$$\begin{cases} \varphi(X^h) = c_1X^h + c_2X^v, \\ \varphi(X^v) = -\dfrac{c_1^2 - c_1 - 1}{c_2}X^h + (1-c_1)X^v, \end{cases} \tag{25}$$

for all non-zero real constants c_1 and c_2 and all vector fields $X \in \mathfrak{X}_0^1(M)$ is a Golden Riemannian structure if and only if $\alpha_3(v^2) = -\frac{\alpha_1(v^2)(c_1^2+c_2^2-c_1-1)}{c_1^2-c_1-1}$ and $\beta_1(v^2) = 0$.

Proof. The proof is completely similar to Theorem 3.7. □

Example 3.11. *Let* $(\mathbb{R}^2, g_{Euc})$ *be the* 2*-dimensional Euclidean manifold and* $T\mathbb{R}^2$ *be the tangent bundle over* $\mathbb{R}^2$ *as we have seen above in the Example 3.4. The Kaluza-Klein metric* G_{KK} *corresponded to* $(\mathbb{R}^2, g_{Euc})$ *on* $T\mathbb{R}^2$ *has the following form.*

$$\begin{cases} G_{KK}(\frac{\partial}{\partial \bar{x}^i}, \frac{\partial}{\partial \bar{x}^j}) = (\alpha_1 + \alpha_3)(1) g_{Euc}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \\ G_{KK}(\frac{\partial}{\partial \bar{x}^i}, \frac{\partial}{\partial y^j}) = 0, \\ G_{KK}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = \alpha(1) g_{Euc}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \end{cases}$$

where $\alpha_1, \alpha_3 : \mathbb{R}^+ \to \mathbb{R}$ *are smooth functions with* $\alpha_3(1) = -\frac{\alpha_1(1)(c_1^2 + c_2^2 - c_1 - 1)}{c_1^2 - c_1 - 1}$ *and* $i, j \in \{1, 2\}$*. Using (25), we define the structure* φ *on* $(T\mathbb{R}^2, G_{KK})$ *by*

$$\begin{cases} \varphi(\frac{\partial}{\partial \bar{x}^i}) = c_1 \frac{\partial}{\partial \bar{x}^i} + c_2 \frac{\partial}{\partial y^i}, \\ \varphi(\frac{\partial}{\partial y^i}) = -\dfrac{c_1^2 - c_1 - 1}{c_2} \frac{\partial}{\partial \bar{x}^i} + (1 - c_1) \frac{\partial}{\partial y^i}, \end{cases} \tag{26}$$

for all non-zero real constants c_1 *and* c_2*. It is easy to check that the condition (2) is satisfied by* φ*. More precisely, we have*

$$\varphi^2(\frac{\partial}{\partial \bar{x}^i}) = \varphi(\frac{\partial}{\partial \bar{x}^i}) + I(\frac{\partial}{\partial \bar{x}^i}) = (1 + c_1)\frac{\partial}{\partial \bar{x}^i} + c_2 \frac{\partial}{\partial y^i},$$

$$\varphi^2(\frac{\partial}{\partial y^i}) = \varphi(\frac{\partial}{\partial y^i}) + I(\frac{\partial}{\partial y^i}) = -\frac{c_1^2 - c_1 - 1}{c_2} \frac{\partial}{\partial \bar{x}^i} + (2 - c_1)\frac{\partial}{\partial y^i},$$

for $i \in \{1, 2\}$*. Also, some standard calculation show that*

$$G_{KK}(\varphi(\frac{\partial}{\partial \bar{x}^1}), \varphi(\frac{\partial}{\partial \bar{x}^1})) = G_{KK}(\varphi(\frac{\partial}{\partial \bar{x}^1}), \frac{\partial}{\partial \bar{x}^1}) + G_{KK}(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^1}) = \frac{-c_1 c_2^2 - c_2^2}{c_1^2 - c_1 - 1}\alpha_1(1),$$

$$G_{KK}(\varphi(\frac{\partial}{\partial \bar{x}^1}), \varphi(\frac{\partial}{\partial \bar{x}^2})) = G_{KK}(\varphi(\frac{\partial}{\partial \bar{x}^1}), \frac{\partial}{\partial \bar{x}^2}) + G_{KK}(\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^2}) = 0,$$

$$G_{KK}(\varphi(\frac{\partial}{\partial y^1}), \varphi(\frac{\partial}{\partial y^1})) = G_{KK}(\varphi(\frac{\partial}{\partial y^1}), \frac{\partial}{\partial y^1}) + G_{KK}(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^1}) = (2 - c_1)\alpha_1(1),$$

$$G_{KK}(\varphi(\frac{\partial}{\partial y^1}), \varphi(\frac{\partial}{\partial y^2})) = G_{KK}(\varphi(\frac{\partial}{\partial y^1}), \frac{\partial}{\partial y^2}) + G_{KK}(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}) = 0.$$

