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A Criterion for Univalent Meromorphic Functions

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Abstract. Let $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ and $\mathcal{A}(p)$ be the set of meromorphic functions in $\mathbb{D}$ possessing only simple pole at the point p with $p \in (0\,,\,1)$.
The aim of this paper is to give a criterion by mean of conditions on the parameters $\alpha, \beta \in \mathbb{C}$, $\lambda > 0$ and $g \in \mathcal{A}(p)$ for functions in the class denoted $\mathcal{P}_{\alpha,\beta\,;h}(p\,;\,\lambda)$ of functions $f \in \mathcal{A}(p)$ satisfying a differential Inequality of the form

$$\left|\alpha\Big(\frac{z}{f(z)}\Big)'' + \beta\Big(\frac{z}{g(z)}\Big)''\right| \leq \lambda\mu,\ z \in \mathbb{D}$$

to be univalent in the disc $\mathbb{D}$, where $\mu = (\frac{1-p}{1+p})^2$.

1. Introduction

Let $\mathcal{M}$ be the set of meromorphic functions in the region $\Delta = \{\zeta \in \mathbb{C}, |\zeta| > 1\} \cup \{\infty\}$ with the following Laurent development

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n},\ \zeta \in \Delta. \tag{1.1}$$

Let Σ be the subset of $\mathcal{M}$ consisting of univalent functions. $\mathcal{A}$ is the set of analytic functions f in the unit disc $\mathbb{D}$ normalized by the conditions $f(0) = f'(0) - 1 = 0$. The subset of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. If $f \in \mathcal{A}$, then the function F defined by

$$F(\zeta) = \frac{1}{f(\frac{1}{\zeta})} \tag{1.2}$$

belongs to $\mathcal{M}$ and f is univalent in $\mathbb{D}$ if and only if F is univalent in Δ. In [1], Aksentév proved that a function F in $\mathcal{M}$ is univalent if its derivative F' satisfies the differential Inequality:

$$\left|F'(\zeta) - 1\right| < 1,\ \zeta \in \Delta. \tag{1.3}$$

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If F and f are as in (1.2) then the condition (1.3) is equivalent to

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1,\ z \in \mathbb{D}. \tag{1.4}$$

Hence, by virtue of the Aksentѷe criterion, a criterion for a function $f \in \mathcal{A}$ with $\frac{f(z)}{z} \neq 0$ for $|z| < 1$ to be univalent is stated as follows:

$$\left|U_f(z)\right| < 1,\ z \in \mathbb{D}, \tag{1.5}$$

where $U_f(z) := \left(\frac{z}{f(z)}\right)^2 f'(z) - 1$.
Ozaki and Nunokawa proved in [11], without using the theorem of Aksentév, that functions in $\mathcal{A}$ satisfying (1.4) are univalent.
For $\lambda \in (0, 1]$, let $\mathcal{U}(\lambda)$ be the subclass of $\mathcal{U} = \mathcal{U}(1)$ defined by

$$\mathcal{U}(\lambda) = \{f \in \mathcal{A},\ \left|U_f(z)\right| < \lambda,\ z \in \mathbb{D}\}. \tag{1.6}$$

The classes $\mathcal{U}(\lambda)$ have been extensively studied by many authors and the results obtained cover a wide range of properties (starlikeness, convexity, coefficients properties, radius properties, etc.). For more details on this subjects see [4] - [8] and references therein.
In their article [7], Obradović and Ponnusamy considered the subclass $\mathcal{P}_{\alpha,\beta;g}(\lambda)$ of functions f in $\mathcal{A}$ such that $\frac{f(z)}{z} \neq 0$ for $z \in \mathbb{D}$ and satisfying the differential inequality

$$\left|\alpha\left(\frac{z}{f(z)}\right)'' + \beta\left(\frac{z}{g(z)}\right)''\right| \leq \lambda,\ z \in \mathbb{D} \tag{1.7}$$

where $\alpha \neq 0, \beta$ are given complex numbers and g is a given function in $\mathcal{A}$ with $\frac{g(z)}{z} \neq 0$ in $\mathbb{D}$. One of their main results was the following theorem:

