Filomat 33:8 (2019), 2249–2255
https://doi.org/10.2298/FIL1908249C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cline's Formula for g-Drazin Inverses in a Ring

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Abstract. It is well known that for an associative ring R, if ab has g-Drazin inverse then ba has g-Drazin inverse. In this case, $(ba)^d = b((ab)^d)^2a$. This formula is so-called Cline's formula for g-Drazin inverse, which plays an elementary role in matrix and operator theory. In this paper, we generalize Cline's formula to the wider case. In particular, as applications, we obtain new common spectral properties of bounded linear operators.

1. Introduction

Let R be an associative ring with an identity. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$.

An element $a \in R$ has Drazin inverse in case there exists $b \in R$ such that

$$b = bab, b \in comm^2(a), a - a^2b \in R^{nil}.$$

The preceding b is unique if exists, we denote it by a^D. Let $a, b \in R$. Then ab has Drazin inverse if and only if ba has Drazin inverse. In this case, $(ba)^D = b((ab)^D)^2a$. This was known as Cline's formula for Drazin inverses. Cline's formula plays an elementary role in matrix and operator theory.

An element $a \in R$ has g-Drazin inverse (i.e., generalized Drazin inverse) in case there exists $b \in R$ such that

$$b = bab, b \in comm^2(a), a - a^2b \in R^{qnil}.$$

The preceding b is unique if exists, we denote it by a^d. Here, $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$, where $U(R)$ is the set of all units in R. We say $a \in R$ is quasi-nilpotent if $a \in R^{qnil}$. For a Banach algebra A it is well known that

$$a \in A^{qnil} \Leftrightarrow \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}} = 0.$$

Let $a, b \in R$. Then ab has g-Drazin inverse if and only if ba has g-Drazin inverse. In this case, $(ba)^d = b((ab)^d)^2a$. This was known as Cline's formula for g-Drazin inverses. Many papers discussed Cline's formula in the setting of matrices, operators, elements of Banach algebras or rings (see [1, 2, 4, 5, 7] and [9]).

2010 *Mathematics Subject Classification*. 15A09; Secondary 32A65, 16E50
Keywords. Drazin inverse; generalized Drazin inverse; spectral property; Banach algebra.
Received: 28 July 2018; Revised: 02 January 2019; Accepted: 02 April 2019
Communicated by Dragana Cvetković Ilić
Research supported by the Natural Science Foundation of Zhejiang Province, China (No. LY17A010018).
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The motivation of this paper is to extend Cline's formula and then apply to common spectral properties of bounded linear operators. In Section 2, we generalize the Cline's formula for generalized Drazin inverses. We prove that for a ring R, if $a(ba)^2 = abaca = acaba = (ac)^2a$, for some $a, b, c \in R$ then, $ac \in R^d$ if and only if $ba \in R^d$. When we choose $b = c$, the known Cline's formula follows as a special case.

In Section 3, we generalize the Jacobson's Lemma and prove that if $a(ba)^2 = abaca = acaba = (ac)^2a$ in a ring R, then

$$1 - ac \in U(R) \Longleftrightarrow 1 - ba \in U(R).$$

Combing this generalized Jacobson's Lemma and the main result in Section 2, we thereby determine the common spectral properties of bounded linear operators. Let $A, B, C \in \mathcal{L}(X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$. We prove that $\sigma_d(AC) = \sigma_d(BA)$, where σ_d is the g-Drazin spectrum.

Throughout the paper, all rings are associative with an identity. We use R^{nil} and R^{qnil} to denote the set of all nilpotents and quasinilpotents of the ring R, respectively. R^D and R^d denote the sets of all elements in R which have Drazin and g-Drazin inverses, respectively. $\mathbb{N}$ stands for the set of all natural numbers.

2. Cline's Formula

In [4, Lemma 2.2] proved that $ab \in R^{qnil}$ if and only if $ba \in R^{qnil}$ for any elements a, b in a ring R. We generalized this fact as follows.

Lemma 2.1. *Let R be a ring, and let $a, b, c \in R$. If $a(ba)^2 = abaca = acaba = (ac)^2a$, then the following are equivalent:*

(1) $ac \in R^{qnil}$.
(2) $ba \in R^{qnil}$.