Hence, the condition (3) is satisfied for φ *and so,* φ *is a Golden structure and the triple* $(T\mathbb{R}^2, \varphi, G_{KK})$ *is a Golden Riemannian manifold.*

Corollary 3.12. *Substituting* $c_1 = \frac{1}{2}$ *and* $c_2 = \frac{\sqrt{5}}{2}$ *into Theorem 3.10 implies that* $\alpha_3(v^2) = 0$ *and consequently, if* $\beta_1(v^2) = \alpha_3(v^2) = 0$*, then the structure* φ *defined by*

$$\begin{cases} \varphi(X^h) = \frac{1}{2}X^h + \frac{\sqrt{5}}{2}X^v, \\ \varphi(X^v) = \frac{\sqrt{5}}{2}X^h + \frac{1}{2}X^v, \end{cases}$$

for all vector fields $X \in \mathfrak{T}_0^1(M)$ *is a Golden Riemannian structure on the tangent bundle* (TM, G_{KK})*.*

Similar to Theorem 3.7 it can be checked that the following assertion is valid.

Theorem 3.13. *Let (M, g) be a Riemannian manifold and let* TM *be its tangent bundle equipped to the Kaluza-Klein type metric G_t. If $\alpha_3(v^2) = \beta_3(v^2) = 0$, then the structure φ defined by*

$$\begin{cases} \varphi(X^h) = \frac{1}{2}X^h + \frac{\sqrt{5}}{2}X^v, \\ \varphi(X^v) = \frac{\sqrt{5}}{2}X^h + \frac{1}{2}X^v, \end{cases} \tag{27}$$

for all vector fields $X \in \mathfrak{T}^1_0$, is a Golden Riemannian structure on the tangent bundle (TM, G_t).

4. Integrable Golden Riemannian Structures on the Tangent bundle

In this section, we investigate the integrability of the Golden Riemannian structure φ on the tangent bundle TM equipped to the g-natural metrics.

Let (M, g) be a Riemannian manifold and let TM be its tangent bundle equipped with the g-natural metric G. The Golden Riemannian structure φ on the tangent bundle (TM, G) is integrable if $\phi_\varphi G = 0$. Also, (TM, G) is a locally decomposable Golden Riemannian manifold if and only if $\phi_\varphi G = 0$ ([9]). We have the following Proposition.

Proposition 4.1. *Let (M, g) be a Riemannian manifold and* TM *be its tangent bundle equipped with the Sasaki metric G_S. The metric G_S is pure with respect to the Golden Riemannian structure φ defined in Theorem 3.2 by*

$$\begin{cases} \varphi(X^h) = cX^h + \sqrt{-c^2 + c + 1}X^v, \\ \varphi(X^v) = \sqrt{-c^2 + c + 1}X^h + (1 - c)X^v, \end{cases}$$

for all non-zero real constants c when $-c^2 + c + 1 > 0$, and all vector fields $X \in \mathfrak{T}^1_0(M)$.

Proof. It is easy to check that $G_S(\varphi\bar{X}, \bar{Y}) - G_S(\bar{X}, \varphi\bar{Y}) = 0$, for all vector fields $\bar{X}, \bar{Y} \in \mathfrak{T}^1_0(\mathrm{TM})$, i.e. G_S is pure concerning the Golden Riemannian structure φ. □

The following Theorem shows that there is a direct correlation between the locally decomposable property of $(\mathrm{TM}, \varphi, G_S)$ and the locally flatness of manifold (M, g), where the Golden Riemannian structure φ is defined by (8).

Theorem 4.2. *Let (M, g) be a Riemannian manifold and* TM *be its tangent bundle equipped to the Golden Riemannian structure φ defined by (8) and the Sasaki metric G_S. The locally flatness of (M, g) is the necessary and sufficient condition for the triple* $(\mathrm{TM}, \varphi, G_S)$ *to be locally decomposable Riemannian manifold.*