Theorem 1.1. *Let $g \in \mathcal{A}$ with $\frac{g(z)}{z} \neq 0$ in $\mathbb{D}$ and $K = \sup_{z\in\mathbb{D}} |(\frac{z}{g(z)})^2 g'(z) - 1|$. Then we have*

$$\mathcal{P}_{\alpha,\beta;g}(2\,\lambda\,|\alpha| - 2\,K\,|\beta|) \subset \mathcal{U}(\lambda). \tag{1.8}$$

In particular, we have

$$\mathcal{P}_{\alpha,\beta;g}(2\,|\alpha| - 2\,K\,|\beta|) \subset \mathcal{U}(1). \tag{1.9}$$

Let $p \in (0, 1)$ and $\mathcal{A}(p)$ be the set of meromorphic functions in $\mathbb{D}$ normalized by $f(0) = f'(0) - 1 = 0$ and possessing only simple pole at the point p. Each function f in $\mathcal{A}(p)$ has a Laurent expansion of the form

$$f(z) = \frac{m}{z-p} + \frac{m}{p} + \left(\frac{m}{p^2} + 1\right)z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathbb{D} \setminus \{p\},\ m \neq 0, \tag{1.10}$$

where m is the residue of f at p ($m \neq 0$). Our investigations will concern functions in $\mathcal{A}(p)$ satisfying the condition

$$\left|1 + \frac{p^2}{m}\right| < 1. \tag{1.11}$$

In a recent paper [2], Bhowmik and Parveen introduced, for $0 < \lambda \leq 1$, a meromorphic analogue of the class $\mathcal{U}(\lambda)$, namely the class $\mathcal{U}_p(\lambda)$ consisting of functions f in $\mathcal{A}(p)$ satisfying

$$|U_f(z)| \leq \lambda\,\mu,\ z \in \mathbb{D}, \tag{1.12}$$

where

$$U_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1,\ z \in \mathbb{D} \text{ and } \mu = \left(\frac{1-p}{1+p}\right)^2 \tag{1.13}$$

They obtained some results for the class $\mathcal{U}_p(\lambda)$, in particular they proved the following theorem :

Theorem 1.2. (Theorem 1, [2]) Let f be of the form (1.10). If

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| \leq \left(\frac{1-p}{1+p}\right)^2,\ z \in \mathbb{D}$$

, then f is univalent in $\mathbb{D}$.

Note that Ponnusamy and Wirths have proved by elegant method (Theorem 2, [12]), that functions in $\mathcal{U}_p(\lambda)$ are univalent on the closure of the disc $\mathbb{D}$.

The main object of the present paper is to give, for the class $\mathcal{A}(p)$, an analog result to the Theorem 1.1 obtained for the class $\mathcal{A}$.

2. Main Results

We start by some "round trip" results between the classes $\mathcal{A}(p)$ and $\mathcal{A}$.

Proposition 2.1. Let $f(z) = \frac{m}{z-p} + \frac{m}{p} + \frac{m+p^2}{p^2}z + \sum_{n=2}^{\infty} a_n z^n$ be a function in $\mathcal{A}(p)$ such that $\frac{f(z)}{z} \neq 0$ in $\mathbb{D}$ and $-c$ be an omitted value by f. Let g be defined by

$$g(z) = \frac{c\, f(z)}{c + f(z)}. \tag{2.1}$$

Then $g \in \mathcal{A}$ and we have

$$g(p) = c,\ g'(p) = -\frac{c^2}{m} = -\frac{g^2(p)}{m}, \tag{2.2}$$

$$U_g(p) = -1 - \frac{p^2}{m}, \tag{2.3}$$

and

$$\lim_{z\to p} U_f(z) = U_g(p) = -1 - \frac{p^2}{m}. \tag{2.4}$$

Proof. Since f is holomorphic in $\mathbb{D} \setminus \{p\}$, g is also holomorphic in $\mathbb{D} \setminus \{p\}$. It is easy to check that $g(0) = g'(0) - 1 = 0$.
For the value of $g(p)$, we have

$$g(p) = \lim_{z\to p} g(z) = \lim_{z\to p} \frac{c\, f(z)}{c + f(z)} = \lim_{z\to p} \frac{c\,(z-p) f(z)}{c(z-p) + (z-p) f(z)} = \frac{c\, m}{m} = c.$$

