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Periodic Solution of the DS-I-A Epidemic Model with Stochastic Perturbations

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Abstract. The paper introduces DS-I-A model with periodical coefficients. First of all, we show that there is a unique positive solution of the stochastic model. Furthermore we deduce the conditions under which the disease will end and continue. At last, we draw a conclusion that there exists nontrivial positive periodic solution for the stochastic system by stochastic Lyapunov functions. Simulations are also carried out to confirm our analytical results.

1. Introduction

Human immunodeficiency virus (HIV) infection is characterized by three different phases, namely the primary infection, clinically asymptomatic stage (chronic infection), and acquired immunodeficiency syndrome (AIDS) or drug therapy. Mathematical modeling is useful for understanding the spread of HIV/AIDS. Thus various models have been developed to describe the spread of this disease according to its characteristics, see [1]-[5]. Many works have focused on the epidemic models with bilinear incidence whereas Anderson and May and De Jong et al. pointed out that the epidemic models with standard incidence provide a more natural description for humankind and gregarious animals [6]-[7]. Among these models, the following DS-I-A model proposed by Hyman et al. [5] describes HIV spreads in multi-groups of susceptibilities:

$$\begin{cases} \frac{dS_k(t)}{dt} &= \mu(S_k^0 - S_k(t)) - \frac{\beta\alpha_k S_k(t)I(t)}{N(t)}, \quad 1 \le k \le n, \\ \frac{dI(t)}{dt} &= \sum_{k=1}^{n} \frac{\beta\alpha_k S_k(t)I(t)}{N(t)} - (\mu+\gamma)I(t), \\ \frac{dA(t)}{dt} &= \gamma I(t) - \delta A(t), \end{cases} \tag{1}$$

in which $N(t) = \sum_{k=1}^{n} S_k(t) + I(t), S_i(t)(i = 1, 2, \dots, n)$ denote the n individuals susceptible to infection subgroups, $I(t)$ the infected individuals; $A(t)$ the AIDS cases; $\mu S_k^0(k = 1, 2, \dots, n)$ the input flow into the n susceptible subgroups; μ the natural mortality rate; γ the removal rate coefficient of the infected individuals and δ the sum of natural mortality rate and mortality due to illness; $\alpha_k(k = 1, 2, \dots, n)$ the susceptibility

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of susceptible individuals in subgroup I and $\frac{\beta I(t)S_k(t)}{N(t)}\alpha_k$ the standard incidence ratio of susceptible subgroups S_k. Since the dynamics of group A has no effect on the disease transmission dynamics, thus we only consider

$$\begin{cases} \frac{dS_k(t)}{dt} &= \mu(S_k^0 - S_k(t)) - \frac{\beta\alpha_k S_k(t)I(t)}{N(t)}, \quad 1 \le k \le n, \\ \frac{dI(t)}{dt} &= \sum_{k=1}^{n} \frac{\beta\alpha_k S_k(t)I(t)}{N(t)} - (\mu+\gamma)I(t), \end{cases} \tag{2}$$

The threshold conditions can be calculated which determine whether an infectious disease will spread in susceptible population when the disease is introduced into the crowed, according to research the disease free equilibrium $E_0(S_1^0, S_2^0, \dots, S_n^0, 0)$ of system (2) in [8].

And they obtain reproductive number

$$R_0 = \frac{\beta\sum_{k=1}^{n}\alpha_k S_k^0}{(\mu+\gamma)\sum_{k=1}^{n} S_k^0},$$

where $R_0 < 1, E_0$ is local asymptotic stabile and disease extinct. When $R_0 > 1$, then E_0 is unstable and the disease will persistent existence (see [5]). The effective contact rate of infected individual in subgroup $S_k(k = 1, 2, \dots, n)$ is $\alpha_k\beta(k = 1, 2, \dots, n)$. So for initial time $(S_i = S_i^0)$, the average effective contact rate of infected individual in subgroup $S_k(k = 1, 2, \dots, n)$ is $\frac{\beta\sum_{k=1}^{n}\alpha_k S_k^0}{\sum_{k=1}^{n} S_k^0}$. $\frac{1}{\mu+\gamma}$ the average disease period of infected individuals. So R_0 is basic reproductive number.

It is well recognized fact that real life is full of randomness and stochasticity. Hence the epidemic models are always affected by the environmental noise (in cite [9]-[16]). In [17]-[22], the stochastic models may be more convenient epidemic models in many situations. To establish the stochastic differential equation(SDE) model, we naturally use the equation in the form of differential

$$dS_k(t) = \Big[\mu(S_k^0 - S_k(t)) - \frac{\beta\alpha_k S_k(t)I(t)}{N(t)}\Big]dt, \quad 1 \le k \le n. \tag{3}$$

Here $[t, t + \Delta t)$ is a small time interval and $d\cdot$ for the small change. For example $dS_k(t) = S_k(t + dt) - S_k(t)$, $1 \le k \le n$ and the change $dS_k(t)$ is described by (3). Consider the effective contact rate constant of infected individual $\beta\alpha_k$, $1 \le k \le n$ in the deterministic model. The total number of newly increased I in the small interval $[t, t + dt)$ is

$$\sum_{k=1}^{n} \frac{\beta\alpha_k S_k(t)I(t)}{N(t)} dt.$$

Now suppose that some stochastic environment factors acts simultaneously on each subgroups in the disease. In this case, $\beta\alpha_k$, $1 \le k \le n$ changes to a random variable $\widetilde{\beta\alpha_k}$, $1 \le k \le n$. More precisely

$$\widetilde{\beta\alpha_k}dt = \beta\alpha_k dt + \sigma_k dB_k(t) \quad 1 \le k \le n.$$

Here $dB_k(t) = B_k(t + dt) - B_k(t)(k = 1, 2, \dots, n)$ is the increment of a standard Brownian motion. And $B_k(t)(k = 1, 2, \dots, n)$ are independent standard Brownian motions with $B_k(0) = 0(k = 1, 2, \dots, n)$ and $\sigma_k^2 > 0(k = 1, 2, \dots, n)$ denote the intensities of the white noise. Thus the number of newly increasing I that each subgroups S_k, $1 \le k \le n$ infected in $[t, t + dt)$ is normally distributed with mean $\beta\alpha_k dt$ and variance $\sigma_k^2 dt$, where $k = 1, 2, \dots, n$.

Therefore we replace $\beta\alpha_k dt$ in equation (3) by $\widetilde{\beta\alpha_k}dt = \beta\alpha_k dt + \sigma_k dB(t)$ to get

$$dS_k(t) = \Big[\mu(S_k^0 - S_k(t)) - \frac{\beta\alpha_k S_k(t)I(t)}{N(t)}\Big]dt - \sigma_k \frac{S_k(t)I(t)}{N(t)}dB_k(t), \quad 1 \le k \le n.$$

Note that $\widetilde{\beta\alpha_i}dt$ now denotes the mean of the stochastic number of S_i infected in the infinitesimally small time interval $[t, t+dt)$. Similarly, the first equation of (2) becomes another SDE. That is, the deterministic infectious diseases model (2) becomes the *Itô* SDE

$$\begin{cases} dS_k(t) &= [\mu(S_k^0 - S_k(t)) - \frac{\beta\alpha_k S_k(t)I(t)}{N(t)}]dt - \sigma_k \frac{S_k(t)I(t)}{N(t)}dB_k(t), \quad 1 \le k \le n, \\ dI(t) &= [\sum_{k=1}^{n} \frac{\beta\alpha_k S_k(t)I(t)}{N(t)} - (\mu+\gamma)I(t)]dt + \sum_{k=1}^{n} \sigma_k \frac{S_k(t)I(t)}{N(t)}dB_k(t), \end{cases} \tag{4}$$

Other parameters are the same as in system (2). On the other hand, many infectious of humans fluctuate over time and often show seasonal patterns of incidence. Taking account of periodic variation in epidemic models and studying the existence of periodic solutions are important and interesting to predict and control the spread of infectious diseases. Many results on the periodic solution of epidemic models have been reported [23–25] by using Has'minskii theory of periodic solutions and constructing suitable Lyapunov functions.