Proof. Taking into account Proposition 4.1 and using (4) and the fact that

$$X^h(g(Y, Z))^v = (Xg(Y, Z))^v, \qquad X^v(g(Y, Z))^v = 0,$$

for all vector fields $X, Y \in \mathfrak{T}^1_0(M)$, we have

$$(\phi_\varphi G_S)(\bar{X}, \bar{Y}, \bar{Z}) = (\varphi\bar{X})(G_S(\bar{Y}, \bar{Z})) - \bar{X}(G_S(\varphi\bar{Y}, \bar{Z})) + G_S((L_{\bar{Y}}\varphi)\bar{X}, \bar{Z}) + G_S(\bar{Y}, (L_{\bar{Z}}\varphi)\bar{X}),$$

for all vector fields $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{T}^1_0(\mathrm{TM})$. Now, it deduces that

$$(\phi_\varphi G_S)(X^h, Y^h, Z^h) = \sqrt{-c^2 + c + 1}G_S((\mathrm{R}(Y, u)X - \mathrm{R}(X, Y)u)^h, Z^h),$$
$$(\phi_\varphi G_S)(X^h, Y^h, Z^v) = 0, \qquad (\phi_\varphi G_S)(X^h, Y^v, Z^v) = 0, \qquad (\phi_\varphi G_S)(X^h, Y^v, Z^h) = 0,$$
$$(\phi_\varphi G_S)(X^v, Y^v, Z^v) = 0, \qquad (\phi_\varphi G_S)(X^v, Y^h, Z^h) = 0,$$
$$(\phi_\varphi G_S)(X^v, Y^v, Z^h) = \sqrt{-c^2 + c + 1}G_S((\mathrm{R}(u, Y)Z)^h, Z^h),$$
$$(\phi_\varphi G_S)(X^v, Y^h, Z^v) = \sqrt{-c^2 + c + 1}G_S((\mathrm{R}(X, Y)u)^v, Z^v),$$

where R denotes the Riemann curvature tensor on M. Therefore, the triple $(\mathrm{TM}, \varphi, G_S)$ is a locally decomposable Riemannian manifold if and only if (M, g) is locally flat manifold. □

Proposition 4.3. *Let (M, g) be a Riemannian manifold and* TM *be its tangent bundle equipped with the Cheeger-Gromoll type metric G_{ml} with $m = m_\lambda$ and $l = 0$. The metric G_{ml} is pure with respect to the Golden Riemannian structure φ defined in Theorem 3.7 by*

$$\begin{cases} \varphi(X^h) = (1 - c_1)X^h + c_2X^v, \\ \varphi(X^v) = -\dfrac{c_1^2 - c_1 - 1}{c_2}X^h + c_1X^v, \end{cases}$$

for all non-zero real constants c_1, and c_2 and all vector fields $X \in \mathfrak{T}_0^1(M)$.

Proof. Standard calculations show that $G_{ml}(\varphi X^h, Y^h) = G_{ml}(X^h, \varphi Y^h)$, $G_{ml}(\varphi X^v, Y^v) = G_{ml}(X^v, \varphi Y^v)$, $G_{ml}(\varphi X^v, Y^h) = G_{ml}(X^v, \varphi Y^h)$ and $G_{ml}(\varphi X^h, Y^v) = G_{ml}(X^h, \varphi Y^v)$ for all vector fields $X, Y \in \mathfrak{T}_0^1(M)$, where $l = 0$ and $m = m_\lambda$, which proves the assertion. □

Theorem 4.4. *Let (M, g) be a Riemannian manifold and* TM *be its tangent bundle equipped to the Golden Riemannian structure φ defined by (23) and the Cheeger-Gromoll type metric G_{ml} with $m = m_\lambda$ and $l = 0$. The triple* $(\mathrm{TM}, \varphi, G_{ml})$ *is a locally decomposable Golden Riemannian manifold if and only if (M, g) is locally flat.*

Proof. First of all, notice that for $l = 0$ and $m = m_\lambda$, the Cheeger-Gromoll type metric G_{ml} has the following form

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = g(X, Y), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = 0, \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(v^2)g(X, Y), \end{cases}$$

where $v^2 = g(u, u)$ and $X, Y \in \mathfrak{T}_0^1(M)$. Using the fact that $X^h(g(Y, Z))^v = (Xg(Y, Z))^v$ and $X^v(g(Y, Z))^v = 0$ for all $X, Y, Z \in \mathfrak{T}_0^1(M)$ and taking into account Proposition 4.3 we have

$$(\phi_\varphi G_{ml})(\bar{X}, \bar{Y}, \bar{Z}) = (\varphi\bar{X})(G_{ml}(\bar{Y}, \bar{Z})) - \bar{X}(G_{ml}(\varphi\bar{Y}, \bar{Z})) + G_{ml}((L_{\bar{Y}}\varphi)\bar{X}, \bar{Z}) + G_{ml}(\bar{Y}, (L_{\bar{Z}}\varphi)\bar{X}),$$