To conclude that $g \in \mathcal{A}$, we have to prove that $g'(p)$ exists.
We have, by (2.1, that

$$\lim_{z\to p} \frac{g(z) - g(p)}{z-p} = \lim_{z\to p} \frac{g(z) - c}{z-p} = \lim_{z\to p} \frac{-c^2}{c(z-p) + (c-p) f(z)} = \frac{-c^2}{m}.$$

Thus $g'(p)$ exists and its value gives (2.2). Now, taking (2.2) in the expression of U_g, we get

$$U_g(p) = -\left(\frac{p}{c}\right)^2 \frac{c^2}{m} - 1 = -1 - \frac{p^2}{m}.$$

To prove (2.4), we have by a little calculation

$$U_f(z) = U_g(z),\ z \in \mathbb{D} \setminus \{p\}. \tag{2.5}$$

Thus we have

$$\lim_{z\to p} U_f(z) = U_g(p)$$

which yields, by (2.3), the desired result. □

Remark 2.2. We obtain from (2.4) that a necessary condition for f in $\mathcal{A}(p)$ to be in $\mathcal{U}_p(\lambda)$ is that $|1 + \frac{p^2}{m}| \leq \lambda\mu$, where m is the residue of f at p.

Proposition 2.3. Let $p \in (0, 1)$ and $g \in \mathcal{A}$ such that $g'(p) \neq 0$ and $g(z) - g(p)$ has no zero in $\mathbb{D} \setminus \{p\}$. We suppose also that g satisfies the following condition

$$|g^2(p) - g'(p)p^2| < |g^2(p)|. \tag{2.6}$$

Then, the function f defined by

$$f(z) = \frac{-g(p)g(z)}{g(z) - g(p)}$$

belongs to $\mathcal{A}(p)$ and satisfies (1.11). If in addition g is univalent, then f is also univalent.

Proof. It is obvious that f is holomorphic in $\mathbb{D} \setminus \{p\}$ and that $f(p) = \infty$. We get by a simple calculation

$$\lim_{z\to p}(z - p)f(z) = -\frac{g^2(p)}{g'(p)}.$$

From (2.6) we have $g(p) \neq 0$. Hence the limit above shows that f has a simple pole with residue $m = -\frac{g^2(p)}{g'(p)}$ at the point p. By the condition (2.6) we have

$$\left|1 - \frac{p^2 g^2(p)}{g'(p)}\right| < 1$$

and hence f satisfies the condition (1.11).
It is easy to verify that f is univalent if g is univalent.

□

Remark 2.4. The condition (2.6) is satisfied when $g \in \mathcal{U}(1)$;

Let $\mathcal{P}_{\alpha,\beta;h}(p ; \lambda)$ be the set of functions f in $\mathcal{A}(p)$ of the form (1.10) such that $\frac{f(z)}{z} \neq 0$ in $\mathbb{D}$ and satisfying the condition

$$\left|\alpha\left(\frac{z}{f(z)}\right)'' + \beta\left(\frac{z}{h(z)}\right)''\right| \leq \lambda\mu,\ z \in \mathbb{D} \tag{2.7}$$

and

$$|1 + \frac{p^2}{m}| \leq \lambda\,\mu, \tag{2.8}$$

where $\alpha \neq 0, \beta$ are given complex numbers and h is a given function in $\mathcal{A}(p)$ with $\frac{h(z)}{z} \neq 0$ in $\mathbb{D}$.

We observe that $\mathcal{P}_{1,0\,;h}(p\,;\,\lambda)$ doesn't depend on the function h and thus will be simply noted $\mathcal{P}(p\,;\,\lambda)$. The particular case where $\lambda = 2$ has been considered by Bhowmik and Parveen in [3].

We need the following Lemma:

Lemma 2.5. Let $0 < \lambda < \mu^{-1}$. If f belongs to $\mathcal{U}_p(\lambda)$ then, f is univalent in $\mathbb{D}$.