Motivated by above facts, in this paper, we will consider the following stochastic DS-I-A model:

$$\begin{cases} dS_k(t) &= [\mu(t)(S_k^0(t) - S_k(t)) - \frac{\beta(t)\alpha_k(t)S_k(t)I(t)}{N(t)}]dt - \sigma_k(t)\frac{S_k(t)I(t)}{N(t)}dB_k(t), \quad 1 \le k \le n, \\ dI(t) &= [\sum_{k=1}^{n} \frac{\beta(t)\alpha_k(t)S_k(t)I(t)}{N(t)} - (\mu(t)+\gamma(t))I(t)]dt + \sum_{k=1}^{n} \sigma_k(t)\frac{S_k(t)I(t)}{N(t)}dB_k(t), \end{cases} \tag{5}$$

in which the parameter functions $\mu, S_k^0, \sigma_k, \beta, \alpha_k, \gamma, k = 1, 2, \ldots, n$, are positive, non-constant and continuous functions of period T. This paper is organized as follows. In Section 2, we show there is a unique positive solution of system (5) by the same way as mentioned in Ref.[26]-[28]. In Section 3, we establish sufficient conditions for extinction of disease. The condition for the disease being persistent is given in Sections 4. In Section 5, we verify that there exists nontrivial positive periodic solution of system (5). In Section 6, outcomes of numerical simulations are also reported in support of analytical results.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions(i.e. it is right continuous and $\mathcal{F}_0$ contains all P -null sets). Denote

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0 \, for \, all \, 1 \le i \le n\}.$$

If $f(t)$ is an integral function on $[0,\infty)$, define $\langle f \rangle_T = \frac{1}{T}\int_0^T f(s)ds$. If $f(t)$ is a bounded function on $[0,\infty)$, define $f^l = \inf_{t\in[0,\infty)} f(t), f^u = \sup_{t\in[0,\infty)} f(t)$. We consider the general d-dimensional stochastic differential equation

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t), \; for \, t \ge t_0 \tag{6}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^n$, where $B(t)$ denotes d-dimensional standard Brownian motions defined on the above probability space.

Define the differential operator $\mathcal{L}$ associated with Eq.(6) by

$$\mathcal{L} = \frac{\partial}{\partial t} + \Sigma f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\Sigma[g^T(x,t)g(x,t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If $\mathcal{L}$ acts on a function $V \in C^{2,1}(\mathbb{R}^n \times \bar{\mathbb{R}}_+; \bar{\mathbb{R}}_+)$, then

$$\mathcal{L}V(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trac[g^T(x,t)V_{xx}(x,t)g(x,t)]$$

where $V_t = \frac{\partial V}{\partial t}, V_x = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{d\times d}$. By Itô's formula, if $x(t)$ is a solution of Eq.(6), then

$$dV(x(t), t) = LV(x(t), t)dt + V_x(x(t), t)g(x(t), t)dB(t).$$

In Eq.(6),we assume that $f(0, t) = 0$ and $g(0, t) = 0$ for all $t \geq t_0$. So $x(t) \equiv 0$ is a solution of Eq.(6), called the trivial solution or equilibrium position.

By the definition of stochastic differential, the equation (6) is equivalent to the following stochastic integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \sum_{r=1}^{d} \int_{t_0}^{t} g_r(x(s), s)dB_r(s),\ for\, t \geq t_0 \tag{7}$$

2. Existence and uniqueness of positive solution

In this section we first show that the solution of system (5) is positive and global. To get a unique global(i.e. no explosion in a finite time) solution for any initial value, the coefficients of the equation are required to satisfy the linear growth condition and the local lipschitz condition. However, the coefficients of system (5) do not satisfy the linear growth condition, as the item $\frac{\beta\alpha_i S_i(t)I(t)}{N(t)}$ is nonlinear. So the solution of system (5) may explore in finite time. In this section, we show that the solution of system (5) is positive and global by using the Lyapunov analysis method.

Theorem 2.1. *There is a unique positive solution $X(t) = (S_1(t), S_2(t), \ldots, S_n(t), I(t))$ of system (5) on $t \geq 0$ for any initial value $(S_1(0), S_2(0), \ldots, S_n(0), I(0)) \in \mathbb{R}_+^{n+1}$, and the solution will remain in $\mathbb{R}_+^{n+1}$ with probability 1, namely, $(S_1(t), S_2(t), \ldots, S_n(t), I(t)) \in \mathbb{R}_+^{n+1}$ for all $t \geq 0$.*

Proof. Since the coefficients of system (5) are locally Lipschitz continuous, then, for given initial value $(S_1(0), S_2(0), \ldots, S_n(0), I(0)) \in \mathbb{R}_+^{n+1}$. There is a unique local solution $(S_1(t), S_2(t), \ldots, S_n(t), I(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [12]. To show the solution is global, we only need to verify that $\tau_e = \infty$ a.s. Let $m_0 \geq 0$ be sufficiently large so that every component of $X(0)$ lies within the interval $[1/m_0, m_0]$. For each $m \geq m_0$, we define the stopping time

$$\tau_m = \inf\{t \in [0, \tau_e) : \min\{S_1(t), S_2(t), \ldots, S_n(t), I(t)\} \leq \frac{1}{m}\ or\ \ \max\{S_1(t), S_2(t), \ldots, S_n(t), I(t)\} \geq m\}$$

where we set $\inf \phi = \infty$(as usual ϕ denotes the empty set) throughout the paper. According to the definition, τ_m is increasing when $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, then $\tau_\infty \leq \tau_e$ a.s. In the following, we need to prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(S_1(t), S_2(t), \ldots, S_n(t), I(t)) \in \mathbb{R}_+^{n+1}$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. If this assertion is violated then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_\infty \leq T\} \geq \varepsilon,\ \ for\ all\ m \geq m_1.$$

For $t \leq \tau_m$, we can see, for each m,

$$\begin{aligned} d(\sum_{k=1}^{n} S_k + I) &= [\mu(t)\sum_{k=1}^{n}(S_k^0(t) - S_k(t)) - (\mu(t) + \gamma(t))I(t)]dt \\ &= [\mu(t)\sum_{k=1}^{n} S_k^0(t) - \mu(t)(\sum_{k=1}^{n} S_k(t) + I(t)) - \gamma(t)I(t)]dt \\ &\leq \mu(t)\sum_{k=1}^{n} S_k^{0u}dt - \mu(t)(\sum_{k=1}^{n} S_k(t) + I(t))dt. \end{aligned}$$

Therefore

$$\sum_{k=1}^{n} S_k(t) + I(t) \leq \begin{cases} \sum_{k=1}^{n} S_k^{0u}, & \text{if } \sum_{k=1}^{n} S_k(0) + I(0) < \sum_{k=1}^{n} S_k^{0u} \\ \sum_{k=1}^{n} S_k(0) + I(0), & \text{if } \sum_{k=1}^{n} S_k(0) + I(0) \geq \sum_{k=1}^{n} S_k^{0u}. \end{cases}$$

Let $C := \max\{\sum_{k=1}^{n} S_k^0 u, \sum_{k=1}^{n} S_k(0) + I(0)\}$. Define a C^2-function $V : \mathbb{R}_+^{n+1} \to \bar{\mathbb{R}}_+$ by

$$V(S_1, S_2, \ldots, S_n, I) = \sum_{k=1}^{n} (S_k - 1 - \ln S_k) + (I - 1 - \ln I).$$