for all vector fields $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{T}_0^1(\mathrm{TM})$. Lemma 2.1 and some direct calculations show that

$$\begin{aligned} &(\phi_\varphi G_{ml})(X^h, Y^h, Z^h) = c_2G_{ml}((\mathrm{R}(Y, u)X - \mathrm{R}(X, Y)u)^h, Z^h), \\ &(\phi_\varphi G_{ml})(X^h, Y^h, Z^v) = 0, \qquad (\phi_\varphi G_{ml})(X^h, Y^v, Z^v) = 0, \qquad (\phi_\varphi G_{ml})(X^h, Y^v, Z^h) = 0, \\ &(\phi_\varphi G_{ml})(X^v, Y^v, Z^v) = 0, \qquad (\phi_\varphi G_{ml})(X^v, Y^h, Z^h) = 0, \\ &(\phi_\varphi G_{ml})(X^v, Y^v, Z^h) = -\frac{c_1^2 - c_1 - 1}{c_2}G_{ml}((\mathrm{R}(u, Y)Z)^h, Z^h), \\ &(\phi_\varphi G_{ml})(X^v, Y^h, Z^v) = -\frac{c_1^2 - c_1 - 1}{c_2}G_{ml}((\mathrm{R}(X, Y)u)^v, Z^v). \end{aligned}$$

Hence, the locally flatness of (M, g) is the necessary and sufficient condition for the triple $(\mathrm{TM}, \varphi, G_{ml})$ with $l = 0$ and $m = m_\lambda$, to be a locally decomposable Golden Riemannian manifold. □

Now, we present the following Proposition which proves that the Kaluza-Klein metric G_{KK} defined in Theorem 3.10, is pure with respect to the Golden Riemannian structure φ determined by (25).

Proposition 4.5. *Let (M, g) be a Riemannian manifold and* TM *be its tangent bundle equipped with the Kaluza-Klein metric G_{KK} determined in Theorem 3.10. The Kaluza-Klein metric G_{KK} is pure with respect to the Golden Riemannian structure φ defined in Theorem 3.10 by*

$$\begin{cases} \varphi(X^h) = c_1X^h + c_2X^v, \\ \varphi(X^v) = -\dfrac{c_1^2 - c_1 - 1}{c_2}X^h + (1 - c_1)X^v, \end{cases}$$

for all non-zero real constants c_1 and c_2 and all vector fields $X \in \mathfrak{T}_0^1(M)$.

Proof. Calculations show that $G_{KK}(\varphi\bar{X}, \bar{Y}) - G_{kk}(\bar{X}, \varphi\bar{Y}) = 0$, for all vector fields $\bar{X}, \bar{Y} \in \mathfrak{T}_0^1(\mathrm{TM})$. Therefore, the Kaluza-Klein metric G_{KK} is pure concerning the Golden Riemannian structure φ defined by (25) . □

Taking into account Proposition 4.5 after some standard calculations, we establish the truthfulness of the following.

Theorem 4.6. *Let (M, g) be a Riemannian manifold and* TM *be its tangent bundle equipped to the Golden structure φ defined by (25) and the Kaluza-Klein metric G_{KK} determined in Theorem 3.10. The triple $(\mathrm{TM}, \varphi, G_{KK})$ is a locally decomposable Golden Riemannian manifold if and only if (M, g) is locally flat.*

It can be proven that the following assertion is valid.

Proposition 4.7. *Let (M, g) be a Riemannian manifold and* TM *be its tangent bundle equipped with the Kaluza-Klein type metric G_t determined in Theorem 3.13. The metric G_t is pure with respect to the Golden Riemannian structure φ defined in Theorem 3.13 by*

$$\begin{cases} \varphi(X^h) = \frac{1}{2}X^h + \frac{\sqrt{5}}{2}X^v, \\ \varphi(X^v) = \frac{\sqrt{5}}{2}X^h + \frac{1}{2}X^v, \end{cases}$$

for all vector fields $X \in \mathfrak{T}_0^1(M)$.

After some direct calculations this fact is demonstrable that

Theorem 4.8. *Let (M, g) be a Riemannian manifold and* TM *be its tangent bundle equipped to the Golden Riemannian structure φ defined by (27) and the Kaluza-Klein type metric G_t determined in Theorem 3.13. The triple $(\mathrm{TM}, \varphi, G_t)$ is a locally decomposable Golden Riemannian manifold if and only if (M, g) is locally flat.*

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