Proof. Let $-c$ be an omitted value for f and let $g = \frac{cf}{c+f}$. As seen above we have

$$U_g(z) = U_f(z)$$

and hence $g \in \mathcal{U}(\lambda\mu)$. Since $\lambda\mu < 1$, g belongs to $\mathcal{U}(1)$ and thus it is univalent. This implies that f is univalent. □

Theorem 2.6. Let $h \in \mathcal{A}(p)$ be such that $\frac{h(z)}{z} \neq 0$ for $z \in \mathbb{D}$ and

$$K = \sup_{z\in\mathbb{D}} \left| \left(\frac{z}{h(z)}\right)^2 h'(z) - 1 \right| < +\infty.$$

If $f \in \mathcal{P}_{\alpha,\beta;h}(p;\ 2\lambda\ |\alpha| - 2K\frac{|\beta|}{\mu})$, then $f \in \mathcal{U}_p(\lambda)$. If in addition $\lambda < \mu^{-1}$, the function f is univalent in the disc $\mathbb{D}$. In particular, we have

$$\mathcal{P}_{\alpha,\beta;h}(p\,;\,2\,\mu\,|\alpha| - 2\,K\,\frac{|\beta|}{\mu}) \subset \mathcal{U}_p(1).$$

Proof. Let $f \in \mathcal{P}_{\alpha,\beta;h}(p\,;\,2\,\lambda\,|\alpha| - 2\,K\,\frac{|\beta|}{\mu})$. Let g and k be defined by

$$g = \frac{c\,f}{c+f} \text{ and } k = \frac{d\,h}{d+h} \tag{2.9}$$

where $-c$ and $-d$ are omitted values respectively by f and h. By Proposition 2.1, g and k belong to $\mathcal{A}$. A little calculation shows that $\frac{g(z)}{z} \neq 0$ and $\frac{k(z)}{z} \neq 0$ in $\mathbb{D}$ and

$$\frac{z}{g(z)} = \frac{z}{f(z)} + \frac{z}{c} \text{ and } \frac{z}{k(z)} = \frac{z}{h(z)} + \frac{z}{d}, \tag{2.10}$$

which gives

$$\left(\frac{z}{g(z)}\right)'' = \left(\frac{z}{f(z)}\right)'',\ \left(\frac{z}{k(z)}\right)'' = \left(\frac{z}{h(z)}\right)''. \tag{2.11}$$

Since f belongs to $\mathcal{P}_{\alpha,\beta;h}(p\,;\,2\,\lambda\,|\alpha| - 2\,K\,\frac{|\beta|}{\mu})$, we have by (2.11)

$$g \in \mathcal{P}_{\alpha,\beta;k}(2\,\lambda\,\mu\,|\alpha| - 2\,K\,|\beta|). \tag{2.12}$$

Applying (2.5) to h and k, we obtain

$$\sup_{z\in\mathbb{D}} \left| \left(\frac{z}{k(z)}\right)^2 k'(z) - 1 \right| = K. \tag{2.13}$$

Moreover (2.12) and (2.13) give, by applying Theorem 1.1 to g and k,

$$g \in \mathcal{U}(\lambda\,\mu)$$

which gives from (2.5) and (2.8) that $f \in \mathcal{U}_p(\lambda)$.
If now $0 < \lambda < \mu^{-1}$, then f is univalent by Lemma 2.5 .
The second assertion of the theorem follows by taking $\lambda = 1$ in the first one. □

Let $p \in (0\,,\,1)$ and let $h(z) = \frac{z}{(z-p)(z-\frac{1}{p})}$. A little calculation yields

$$\sup\left|\left(\frac{z}{h(z)}\right)^2 h'(z) - 1\right| = 1 \text{ and } \left(\frac{z}{h(z)}\right)'' = 2,\ z \in \mathbb{D}$$

Corollary 2.7. Let $0 < p < 1$ and $f \in \mathcal{A}(p)$ with $\frac{f(z)}{z} \neq 0$ for $z \in \mathbb{D}$. Let $\alpha \neq 0$ and β be two complex numbers. If f satisfies

$$\left|\alpha\left(\frac{z}{f(z)}\right)'' + \beta\right| \leq 2\left(\lambda\mu\ |\alpha| - \frac{|\beta|}{2}\right),\ z \in \mathbb{D} \tag{2.14}$$

then $f \in \mathcal{U}_p(\lambda)$. If in addition $0 < \lambda < \mu^{-1}$, then f is univalent in $\mathbb{D}$.