The non-negativity of this function can be see from $u - 1 - \log u \geq 0$, $\forall u > 0$. Let $m \geq m_0$ and $T > 0$ be arbitrary then by Itô's formula one obtains

$$\begin{aligned} dV(S_1, S_2, \ldots, S_n, I) &= \mathcal{L}V(S_1, S_2, \ldots, S_n, I)dt - \textstyle\sum_{k=1}^{n} \sigma_k(t)(S_k(t) - 1)\frac{I(t)}{N(t)} dB_k(t) \\ &\quad + \sum_{k=1}^{n} \sigma_k(t)(I(t) - 1)\frac{S_k(t)}{N(t)} dB_k(t), \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}V &= \sum_{k=1}^{n} (1 - \frac{1}{S_k(t)})[\mu(t)(S_k^0(t) - S_k(t)) - \frac{\beta(t)\alpha_k(t)S_k(t)I(t)}{N(t)}] + (1 - \frac{1}{I(t)}) \\ &\quad \times [\sum_{k=1}^{n} \frac{\beta(t)\alpha_k(t)S_k(t)I(t)}{N(t)} - (\mu(t) + \gamma(t))I(t)] + \sum_{k=1}^{n} \frac{\sigma_k^2(t)}{2} \frac{I^2(t)}{N^2(t)} + \sum_{k=1}^{n} \frac{\sigma_k^2(t)}{2} \frac{S_k^2(t)}{N^2(t)} \\ &= \mu(t)\sum_{k=1}^{n} S_k^0(t) - \mu(t)(\sum_{k=1}^{n} S_k(t) + I(t)) - \gamma(t)I(t) - \mu(t)\sum_{k=1}^{n} \frac{S_k^0(t)}{S_k(t)} + (n+1)\mu(t) \\ &\quad + \gamma(t) + \tfrac{\beta(t)I(t)}{N(t)} \sum_{k=1}^{n} \alpha_k(t) - \sum_{k=1}^{n} \frac{\beta(t)\alpha_k(t)S_k(t)}{N(t)} + \sum_{k=1}^{n} \frac{\sigma_k^2(t)}{2} \frac{I^2(t)}{N^2(t)} + \sum_{k=1}^{n} \frac{\sigma_k^2(t)}{2} \frac{S_k^2(t)}{N^2(t)} \\ &< \mu^u \sum_{k=1}^{n} S_k^{0u} + (n+1)\mu^u + \gamma^u + \beta^u \sum_{k=1}^{n} \alpha_k^u + \sum_{k=1}^{n} (\sigma_k^u)^2 := M, \end{aligned} \tag{8}$$

where M is a positive constant which is independent of $S_1(t), S_2(t), \ldots, S_n(t), I(t)$ and t. The remainder of the proof follows that in ref. [29].

Remark 2.2. *From Theorem (2.1) there is a unique global solution $(S_1(t), S_2(t), \ldots, S_n(t), I(t)) \in \mathbb{R}_+^{n+1}$ almost surely of system (5), for any initial value $(S_1(0), S_2(0), \ldots, S_n(0), I_0) \in \mathbb{R}_+^{n+1}$. Hence*

$$d(\sum_{k=1}^{n} S_k(t) + I(t)) \leq \mu(t)\sum_{k=1}^{n} S_k^{0u} dt - \mu(t)(\sum_{k=1}^{n} S_k(t) + I(t))dt,$$

and

$$\sum_{k=1}^{n} S_k(t) + I(t) \leq \sum_{k=1}^{n} S_k^{0u} + e^{-\int_0^t \mu(s)ds}[\sum_{k=1}^{n} S_k(0) + I(0) - \sum_{k=1}^{n} S_k^{0u}].$$

If $\sum_{k=1}^{n} S_k(0) + I(0) < \sum_{k=1}^{n} S_k^{0u}$, then $\sum_{k=1}^{n} S_k(t) + I(t) < \sum_{k=1}^{n} S_k^{0u}$ a.s.. Thus the region

$$\Gamma^* = \left\{ (S_1, S_2, \ldots, S_n, I) \in \mathbb{R}_+^{n+1}, \sum_{k=1}^{n} S_k(t) + I(t) < \sum_{k=1}^{n} S_k^{0u} \right\}$$

is a positively invariant set of system (5).

3. Extinction

The other main concern in epidemiology is how we can regulate the disease dynamics so that the disease will be eradicated in a long term. In this section, we shall give a sharp result of the extinction of disease in the stochastic model (5).

Theorem 3.1. *Assume $J = \{1, 2, \dots, n\}$, and $J = N_1 \bigoplus N_2$, where $N_1 = \{i | (\sigma_i^l)^2 \geq \beta^u \alpha_i^u\}$, and $N_2 = \{i | (\sigma_i^l)^2 < \beta^u \alpha_i^u\}$. If $\hat{R}_0^* := \dfrac{\sum_{i\in N_1} \frac{(\beta^u)^2(\alpha_i^u)^2}{2(\sigma_i^l)^2} + \sum_{j\in N_2}\big(\beta^u\alpha_j^u - \frac{(\sigma_j^l)^2}{2}\big)}{\langle \mu+\gamma\rangle_T} < 1$, then the disease $I(t)$ will die out exponentially with probability one, i.e.,*

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq \langle \mu + \gamma\rangle_T(\hat{R}_0^* - 1) < 0 \quad a.s..$$

Proof. Making use of the Itô's formula to $\ln I(t)$, one has

$$\begin{aligned}
d\ln I &= \tfrac{1}{I(t)}\Big[\frac{\beta(t)I(t)\sum_{k=1}^{n}\alpha_k(t)S_k(t)}{N(t)} - (\mu(t)+\gamma(t))I(t)\Big]dt - \tfrac{1}{I^2(t)}\sum_{k=1}^{n}\frac{\sigma_k^2(t)}{2}\frac{S_k^2(t)I^2(t)}{N^2(t)}dt \\
&\quad + \sum_{k=1}^{n}\sigma_k(t)\frac{S_k(t)}{N(t)}dB_k(t) \\
&= \Big[\frac{\beta(t)\sum_{k=1}^{n}\alpha_k(t)S_k(t)}{N(t)} - \sum_{k=1}^{n}\frac{\sigma_k^2(t)}{2}\frac{S_k^2(t)}{N^2(t)} - (\mu(t)+\gamma(t))\Big]dt + \sum_{k=1}^{n}\sigma_k(t)\frac{S_k(t)}{N(t)}dB_k(t) \\
&= \sum_{k=1}^{n}\left(\beta(t)\alpha_k(t)\frac{S_k(t)}{N(t)} - \frac{\sigma_k^2(t)}{2}\frac{S_k^2(t)}{N^2(t)}\right)dt - (\mu(t)+\gamma(t))dt + \sum_{k=1}^{n}\sigma_k(t)\frac{S_k(t)}{N(t)}dB_k(t) \\
&\leq \sum_{k=1}^{n}\left(\beta^u\alpha_k^u\frac{S_k(t)}{N(t)} - \frac{(\sigma_k^l)^2}{2}\frac{S_k^2(t)}{N^2(t)}\right)dt - (\mu(t)+\gamma(t))dt + \sum_{k=1}^{n}\sigma_k^u\frac{S_k(t)}{N(t)}dB_k(t) \\
&= -\sum_{k=1}^{n}\Big[\frac{(\sigma_k^l)^2}{2}\frac{S_k^2(t)}{N^2(t)} - \beta^u\alpha_k^u\frac{S_k(t)}{N(t)} + \Big(\frac{\sqrt{2}\beta^u\alpha_k^u}{2\sigma_k^l}\Big)^2\Big]dt + \sum_{k=1}^{n}\frac{\beta^u\alpha_k^u}{2(\sigma_k^l)^2}dt \\
&\quad -(\mu(t)+\gamma(t))dt + \sum_{k=1}^{n}\sigma_k^u\frac{S_k(t)}{N(t)}dB_k(t) \\
&= -\sum_{k=1}^{n}\Big(\frac{\sigma_k^l}{\sqrt{2}}\frac{S_k(t)}{N(t)} - \frac{\sqrt{2}\beta^u\alpha_k^u}{2\sigma_k^l}\Big)^2dt + \sum_{k=1}^{n}\frac{(\beta^u)^2(\alpha_k^u)^2}{2(\sigma_k^l)^2}dt - (\mu(t)+\gamma(t))dt \\
&\quad + \sum_{k=1}^{n}\sigma_k^u\frac{S_k(t)}{N(t)}dB_k(t).
\end{aligned} \tag{9}$$