Proof. $\Longrightarrow$ By hypothesis, $a(ba)^2 = (ac)^2a$ and $a(ba)^3 = (ac)^3a$. Suppose that $ac \in R^{qnil}$. Let $y \in comm(ba)$. Then $(1 - yba)(1 + yba - y^2baba) = 1 - y^3bababa$, and so

$$\begin{array}{rl} & (1 - yba)(1 + yba + y^2baba)(1 + y^3bababa) \\ = & 1 - y^6babababababa \\ = & 1 - y^6b(acaca)bababa \\ = & 1 - y^6b(acac)(ababa)ba \\ = & 1 - y^6b(acac)(acaca)ba. \end{array}$$

In view of Jacobson's Lemma (see [6, Theorem 2.1]), we will suffice to prove

$$1 - abay^6bacacac(ac) \in U(R).$$

As $ac \in R^{qnil}$, we will suffice to check

$$abay^6bacacac(ac) = (ac)abay^6bacacac.$$

One easily checks that

$$\begin{array}{rcl} abay^6bacacac(ac) & = & abay^6b(acacac)ac \\ & = & ay^6bababababac; \\ (ac)abay^6bacacac & = & (ac)ababay^6cacac \\ & = & (acacaca)y^6cacac \\ & = & (abababa)y^6cacac \\ & = & ay^6bababababac. \end{array}$$

Hence $1 + yba \in U(R)$. This shows that $ba \in R^{qnil}$.

$\Longleftarrow$ If $ba \in R^{qnil}$, by the preceding discussion, we see that $ab \in R^{qnil}$. With the same argument as above we get $ca \in R^{qnil}$, and therefore $ac \in R^{qnil}$. □

We come now to the main result of this paper

Theorem 2.2. *Let R be a ring, and let $a,b,c \in R$. If $a(ba)^2 = abaca = acaba = (ac)^2a$, then the following are equivalent:*

(1) $ac \in R^d$.
(2) $ba \in R^d$.

In this case, $(ac)^d = a((ba)^d)^2c$ and $(ba)^d = b((ac)^d)^2a$.

Proof. Suppose that ac has g-Drazin inverse and $(ac)^d = d$. Let $e = bd^2a$ and $f \in comm(ba)$. Note that $b(ac)^4 = (ba)^4c$ and $(ba)^4 = b(ac)^3a$. Then

$$fe = fb((ac)^2d^3)^2a = fb(ac)^4d^6a = (ba)^4fcd^6a = b((ac)^3afc)d^6a.$$

Also we have

$$\begin{aligned} ac((ac)^3afc) &= (ac)^4afc = af(ba)^4c = af(ba)^3cac \\ &= ((ab)^3afc)ac = ((ac)^3afc)ac. \end{aligned}$$

Since $d \in comm^2(ac)$, we get $((ac)^3afc)d = d((ac)^3afc)$. Thus, we conclude that

$$\begin{aligned} fe &= b((ac)^3afc)d^6a = bd^6((ac)^3afc)a \\ &= bd^6(ab)^3afc = bd^6af(ba)^3ca \\ &= bd^6af(ba)^4 = bd^6a(ba)^4f \\ &= bd^6a(ca)^4f = bd^2af = ef. \end{aligned}$$

This implies that $e \in comm^2(ba)$. We have

$$\begin{aligned} e(ba)e &= bd^2a(ba)bd^2a = bd^2ababacd^3a \\ &= bd^2(ac)^3d^3a = bd^2a = e. \end{aligned}$$

Let $p = 1 - acd$, then,

$$pac = ac - acdac = ac - (ac)^2d$$

that is contained in R^{qnil}. Moreover, we have

$$\begin{aligned} ba - (ba)^2e &= ba - bababd^2a = ba - bababacd^2da \\ &= ba - bacacacd^2da = b(1 - acd)a = bpa. \end{aligned}$$

One easily checks that

$$\begin{aligned} abpabpa &= ab(1 - acd)ab(1 - acd)a \\ &= ab(1 - dac)aba(1 - cda) \\ &= (ababa - abdacaba)(1 - cda) \\ &= (abaca - abdacaca)(1 - cda) \\ &= ab(1 - dac)aca(1 - cda) \\ &= ab(1 - dac)ac(1 - acd)a \\ &= abpacpa, \end{aligned}$$

and so

$$(pa)b(pa)b(pa) = (pa)b(pa)c(pa).$$

Likewise, we verify

$$(pa)b(pa)b(pa) = (pa)c(pa)b(pa) = (pa)c(pa)c(pa).$$

Then by Lemma 2.1., $bpa \in R^{qnil}$. Hence ba has g-Drazin inverse e. That is, $e = bd^2a = (ba)^d$. Moreover, we check

$$\begin{aligned} a((ba)^d)^2c &= abd^2abd^2(ac) \\ &= abd^3(acabac)d^2 \\ &= abd^3(acacac)d^2 \\ &= ab(acacac)d^5 \\ &= (ac)^4d^5 \\ &= (ac)^d, \end{aligned}$$

as required. □

Corollary 2.3. *Let R be a ring, let $k \in \mathbb{N}$, and let $a, b, c \in R$. If $a(ba)^2 = abaca = acaba = (ac)^2a$, If $(ac)^k$ has g-Drazin inverse if and only if $(ba)^k$ has g-Drazin inverse.*

Proof. Case 1. $k = 1$. This is obvious by Theorem 2.2.