Proof. Let $h(z) = \frac{z}{(z-p)(z-\frac{1}{p})}$. We have, as shown above, that

$$\sup\left|\left(\frac{z}{h(z)}\right)^2 h'(z) - 1\right| = 1 \text{ and } \left(\frac{z}{h(z)}\right)'' = 2.$$

Now, if f satisfies (2.14) then $f \in \mathcal{P}_{\alpha,\frac{\beta}{2};h}(p\,;\,2\,(\,\lambda\,|\alpha| - \frac{|\beta|}{2\mu}))$ and hence, by taking $K = 1$ in the first statement of Theorem 2.6, we get the desired conclusion. □

If we take $|\alpha| = 1$ and $\beta = 0$ in Corollary 2.7, we obtain the following

Corollary 2.8. Let $0 < p < 1$ and $f \in \mathcal{A}(p)$ with $\frac{f(z)}{z} \neq 0$ for $z \in \mathbb{D}$. If f satisfies

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq 2\,\lambda\,\mu,\ z \in \mathbb{D}, \tag{2.15}$$

then $f \in \mathcal{U}_p(\lambda)$, in other words, we have $\mathcal{P}(p; 2\,\lambda) \subset \mathcal{U}_p(\lambda)$. If in addition $0 < \lambda < \mu^{-1}$, then functions in $\mathcal{P}(p; 2\,\lambda)$ are univalent .

Corollary 2.9. If $0 < \lambda \leq 2$, then $\mathcal{P}(p; \lambda) \subset \mathcal{U}_p(1)$ and hence functions in $\mathcal{P}(p; \lambda)$ are univalent.

Proof. Since $\mu^{-1} > 1$, $0 < \frac{\lambda}{2} < \mu^{-1}$. Hence, the desired conclusion follows by applying Corollary 2.8 to $\frac{\lambda}{2}$. □

Remark 2.10. If we take $\lambda = 2$ in Corollary 2.9, we obtain Theorem 2 in [3].

We need the two followings lemmas :

Lemma 2.11. Let $g \in \mathcal{P}_{\alpha,\beta;k}(\lambda)$. Then there exists a Schwarz function w in $\mathbb{D}$ such that

$$\frac{z}{g(z)} - 1 = -\frac{\beta}{\alpha}\left(\frac{z}{k(z)} + \frac{k''(0)}{2}z - 1\right) - \frac{g''(0)}{2}z + \frac{\lambda z}{\alpha}\int_0^1 \frac{w(tz)}{t}(1-t)dt.$$

Proof. The proof can be extracted of the proof of Theorem 1.3 ([7], p.186). □

Lemma 2.12. Let $h \in \mathcal{A}(p)$, $-c$ be an omitted value for h and $k = \frac{c\,h}{c+h}$. Then,

$$1 - \frac{z}{k(z)} - \frac{k''(0)}{2}z = 1 - \frac{z}{h(z)} - \frac{h''(0)}{2}z,\ z \in \mathbb{D}. \tag{2.16}$$

Proof. We have

$$\frac{z}{k(z)} = \frac{z}{h(z)} + \frac{z}{c} \tag{2.17}$$

and

$$k''(0) = h''(0) - \frac{2}{c} \tag{2.18}$$

Taking (2.17) and (2.18) in the left side of (2.17), we get the desired conclusion. □

The following theorem is an analogue result of Corollary 1.8 in [7].

Theorem 2.13. Let $f \in \mathcal{P}_{\alpha,\beta;h}(p\,;\,\lambda)$ and $M = \sup_{z\in\mathbb{D}} |1 - \frac{z}{h(z)} - \frac{h''(0)}{2}z|$. Then

$$\left|\frac{z}{f(z)} - 1\right| \leq \left|\frac{\beta}{\alpha}\right| M + \frac{|f''(0)|}{2}|z| + \frac{\lambda\mu}{2|\alpha|}|z|^2. \tag{2.19}$$

Proof. Let $-c$ and $-d$ be omitted values by f and h, respectively.Furthermore let g and k be defined by

$$g = \frac{cf}{c+f}, \text{ and } k = \frac{dh}{d+h},$$

respectively. We have

$$\frac{z}{f(z)} - 1 = \frac{z}{g(z)} - 1 - \frac{z}{c} \tag{2.20}$$

and

$$\frac{g''(0)}{2} = \frac{f''(0)}{2} - \frac{1}{c}. \tag{2.21}$$

Since $f \in \mathcal{P}_{\alpha,\beta;h}(p\,,\,\lambda\mu)$, we have $g \in \mathcal{P}_{\alpha,\beta;k}(\lambda\mu)$. Applying Lemma 2.11, we obtain