Let $\frac{S_k}{N} = z_k$, $k = 1, 2, \dots, n$, and $0 < z_k \leq 1$, we can obtain

$$\begin{aligned}
f(z_k) &:= \left(\beta^u\alpha_k^u z_k - \tfrac{(\sigma_k^l)^2}{2}z_k^2\right) \\
&= -\Big(\tfrac{\sigma_k^l}{\sqrt{2}}z_k - \tfrac{\sqrt{2}\beta^u\alpha_k^u}{2\sigma_k^l}\Big)^2 + \tfrac{(\beta^u)^2(\alpha_k^u)^2}{2(\sigma_k^l)^2}.
\end{aligned}$$

Case 1: When $\frac{\sigma_k^l}{\sqrt{2}} \geq \frac{\sqrt{2}\beta^u\alpha_k^u}{2\sigma_k^l}$, that is $(\sigma_k^l)^2 \geq \beta^u\alpha_k^u$, then $f(z_k) \leq f(\frac{\beta^u\alpha_k^u}{(\sigma_k^l)^2})$, we obtain:

$$f(z_k) \leq \frac{(\beta^u)^2(\alpha_k^u)^2}{2(\sigma_k^l)^2}, \tag{10}$$

where $k = 1, 2, \ldots, n$.

Case 2: When $\frac{\sigma_k^l}{\sqrt{2}} < \frac{\sqrt{2}\beta^u\alpha_k^u}{2\sigma_k^l}$, that is $(\sigma_k^l)^2 < \beta^u\alpha_k^u$, then $f(z_k) \le f(1)$, we can obtain:

$$f(z_k) \le \beta^u\alpha_k^u - \frac{(\sigma_k^l)^2}{2}, \tag{11}$$

where $k = 1, 2, \ldots, n$.

Assume $J = \{1, 2, \ldots, n\}$, and $J = N_1 \bigoplus N_2$, where $N_1 = \{i|(\sigma_i^l)^2 \ge \beta^u\alpha_i^u\}$, and $N_2 = \{i|(\sigma_i^l)^2 < \beta^u\alpha_i^u\}$ then

$$d\ln I \le \sum_{i\in N_1} \frac{(\beta^u)^2(\alpha_i^u)^2}{2(\sigma_i^l)^2}dt + \sum_{j\in N_2}\Big(\beta^u\alpha_j^u - \frac{(\sigma_j^l)^2}{2}\Big)dt - (\mu(t)+\gamma(t))dt + \sum_{k=1}^{n}\sigma_k^u\frac{S_k}{N}dB_k(t) \tag{12}$$

Integrating (12) from 0 to t and dividing by t, we obtain

$$\frac{\ln I(t) - \ln I(0)}{t} \le \langle\mu+\gamma\rangle_T(\hat{R}_0^* - 1) + \sum_{k=1}^{n}\sigma_k^u\frac{1}{t}\int_0^t \frac{S_k(t)}{N(t)}dB_k(t). \tag{13}$$

An application of the strong law of large numbers (in [12]) we can obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{S_k}{N}dB_k(t) = 0, \quad 1 \le k \le n \quad a.s.. \tag{14}$$

Taking the superior limit on both side of (13) and combining with (14), one arrives at

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \le \langle\mu+\gamma\rangle_T(\hat{R}_0^* - 1) < 0 \quad a.s.,$$

which implies that $\lim_{t\to\infty} I(t) = 0$ *a.s*. Thus the disease $I(t)$ will tend to zero exponentially with probability one.

By system (5) and (1), it is easy to see that when $\lim_{t\to\infty} I(t) = 0$ *a.s.*, then $\lim_{t\to\infty} A(t) = 0$ *a.s*. This completes the proof.

4. Persistence

Definition 4.1. *System (5) is said to be persistence in the mean if*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \frac{I(r)}{N(r)}dr > 0 \quad a.s..$$

We define a parameter

$$R_0^s := \sum_{k=1}^{n}\frac{\langle(\mu^2\beta\alpha_k S_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu + \frac{\sigma_k^2}{2}\rangle_T\langle\mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T\langle\mu\sum_{k=1}^{n}S_k^0\rangle_T}. \tag{15}$$

Theorem 4.2. *Assume that $R_0^s > 1$, then for any initial value $(S_1(0), S_2(0), \ldots, S_n(0), I_0) \in \Gamma^*$ the solution $(S_1(t), S_2(t), \ldots, S_n(t), I(t))$ of system (5) has the following property:*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\frac{I(r)}{N(r)}dr \ge \frac{(\mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2})(R_0^s - 1)}{\beta^u\sum_{k=1}^{n}\frac{\langle(\mu^2\beta\alpha_k S_k^0)^{\frac{1}{3}}\rangle_T^{\frac{1}{3}}}{\langle\mu+\frac{\sigma_k^2}{2}\rangle_T^2\langle\mu\sum_{k=1}^{n}S_k^0\rangle_T}\alpha_k^u}, \tag{16}$$

where $k = 1, 2, \ldots, n$.

Proof.

$$\mathcal{L}\left(\sum_{k=1}^{n} S_k + I\right) = \mu(t)\sum_{k=1}^{n}(S_k^0(t) - S_k(t)) - (\mu(t)+\gamma(t))I(t) = \mu(t)\sum_{k=1}^{n} S_k^0(t)$$
$$-\mu(t)(\sum_{k=1}^{n} S_k(t) + I(t)) - \gamma(t)I(t)$$
$$= \mu(t)\sum_{k=1}^{n} S_k^0(t) - \mu(t)N(t) - \gamma(t)I(t)$$

$$\mathcal{L}(-\ln S_k) = -\frac{\mu(t)S_k^0(t)}{S_k(t)} + \mu(t) + \frac{\beta(t)\alpha_k(t)I(t)}{N(t)} + \frac{\sigma_k^2(t)}{2}\frac{I^2(t)}{N^2(t)},$$

where $k = 1, 2, \dots, n$,

$$\mathcal{L}(-\ln I) = -\frac{\beta(t)\sum_{k=1}^{n}\alpha_k(t)S_k(t)}{N(t)} + (\mu(t)+\gamma(t)) + \sum_{k=1}^{n}\frac{\sigma_k^2(t)}{2}\frac{S_k^2(t)}{N^2(t)}.$$

Hence we define

$$U(S_1, S_2, \dots, S_n, I) = -\ln I(t) - \sum_{k=1}^{n} c_k \ln S_k(t) + \sum_{k=1}^{n} a_k\left(\sum_{k=1}^{n} S_k(t) + I(t)\right),$$

with

$$c_k = \frac{\langle(\mu^2\beta\alpha_k S_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu + \frac{\sigma_k^2}{2}\rangle_T^2\langle\mu\sum_{k=1}^{n} S_k^0\rangle_T}, \quad a_k = \frac{\langle(\mu^2\beta\alpha_k S_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu + \frac{\sigma_k^2}{2}\rangle_T\langle\mu\sum_{k=1}^{n} S_k^0\rangle_T^2},$$

in which $k = 1, 2, \dots, n$.