Case 2. $k = 2$. We easily check that

$$\begin{aligned} a(bab)a(bab)a &= a(bab)a(cac)a \\ &= a(cac)a(bab)a \\ &= a(cac)a(cac)a. \end{aligned}$$

The result follows by Theorem 2.2.

Case 3. $k \geq 3$. Then $(ac)^k = (ab)^{k-1}ac$. Hence, $(ac)^k$ has g-Drazin inverse if and only if $(ab)^k = (ac)(ab)^{k-1}$ has g-Drazin inverse. This completes the proof. □

Corollary 2.4. *Let R be a ring, and let $a, b, c \in R$. If $aba = aca$, then $ac \in R^d$ if and only if $ba \in R^d$. In this case, $(ba)^d c = b(ac)^d$.*

Proof. In view of Theorem 2.2., $ac \in R^d$ if and only if $ba \in R^d$. Moreover, $(ac)^d = a((ba)^d)^2c$ and $(ba)^d = b((ac)^d)^2a$. Therefore $(ba)^d c = b((ac)^d)^2ac = b(ac)^d$, as required. □

Lemma 2.5. *Let R be a ring, and let $a \in R$. If $a \in R^D$, then $a \in R^d$ and $a^D = a^d$.*

Proof. This is obvious as the g-Drazin inverse of a is unique. □

Lemma 2.6. *Let R be a ring, and let $a, b, c \in R$. If $a(ba)^2 = abaca = acaba = (ac)^2a$, then $ac \in R^{nil}$ if and only if $ba \in R^{nil}$.*

Proof. $\Longrightarrow$ Let $ac \in R^{nil}$, then there exists some $n \in \mathbb{N}$ such that $(ac)^n = 0$. We may assume that n is even. Hence $(ac)^n a = (ac)^{n-2}(ac)^2a = (ac)^{n-2}a(ba)^2 = (ac)^{n-4}(ac)^2a(ba)^2 = (ac)^{n-4}a(ba)^4 = \cdots = (ac)^2a(ba)^{n-2} = a(ba)^n = 0$ and so $(ba)^{n+1} = 0$.

$\Longleftarrow$ It can be proved in the similar way. □

Theorem 2.7. *Let R be a ring, and let $a, b, c \in R$. If $a(ba)^2 = abaca = acaba = (ac)^2a$, then $ac \in R^D$ if and only if $ba \in R^D$. In this case, we have*

$$(ba)^D = b(((ac)^D)^2)a, (ac)^D = a(((ba)^D)^2)c.$$

Proof. Suppose that $ac \in R^D$. Then $ac \in R^d$ by Lemma 2.1. In view of Theorem 2.2, we see that $ba \in R^d$, and $(ba)^d = b((ac)^d)^2a$. Let $p = 1 - (ac)(ac)^d$. As in the proof of Theorem 2.2, we have

$$(pa)b(pa)b(pa) = (pa)b(pa)c(pa) = (pa)c(pa)b(pa) = (pa)c(pa)c(pa);$$
$$(pa)c = ac - (ac)^2(ac)^D \in R^{nil}.$$

In light of Lemma 2.6, $bpa \in R^{nil}$. Therefore

$$\begin{aligned} ba - (ba)^2(ba)^d &= ba - babab((ac)^d)^2a \\ &= ba - bababac((ac)^d)^3a \\ &= ba - bacacac((ac)^d)^3a \\ &= ba - b(ac)(ac)^d a \\ &= bpa \in R^{nil}. \end{aligned}$$

Therefore $ba \in R^D$ and $(ac)^D = a(((ba)^D)^2)c$. Moreover, $(ba)^D = b(((ac)^D)^2)a$. Conversely if $ba \in R^D$, then by [5, Theorem 2.1], $ab \in R^D$. With the same argument we get $ca \in R^D$ and so $ac \in R^D$. □

Recall that a has the group inverse if a has Drazin inverse with index 1, and denote the group inverse by $a^{\#}$. As an immediate consequence of Theorem 2.7., we now derive

Corollary 2.8. *Let R be a ring, and let $a, b, c \in R$. If $a(ba)^2 = abaca = acaba = (ac)^2 a$, then ac has group if and only if*

(1) $ba \in U(R)$; *or*
(2) ba *has group inverse and* $(ba)^{\#} = b((ac)^{\#})^2 a$; *or*
(3) $ba \in R^D$ *and* $(ba)^D = b((ac)^{\#})^2 a$.