$$\frac{z}{g(z)} - 1 = -\frac{\beta}{\alpha}\left(\frac{z}{k(z)} + \frac{k''(0)}{2}z - 1\right) - \frac{g''(0)}{2}z + \frac{\lambda\mu z}{\alpha}\int_0^1 \frac{w(tz)}{t}(1-t)dt, \tag{2.22}$$

where w is a Schwarz function in $\mathbb{D}$. Taking (2.22) in (2.20), we obtain

$$\frac{z}{f(z)} - 1 = -\frac{\beta}{\alpha}\left(\frac{z}{k(z)} + \frac{k''(0)}{2}z - 1\right) - \frac{g''(0)}{2}z - \frac{z}{c} + \frac{\lambda\mu z}{\alpha}\int_0^1 \frac{w(tz)}{t}(1-t)dt. \tag{2.23}$$

Now, taking (2.21) in (2.23), we get

$$\frac{z}{f(z)} - 1 = -\frac{\beta}{\alpha}\left(\frac{z}{k(z)} + \frac{k''(0)}{2}z - 1\right) - \frac{f''(0)}{2}z + \frac{\lambda\mu z}{\alpha}\int_0^1 \frac{w(tz)}{t}(1-t)dt. \tag{2.24}$$

The last equality gives us, using the fact that $|w(z)| \leq |z|$ in $\mathbb{D}$,

$$\left|\frac{z}{f(z)} - 1\right| \leq \left|\frac{\beta}{\alpha}\right| \sup_{z\in\mathbb{D}}\left|\frac{z}{k(z)} + \frac{k''(0)}{2}z - 1\right| + \frac{|f''(0)|}{2}|z| + \frac{\lambda\mu}{2|\alpha|}|z|^2. \tag{2.25}$$

We have, by Lemma 2.12,

$$\sup_{z\in\mathbb{D}}\left|\frac{z}{k(z)} + \frac{k''(0)}{2}z - 1\right| = \sup_{z\in\mathbb{D}}\left|\frac{z}{h(z)} + \frac{h''(0)}{2}z - 1)\right| = M \tag{2.26}$$

Taking (2.26) in (2.25), we get the desired result.

□

As a consequence of Theorem 2.13, we have the following corollary:

Corollary 2.14. If z is a given point in $\mathbb{D}$ then, we have

(1) $\left|\frac{z}{f(z)} - 1\right| \leq (\frac{1}{p} + \frac{\lambda\mu p^2}{2})|z| + \frac{\lambda\mu}{2}|z|^2,\ \forall f \in \mathcal{P}(p;\lambda);$

(2) $\left|\frac{z}{f(z)} - 1\right| \leq \frac{1}{p} + \frac{\lambda\mu p^2}{2} + \frac{\lambda\mu}{2},\ \forall f \in \mathcal{P}(p;\lambda).$

Proof. Let $f \in \mathcal{P}(p;\lambda)$. Taking $\alpha = 1$, $\beta = 0$ and $h(z) = \frac{pz}{pz^2+(1+p^2)z+p}$, the formula (2.24) gives

$$\frac{z}{f(z)} - 1 = -\frac{f''(0)}{2}z + \lambda\mu z\int_0^1 \frac{w(tz)}{t}(1-t)dt. \tag{2.27}$$

Putting $z = p$ in the last equality, we obtain

$$\frac{f''(0)}{2} = \frac{1}{p}(1 + \lambda\mu p\int_0^1 \frac{w(tp)}{t}(1-t)dt). \tag{2.28}$$

Since w is a Schwarz function, the modulus of the integral in (2.28) is majored by $\frac{p^2}{2}$ and hence we have

$$|\frac{f''(0)}{2}| \leq \frac{1}{p} + \frac{\lambda\mu p^2}{2}. \tag{2.29}$$

Now, taking (2.29) in (2.19), where α, β and h as above, we obtain the estimation

$$|\frac{z}{f(z)} - 1| \leq (\frac{1}{p} + \frac{\lambda\mu p^2}{2})|z| + \frac{\lambda\mu}{2}|z|^2. \tag{2.30}$$

This achieves the proof of (1). The estimation (2) is an immediate consequence of (1). □

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