Using Itô's formula and Basic inequality $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ one can write

$$\begin{aligned}
\mathcal{L}U &= -\sum_{k=1}^{n}\frac{\beta(t)\alpha_k(t)S_k(t)}{N(t)} + (\mu(t)+\gamma(t)) - \sum_{k=1}^{n}\frac{c_k\mu(t)S_k^0(t)}{S_k(t)} - \sum_{k=1}^{n} a_k\mu(t)N(t) \\
&\quad + \frac{\beta(t)I(t)\sum_{k=1}^{n} c_k\alpha_k(t)}{N(t)} + \sum_{k=1}^{n} c_k(\mu(t) + \frac{\sigma_k^2(t)}{2}\frac{I^2(t)}{N^2(t)}) + \mu(t)\sum_{k=1}^{n} a_k(\sum_{k=1}^{n} S_k^0(t)) - \sum_{k=1}^{n} a_k\gamma(t)I(t) \\
&\quad + \frac{\beta(t)\alpha_k(t)I(t)}{N(t)} + \sum_{k=1}^{n}\frac{\sigma_k^2(t)}{2}\frac{S_k^2(t)}{N^2(t)} \\
&\leq -\sum_{k=1}^{n}\frac{\beta(t)\alpha_k(t)S_k(t)}{N(t)} + (\mu(t)+\gamma(t) + \sum_{k=1}^{n}\frac{\sigma_k^2(t)}{2}) - \sum_{k=1}^{n}\frac{c_k\mu(t)S_k^0(t)}{S_k(t)} - \sum_{k=1}^{n} a_k\mu(t)N(t) \\
&\quad + \frac{\beta(t)I(t)\sum_{k=1}^{n} c_k\alpha_k(t)}{N(t)} + \sum_{k=1}^{n} c_k(\mu(t) + \frac{\sigma_k^2(t)}{2}) + \mu(t)\sum_{k=1}^{n} a_k(\sum_{k=1}^{n} S_k^0(t)) - \sum_{k=1}^{n} a_k\gamma(t)I(t) \\
&= \sum_{k=1}^{n}[-\frac{\beta(t)\alpha_k(t)S_k(t)}{N(t)} - \frac{c_k\mu S_k^0(t)}{S_k(t)} - a_k\mu(t)N(t)] + (\mu(t)+\gamma(t) + \sum_{k=1}^{n}\frac{\sigma_k^2(t)}{2}) \\
&\quad + \frac{\beta(t)I(t)\sum_{k=1}^{n} c_k\alpha_k(t)}{N(t)} + \sum_{k=1}^{n} c_k(\mu(t) + \frac{\sigma_k^2(t)}{2}) + \mu(t)\sum_{k=1}^{n} a_k(\sum_{k=1}^{n} S_k^0(t)) - \sum_{k=1}^{n} a_k\gamma(t)I(t)
\end{aligned}$$

$$
\begin{aligned}
&\leq -3\sum_{k=1}^{n}(c_k\beta(t)\mu^2(t)\alpha_k(t)S_k^0(t)a_k)^{\frac{1}{3}}+\sum_{k=1}^{n}c_k(\mu(t)+\frac{\sigma_k^2(t)}{2})+\mu(t)\sum_{k=1}^{n}a_k(\sum_{k=1}^{n}S_k^0(t))\\
&\quad+(\mu(t)+\gamma(t)+\sum_{k=1}^{n}\frac{\sigma_k^2(t)}{2})+\frac{\beta(t)\sum_{k=1}^{n}c_k\alpha_k(t)}{N(t)}I(t)\\
&:= R_0(t)+\frac{\beta(t)\sum_{k=1}^{n}c_k\alpha_k(t)}{N(t)}I(t).
\end{aligned}
$$

Define the T−periodic function $w(t)$ which satisfies

$$w'(t)=\langle R_0\rangle_T-R_0(t). \tag{17}$$

By $c_k, a_k, k=1,2,\dots,n,$ we obtain

$$c_k\langle\mu+\frac{\sigma_k^2}{2}\rangle_T = a_k\langle\mu\sum_{k=1}^{n}S_k^0\rangle_T = \frac{\langle(\mu^2\beta\alpha_kS_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu+\frac{\sigma_k^2}{2}\rangle_T\langle\mu\sum_{k=1}^{n}S_k^0\rangle_T},$$

in which $k=1,2,\dots,n$.

Then we get

$$
\begin{aligned}
\mathcal{L}(U+w(t)) &\leq \langle R_0\rangle_T+\frac{\beta^u\sum_{k=1}^{n}c_k\alpha_k^u}{N(t)}I(t)\\
&\leq \sum_{k=1}^{n}\frac{-3\langle(\mu^2\beta\alpha_kS_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu+\frac{\sigma_k^2}{2}\rangle_T\langle\mu\sum_{k=1}^{n}S_k^0\rangle_T}+\sum_{k=1}^{n}\frac{2\langle(\mu^2\beta\alpha_kS_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu+\frac{\sigma_k^2}{2}\rangle_T\langle\mu\sum_{k=1}^{n}S_k^0\rangle_T}\\
&\quad+\langle\mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T+\frac{\beta^u\sum_{k=1}^{n}c_k\alpha_k^u}{N(t)}I(t)\\
&= -\sum_{k=1}^{n}\frac{\langle(\mu^2\beta\alpha_kS_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu+\frac{\sigma_k^2}{2}\rangle_T\langle\mu\sum_{k=1}^{n}S_k^0\rangle_T}+\langle\mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T+\frac{\beta^u\sum_{k=1}^{n}c_k\alpha_k^u}{N(t)}I(t)\\
&= -\langle\mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T\Big[\sum_{k=1}^{n}\frac{\langle(\mu^2\beta\alpha_kS_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu+\frac{\sigma_k^2}{2}\rangle_T\langle\mu+\gamma+\frac{\sigma_{n+1}^2}{2}\rangle_T\langle\mu\sum_{k=1}^{n}S_k^0\rangle_T}-1\Big]\\
&\quad+\frac{\beta^u\sum_{k=1}^{n}c_k\alpha_k^u}{N(t)}I(t)\\
&\leq -\langle\mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T(R_0^s-1)+\frac{\beta^u\sum_{k=1}^{n}c_k\alpha_k^u}{N(t)}I(t),
\end{aligned}
$$

in which R_0^s is defined in (15).

Thus we can obtain

$$d(U+w(t)) \leq -\langle \mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T(R_0^s-1)+\frac{\beta^u\sum_{k=1}^{n}c_k\alpha_k^u}{N}I+\sum_{k=1}^{n}\sigma_k^u\frac{I}{N}dB_k(t)-\sum_{k=1}^{n}\sigma_k^l\frac{S_k}{N}dB_k(t) \tag{18}$$

As $w(t)$ is a $T-$periodic function so we obtain:

$$\langle R_0\rangle_T = \lim_{t\to+\infty}\frac{\int_0^t R_0(t)dt}{t}.$$

and integrating (17) from 0 to t and dividing by t, we can get

$$\frac{w(t)-w(0)}{t} = \langle R_0\rangle_T - \frac{\int_0^t R_0(t)dt}{t}$$

Integrating (18) from 0 to t and dividing by t, we can get

$$\begin{aligned}\tfrac{\ln U(t)-\ln U(0)}{t} \quad \leq \quad & -\langle \mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T(R_0^s-1)t+\beta^u\sum_{k=1}^{n}c_k\alpha_k^u\frac{1}{t}\int_0^t\frac{I(r)}{N(r)}dr \\ & +\sum_{k=1}^{n}\sigma_k^u\frac{1}{t}\int_0^t\frac{I(r)}{N(r)}dB_k(t)-\sum_{k=1}^{n}\sigma_k^l\frac{1}{t}\int_0^t\frac{S_k(r)}{N(r)}dB_k(t).\end{aligned} \tag{19}$$

Since $\sum_{k=1}^{n}S_k(t)+I(t)\leq C$, we can obtain

$$\begin{aligned} W(t) \quad &= \quad -\ln I(t)-\sum_{k=1}^{n}c_k\ln S_k(t)+\sum_{k=1}^{n}a_k\left(\sum_{k=1}^{n}S_k(t)+I(t)\right) \\ &\geq \quad -\ln I(t)-\sum_{k=1}^{n}c_k\ln S_k(t) \\ &\geq \quad -\ln C-\sum_{k=1}^{n}c_k\ln C := \bar{M}.\end{aligned} \tag{20}$$

An application of the strong law of large numbers (in [12]) we can obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{S_i(r)}{N(r)}dB_i(t)=0 \quad 1\leq i\leq n \quad a.s.. \tag{21}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{I(r)}{N(r)}dB_i(t)=0 \tag{22}$$

Taking the superior limit on both side of (19) and combining with (20), (21) and (22) one arrives at

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\frac{I(r)}{N(r)}dr \geq \frac{\langle \mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T(R_0^s-1)}{\beta^u\sum_{k=1}^{n}c_k\alpha_k^u}.$$

Therefore, by the condition $R_0^s > 1$, we have assertion (16). This complete the proof of Theorem (4.2).