We note that if $aba = aca$ in a ring R then $a(ba)^2 = abaca = acaba = (ac)^2 a$, but the converse is not true.

Example 2.9. *Let $R = M_6(\mathbb{Z}_2), x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{Z}_2)$. Then $x^2 \neq 0$ and $x^3 = 0$. Choose*

$$a = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $a(ba)^2 = abaca = acaba = (ac)^2 a$, but $aba \neq aca$. In this case, $ac \in R^D$.

3. Common spectral properties of bounded linear operators

Let X be Banach space, and let $\mathcal{L}(X)$ denote the set of all bounded linear operators from Banach space to itself, and let $a \in \mathcal{L}(X)$. The Drazin spectrum $\sigma_D(a)$ and g-Drazin spectrum $\sigma_d(a)$ are defined by

$$\sigma_D(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \mathcal{L}(X)^D\};$$
$$\sigma_d(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \mathcal{L}(X)^d\}.$$

The goal of this section is concern on common spectrum properties of $\mathcal{L}(X)$. For any $a, b \in R$, Jacobson's Lemma states that $1 + ab \in U(R)$ if and only if $1 + ba \in U(R)$. We now extend this known result as follows.

Lemma 3.1. *Let R be a ring, and let $a, b, c \in R$. If $a(ba)^2 = abaca = acaba = (ac)^2 a$, then*

$$1 - ac \in U(R) \Longleftrightarrow 1 - ba \in U(R).$$

Proof. $\Longrightarrow$ Write $s(1 - ac) = (1 - ac)s = 1$ for some $s \in R$. Then $sac = s - 1$. We see that

$$\begin{array}{rl} & \big((1 + bsa)(1 + ba) - bsa\big)(1 - ba) \\ = & (1 + bsa)(1 - baba) - bsa(1 - ba) \\ = & 1 - baba + bsa - bsababa - bsa(1 - ba) \\ = & 1 - baba + bsa - bsacaba - bsa(1 - ba) \\ = & 1 - baba + bsa - b(s - 1)aba - bsa(1 - ba) \\ = & 1. \end{array}$$

Thus, $1 - ba \in R$ is left invertible.

In light of Jacobson's Lemma, we have $1 - ca \in U(R)$. Set $t = (1 - ca)^{-1}$. Then $a = a(1 - ca)t = (a - aca)t$; hence,

$$\begin{array}{rcl} aba & = & ab(a - aca)t \\ & = & (aba - abaca)t \\ & = & (aba - ababa)t \\ & = & (1 - ab)abat. \end{array}$$

It follows that $(1 - ab)a = a - (1 - ab)abat$, and so $(1 - ab)(a + abat) = a$. This implies that $(1 - ab)(a + abat)b = ab$; whence,

$$(1 - ab)\big(1 + (a + abat)b\big) = (1 - ab) + (1 - ab)(a + abat)b = 1.$$

Thus, $1 - ab \in R$ is right invertible. In light of [6, Theorem 2.1], $1 - ba \in R$ is right invertible.

Therefore $1 - ba \in U(R)$. In this case, $(1 - ba)^{-1} = \big(1 + b(1 - ac)^{-1}a\big)(1 + ba) - b(1 - ac)^{-1}a$.

$\Longleftarrow$ This is symmetric. $\square$

Theorem 3.2. *Let $A, B, C \in \mathcal{L}(X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$, then*

$$\sigma_d(AC) = \sigma_d(BA).$$

Proof. Case 1. $0 \in \sigma_d(AC)$. Then $AC \notin \mathcal{L}(X)^d$. In view of Theorem 2.2., $BA \notin \mathcal{L}(X)^d$. Thus $0 \in \sigma_d(BA)$.