5. Existence of nontrivial positive periodic solution of system (1.5)

Definition 5.1. *A stochastic process $x(t, \varpi)$ is said to be periodic with period T if its finite dimensional distributions are periodic with period T, i.e., for any positive integer m and any moments of time $t_1, t_2, \ldots, t_m$, the joint distributions of the random variables $x(t_{1+kT}, \varpi), \ldots, x(t_{m+kT}, \varpi)$ are independent of k ($k = \pm 1, \pm 2, \cdots$).*

Consider the following periodic stochastic equation

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t), \quad x \in \mathbb{R}^n, \tag{23}$$

where functions f and g are $T-$periodic in t.

Lemma 5.2. *([30]). Assume that system (23) admits a unique global solution. Suppose further that there exists a function $V(t, x) \in C^2$ in $\mathbb{R}$ which is $T-$periodic in t, and satisfies the following conditions*

$$\inf_{|x|>R} V(t, x) \to \infty \text{ as } R \to \infty, \tag{24}$$

and

$$\mathcal{L}V(t, x) \leq -1 \text{ outside some compact set}, \tag{25}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2}trace(g^T(t, x)V_{xx}(t, x)g(t, x)). \tag{26}$$

Then the system (23) has a $T-$periodic solution.

By Theorem (2.1), we can obtain that system (5) has a unique globally positive solution $(S_1(t), S_2(t), \ldots, S_n(t), I(t)) \in \mathbb{R}_+^{n+1}$ on $t \geq 0$ for any initial value $(S_1(0), S_2(0), \ldots, S_n(0), I(0)) \in \mathbb{R}_+^{n+1}$. Based on this result we will give conditions which guarantees the existence of periodic solutions.

Theorem 5.3. *Assume that $R_0^s > 1$ (defined by Section 4), then system (5) admits a nontrivial positive $T-$periodic solution.*

Proof. Since the coefficients of (5) are constants, it is not difficult to show that they satisfy (5.1), (5.2). For all initial value $(S_1(0), S_2(0), \ldots, S_n(0), I_0) \in \Gamma^*$, the solution of (5) is regular by Theorem (2.1). It is clear that coefficients of system (5) satisfy the local Lipschitz condition. According to Lemma (5.2), to prove this result, it only need to construct a C^2-periodic function $V(x, t)$ and a compact set such that (24) and (25) are satisfied. Defining a C^2-function

$$\widehat{V}(S_1, S_2, \ldots, S_n, I, t) = M(U + w(t)) - \sum_{k=1}^{n} \ln S_k - \ln(\sum_{k=1}^{n} S_k^{0u} - \sum_{k=1}^{n} S_k - I),$$

in which $U(t)$ is defined by section 4. And the following condition for $M > 0$ is satisfied

$$-M\lambda + \sum_{k=1}^{n} \frac{(\sigma_k^u)^2}{2} + (n+1)\mu^u = -2, \tag{27}$$

$$\lambda = \langle \mu + \gamma + \sum_{k=1}^{n} \frac{\sigma_k^2}{2} \rangle_T (R_0^s - 1) > 0.$$

It is easy to check that

$$\liminf_{\substack{\sum_{k=1}^{n} S_k(t) + I(t) \to \sum_{k=1}^{n} S_k^{0u} \\ t \to +\infty}} \widehat{V}(S_1, S_2, \ldots, S_n, I, t) = +\infty.$$

In addition, $\widehat{V}(S_1, S_2, \dots, S_n, I, t)$ is a continuous function on $\bar{U}_k$. Therefore $\widehat{V}(S_1, S_2, \dots, S_n, I, t)$ has a minimum value point $(\bar{S_1}, \bar{S_2}, \dots, \bar{S_n}, \bar{I}, t)$ in the interior of Γ^*. Then we define a nonnegative C^2-function V: $\Gamma^* \to \mathbb{R}$ as follows

$$V(S_1, S_2, \dots, S_n, I, t) = \widehat{V}(S_1, S_2, \dots, S_n, I, t) - \widehat{V}(\bar{S_1}, \bar{S_2}, \dots, \bar{S_n}, \bar{I}, t).$$

The differential operator $\mathcal{L}$ acting on the function V leads to

$$\begin{aligned}
\mathcal{L}V \leq\ & M\Big[-\langle \mu + \gamma + \sum_{k=1}^{n} \frac{\sigma_k^2}{2} \rangle_T (R_0^s - 1) + \frac{\beta^u \sum_{k=1}^{n} c_k \alpha_k^u}{N(t)} I(t)\Big] + \frac{\beta(t) \sum_{k=1}^{n} \alpha_k(t)}{N(t)} I(t) - \sum_{k=1}^{n} \frac{\mu(t) S_k^0(t)}{S_k(t)} \\
& + \sum_{k=1}^{n} \frac{\sigma_k^2(t)}{2} \frac{I^2(t)}{N^2(t)} + (n+1)\mu(t) - \frac{\gamma(t) I(t)}{\sum_{k=1}^{n} S_k^0(t) - N(t)} \\
\leq\ & M\Big[-\langle \mu + \gamma + \sum_{k=1}^{n} \frac{\sigma_k^2}{2} \rangle_T (R_0^s - 1) + \frac{\beta^u \sum_{k=1}^{n} c_k \alpha_k^u}{N(t)} I(t)\Big] + \frac{\beta^u \sum_{k=1}^{n} \alpha_k^u}{N(t)} I(t) - \sum_{k=1}^{n} \frac{\mu^l S_k^{0l}}{S_k(t)} + \sum_{k=1}^{n} \frac{(\sigma_k^u)^2}{2} \\
& + (n+1)\mu^u - \frac{\gamma^l I(t)}{\sum_{k=1}^{n} S_k^{0u} - N(t)} \\
:=\ & -M\lambda + \frac{\beta^u \left(M \sum_{k=1}^{n} c_k \alpha_k^u + \sum_{k=1}^{n} \alpha_k^u \right)}{N(t)} I(t) - \sum_{k=1}^{n} \frac{\mu^l S_k^{0l}}{S_k(t)} - \frac{\gamma^l I(t)}{\sum_{k=1}^{n} S_k^{0u} - N(t)} + (n+1)\mu^u + \sum_{k=1}^{n} \frac{(\sigma_k^u)^2}{2}.
\end{aligned}$$

Consider the bounded open subset

$$D = \{(S_1, S_2, \dots, S_n, I) \in \Gamma^*, 0 < \sum_{k=1}^{n} S_k + I < \sum_{k=1}^{n} S_k^{0u},\ 1 \leq i \leq n\},$$

and $\varepsilon_i > 0 (i = 1, 2, 3)$ are sufficiently small constants. In the set $\Gamma^* \setminus D$, we can get $\varepsilon_i (i = 1, 2, 3)$ sufficiently small such that the following conditions hold

$$-\frac{\mu^l S_k^{0l}}{\varepsilon_1} + \widehat{K} \leq -1 \quad k = 1, 2, \dots, n, \tag{28}$$

$$\varepsilon_2 = (n\varepsilon_1)^2. \tag{29}$$

$$\varepsilon_3 = \varepsilon_2^2. \tag{30}$$

$$\beta^u \left(M \sum_{k=1}^{n} c_k \alpha_k^u + \sum_{k=1}^{n} \alpha_k^u \right) n\varepsilon_1 \leq 1. \tag{31}$$

$$\widehat{K} = \beta^u \left(M \sum_{k=1}^{n} c_k \alpha_k^u + \sum_{k=1}^{n} \alpha_k^u \right) - 2. \tag{32}$$

$$\widehat{K} - \frac{\gamma^l}{\varepsilon_2} \leq -1. \tag{33}$$

For the purpose of convenience, we can divide $\Gamma^* \setminus D$ into the following $2n+2$ domains,

$$D_k = \{0 < S_k \leq \varepsilon_1\}, \quad k = 1, 2, \ldots, n.$$
$$D_{n+1} = \{0 < I \leq \varepsilon_2, \varepsilon_1 \leq S_k \, 1 \leq k \leq n\}.$$
$$D_{n+2} = \{\varepsilon_2 \leq I \leq \sum_{k=1}^{n} S_k^{0u} - \varepsilon_3, \sum_{k=1}^{n} S_k^{0u} - \varepsilon_3 \leq \sum_{k=1}^{n} S_k + I\}.$$

Clearly, $D^C = D_1 \cup D_2 \cup D_3 \cup \ldots \cup D_{n+2}$. Next we will prove that $\mathcal{L}V(S_1, S_2, \ldots, S_n, I) \leq -1$ on D^C, which is equivalent to show it on the above $n+2$ domains.

Case 1: If $(S_1, S_2, \ldots, S_n, I) \in D_k, (k = 1, 2, \ldots, n)$, then

$$\begin{aligned}
\mathcal{L}V &\leq -M\lambda + \frac{\beta^u\left(M\sum_{k=1}^{n} c_k\alpha_k^u + \sum_{k=1}^{n}\alpha_k^u\right)}{N(t)} I(t) - \frac{\mu^l S_k^{0l}}{S_k(t)} \\
&\quad +(n+1)\mu^u + \sum_{k=1}^{n} \frac{(\sigma_k^u)^2}{2} \\
&\leq \widehat{K} - \frac{\mu^l S_k^{0l}}{S_k(t)} \\
&\leq \widehat{K} - \frac{\mu^l S_k^{0l}}{\varepsilon_1}.
\end{aligned} \tag{34}$$

In view of (28), one has

$$\mathcal{L}V \leq -1 \quad \text{for any} \quad (S_1, S_2, \ldots, S_n, I) \in D_k, (k = 1, 2, \ldots, n).$$

Case 2: If $(S_1, S_2, \ldots, S_n, I) \in D_{n+1}$, then

$$\begin{aligned}
\mathcal{L}V &\leq -M\lambda + \frac{\beta^u\left(M\sum_{k=1}^{n} c_k\alpha_k^u + \sum_{k=1}^{n}\alpha_k^u\right)}{N(t)} I(t) + (n+1)\mu^u + \sum_{k=1}^{n} \frac{(\sigma_k^u)^2}{2} \\
&\leq -M\lambda + \frac{\beta^u\left(M\sum_{k=1}^{n} c_k\alpha_k^u + \sum_{k=1}^{n}\alpha_k^u\right)\varepsilon_2}{n\varepsilon_1} + (n+1)\mu^u + \sum_{k=1}^{n} \frac{(\sigma_k^u)^2}{2}.
\end{aligned}$$

According to (29) and (31) one can see that

$$\mathcal{L}V \leq -M\lambda + \beta^u \left(M\sum_{k=1}^{n} c_k\alpha_k^u + \sum_{k=1}^{n}\alpha_k^u\right) n\varepsilon_1 + (n+1)\mu^u + \sum_{k=1}^{n} \frac{(\sigma_k^u)^2}{2}. \tag{35}$$

Combining with (27), one has for sufficiently small ε_1,

$$\mathcal{L}V \leq -1 \quad \text{for any} \quad (S_1, S_2, \ldots, S_n, I) \in D_{n+1}.$$

Case 3: If $(S_1, S_2, \ldots, S_n, I) \in D_{n+2}$, then

$$\begin{aligned}
\mathcal{L}V &\leq -M\lambda + \frac{\beta^u\left(M\sum_{k=1}^{n} c_k\alpha_k^u + \sum_{k=1}^{n}\alpha_k^u\right)}{N(t)} I(t) + (n+1)\mu^u + \sum_{k=1}^{n} \frac{(\sigma_k^u)^2}{2} - \frac{\gamma^l I(t)}{\sum_{k=1}^{n} S_k^{0u} - N(t)} \\
&\leq \widehat{K} - \frac{\gamma^l I}{\sum_{k=1}^{n} S_k^{0u} - N(t)} \\
&\leq \widehat{K} - \frac{\gamma^l \varepsilon_2}{\varepsilon_3} \\
&\leq \widehat{K} - \frac{\gamma^l}{\varepsilon_2}.
\end{aligned} \tag{36}$$

In view of (33), one has

$$\mathcal{L}V \leq -1 \quad \text{for any} \quad (S_1, S_2, \ldots, S_n, I) \in D_{n+2}.$$

Obviously, from (34), (35), and (36) one can obtain that for a sufficiently small $\varepsilon_i (i = 1, 2, 3)$,

$$\mathcal{L}V \leq -1 \quad \text{for any} \quad (S_1, S_2, \ldots, S_n, I) \in D^C.$$

Therefore, there is a $T-$periodic solution of system (5) according to Lemma (5.2).

6. Simulation

In this section, we will test our theory conclusion by simulations. In the following simulations, we all use the Milstein's Higher Order Method in [31].

Example 6.1. *Assume that the parametric values in the model (5) are given by* $\alpha_1(t) = 1.2 + 1.1\sin(t), \alpha_2(t) = 1 + 0.9\sin(t), S_1^0(t) = 1.5 + 1.3\sin(t), S_2^0(t) = 1.4 + 1.2\sin(t), \mu(t) = 1.2 + 1.1\sin(t), \gamma(t) = 1.4 + 1.1\sin(t)$ *and* $\beta = 1.5 + \sin(t)$. *The condition of Theorem (3.1) is* $\hat{R}_0^s := \frac{1}{\langle \mu+\gamma \rangle_T} \sum_{k=1}^{n} \frac{\langle \beta^2 \alpha_k^2 \rangle_T}{2\langle \sigma_k^2 \rangle_T} < 1$. *If we choose* $\sigma_1 = 5 + 4.4\cos(t), \sigma_2 = 2.5 + 2.4\sin(t)$, *we can have*

$$\hat{R}_0^* = \frac{\displaystyle\sum_{i \in N_1} \frac{(\beta^u)^2 (\alpha_i^u)^2}{2(\sigma_i^l)^2} + \sum_{j \in N_2} \Big(\beta^u \alpha_j^u - \frac{(\sigma_j^l)^2}{2} \Big)}{\langle \mu + \gamma \rangle_T} < 1,$$

then by Theorem (3.1), we can obtain that $I(t)$ *will tends to zero exponentially with probability one.*

Using the Milstein's Higher Order Method (in [31]), we give the simulations shown in Fig.1 to support our results.