Case 2. $0 \notin \lambda \in \sigma_d(AC)$. Then $\lambda \in acc\sigma(AC)$. Thus, we see that

$$\lambda = \lim_{n\to\infty}\{\lambda_n \mid \lambda_n I - AC \notin \mathcal{L}(X)^{-1}\}.$$

For $\lambda_n \neq 0$, it follows by Lemma 3.1 that $I - (\frac{1}{\lambda_n}A)C \in \mathcal{L}(X)^{-1}$ if and only if $I - B(\frac{1}{\lambda_n}A) \in \mathcal{L}(X)^{-1}$. Therefore

$$\lambda = \lim_{n\to\infty}\{\lambda_n \mid \lambda_n I - BA \notin \mathcal{L}(X)^{-1}\} \in acc\sigma(BA) = \sigma_d(BA),$$

where $acc\sigma(BA)$ denotes the set of all accumulations points of $\sigma(BA)$ (see [6, Theorem 6.3]). Therefore $\sigma_d(AC) \subseteq \sigma_d(BA)$. Likewise, $\sigma_d(BA) \subseteq \sigma_d(AC)$, as required. □

Corollary 3.3. *Let $A, B, C \in \mathcal{L}(X)$ such that $ABA = ACA$, then*

$$\sigma_d(AC) = \sigma_d(BA).$$

Proof. This is obvious by Theorem 3.2. □

Example 3.4. *Let A, B, C be operators, acting on separable Hilbert space $l_2(\mathbb{N})$, defined as follows respectively:*

$$\begin{array}{rcl} A(x_1, x_2, x_3, x_4, \cdots) & = & (0, x_2, 0, x_4, \cdots), \\ B(x_1, x_2, x_3, x_4, \cdots) & = & (0, x_1, x_2, x_4, \cdots), \\ C(x_1, x_2, x_3, x_4, \cdots) & = & (0, 0, x_1, x_4, \cdots). \end{array}$$

Then $ABA = ACA$, and so $\sigma_d(AC) = \sigma_d(BA)$ by Corollary 3.3.

For the Drazin spectrum $\sigma_D(a)$, we now derive

Theorem 3.5. *Let $A, B, C \in \mathcal{L}(X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$, then*

$$\sigma_D(AC) = \sigma_D(BA).$$

Proof. In view of Theorem 2.7, $AC \in \mathcal{L}(X)^D$ if and only if $BA \in \mathcal{L}(X)^D$, and therefore we complete the proof by [7, Theorem 3.1]. □

A bounded linear operator $T \in \mathcal{L}(X)$ is Fredholm operator if $dimN(T)$ and $codimR(T)$ are finite, where $N(T)$ and $R(T)$ are the null space and the range of T respectively. If furthermore the Fredholm index $ind(T) = 0$, then T is said to be Weyl operator. For each nonnegative integer n define $T_{|n|}$ to be the restriction of T to $R(T^n)$. If for some n, $R(T^n)$ is closed and $T_{|n|}$ is a Fredholm operator then T is called a B-Fredholm operator. T is said to be a B-Weyl operator if $T_{|n|}$ is a Fredholm operator of index zero (see [1]). The B-Fredholm and B-Weyl spectrums of T are defined by

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a } B-\text{Fredholm operator}\};$$
$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a } B-\text{Weyl operator}\}.$$

Corollary 3.6. *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

Let $A, B, C \in \mathcal{L}(X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$, then

$$\sigma_{BF}(AC) = \sigma_{BF}(BA).$$

Proof. Let $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/F(X)$ be the canonical map and $F(X)$ be the ideal of finite rank operators in $\mathcal{L}(X)$. As in well known, $T \in \mathcal{L}(X)$ is B-Fredholm if and only if $\pi(T)$ had Drazin inverse. By hypothesis, we see that

$$\begin{array}{rcl}\pi(A)(\pi(B)\pi(A))^2 & = & \pi(A)\pi(B)\pi(A)\pi(C)\pi(A) \\ & = & \pi(A)\pi(C)\pi(A)\pi(B)\pi(A) \\ & = & (\pi(A)\pi(C))^2\pi(A).\end{array}$$

According to Theorem 3.5., for every scalar λ, we have

$$\lambda I - \pi(AC) \text{ has Drazin inverse } \Longrightarrow \lambda I - \pi(BA) \text{ has Drazin inverse.}$$

This completes the proof. □

Corollary 3.7. *Let $A, B, C \in \mathcal{L}(X)$ such that $A(BA)^2 = ABACA = ACABA = (AC)^2A$, then*

$$\sigma_{BW}(AC) = \sigma_{BW}(BA).$$

Proof. If T is B-Fredholm then for $\lambda \neq 0$ small enough, $T - \lambda I$ is Fredholm and $ind(T) = ind(T - \lambda I)$. As in the proof of [7, Lemma 2.3, Lemma 2.4], we see that $I - AC$ is Fredholm if and only if $I - BA$ is Fredholm and in this case, $ind(I - AC) = ind(I - BA)$. Therefore we complete the proof by by Corollary 3.6. □

Acknowledgement

The authors would like to thank the referee for his/her careful reading and valuable remarks. H. Chen was supported by the Natural Science Foundation of Zhejiang Province, China (No. LY17A010018).

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