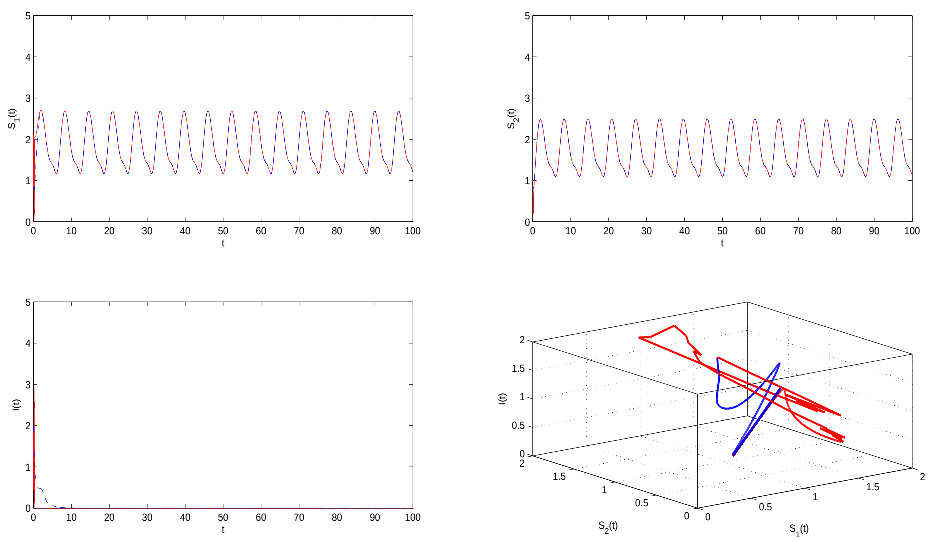

Figure 1: Computer simulation of the path S_1, S_2, I for the SDE DS-I-A epidemic model (5) for $\sigma_1 = 5 + 4.4\cos(t), \sigma_2 = 2.5 + 2.4\sin(t)$. We employ the Milstein's Higher Order Method with initial value $(S_1(0), S_2(0), I(0)) = (0.8, 0.8, 2)$.

Example 6.2. *Assume that the parametric values in the deterministic model (2) are given by* $\alpha_1(t) = 1.2 + 1.1\sin(t), \alpha_2(t) = 1 + 0.9\sin(t), S_1^0(t) = 1.5 + 1.3\sin(t), S_2^0(t) = 1.4 + 1.2\sin(t), \mu(t) = 1.2 + 1.1\sin(t), \gamma(t) = 1.4 + 1.1\sin(t)$ *and* $\beta = 3 + 1.2\sin(t)$. *Then computer simulation of the path* S_1, S_2, I *for the SDE DS-I-A epidemic model (5).*

The condition of Theorem (4.2) is $R_0^s > 1$. *If we choose* $\sigma_1(t) = 0.4 + 0.2\sin(t), \sigma_2(t) = 0.4 + 0.2\sin(t)$ *then by Theorem (4.2), the solution* $(S_1(t), S_2(t), I(t))$ *of system (5) with any initial value* $(S_1(0), S_2(0), I(0)) = (0.8, 0.8, 2) \in \Gamma^*$. *That is to say, the disease will proceed. For*

$$R_0^s := \sum_{k=1}^{n} \frac{\langle(\mu^2\beta\alpha_k S_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu + \frac{\sigma_k^2}{2}\rangle_T \langle\mu + \gamma + \sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T \langle\mu \sum_{k=1}^{n} S_k^0\rangle_T} > 1.$$

Using the Milstein's Higher Order Method (in [31]), we give the simulations shown in Fig.2 to support our results.

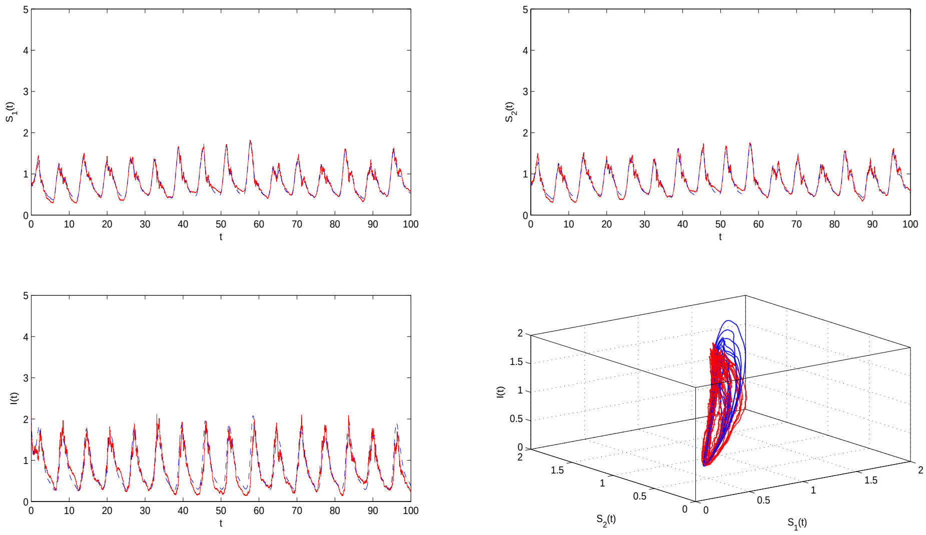

Figure 2: Computer simulation of the path S_1, S_2, I for the SDE DS-I-A epidemic model (5) for $\sigma_1 = 0.4 + 0.2\sin(t), \sigma_2 = 0.4 + 0.2\sin(t)$. We employ the Milstein's Higher Order Method with initial value $(S_1(0), S_2(0), I(0)) = (0.8, 0.8, 2)$.

7. Conclusion

In this paper, the sufficient condition of extinction is given in the almost sure situation, and this value is less than the value of the corresponding deterministic system. At some level, we can consider that the large white noise will control the disease to prevail, which never happen in the deterministic system. Besides, as the solutions of stochastic differential equations are stochastic processes, it is absolutely impossible for stochastic differential equations with periodic coefficients to have periodic solutions. In order to show the stochastic system has the similar property as the deterministic system, we show the transition probability function of the solution is periodic. Thus, we discuss the long time behaviour of system (5) and get following results.

(1) Assume $J = \{1, 2, \ldots, n\}$, and $J = N_1 \bigoplus N_2$, where $N_1 = \{i|(\sigma_i^l)^2 \geq \beta^u\alpha_i^u\}$, and $N_2 = \{i|(\sigma_i^l)^2 < \beta^u\alpha_i^u\}$. If $\hat{R}_0^* := \dfrac{\sum_{i\in N_1}\frac{(\beta^u)^2(\alpha_i^u)^2}{2(\sigma_i^l)^2} + \sum_{j\in N_2}\left(\beta^u\alpha_j^u - \frac{(\sigma_j^l)^2}{2}\right)}{\langle\mu+\gamma\rangle_T} < 1$, then the disease $I(t)$ will die out exponentially with

probability one, i.e.,

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\le\langle\mu+\gamma\rangle_T(\hat{R}_0^*-1)<0\quad a.s..$$

(2) If

$$R_0^s:=\sum_{k=1}^{n}\frac{\langle(\mu^2\beta\alpha_k S_k^0)^{\frac{1}{3}}\rangle_T^3}{\langle\mu+\frac{\sigma_k^2}{2}\rangle_T\langle\mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T\langle\mu\sum_{k=1}^{n}S_k^0\rangle_T}>1, \tag{37}$$

then

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\frac{I(r)}{N(r)}dr\ge\frac{\langle\mu+\gamma+\sum_{k=1}^{n}\frac{\sigma_k^2}{2}\rangle_T(R_0^s-1)}{\beta^u\sum_{k=1}^{n}c_k\alpha_k^u}.$$

and there exists a $T-$periodic solution of (5).

Some interesting topics deserve further consideration. On the one hand, one may propose some more realistic but complex models, such as considering the effects of impulsive perturbations on system (5). On the other hand, it is necessary to reveal that the methods used in this paper can be also applied to investigate other interesting epidemic models. We leave these as our future work.

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