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# On Coefficients of Some $p$-Valent Starlike Functions

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**Abstract.** We consider the class $\mathcal{A}_p$ of functions $f$ analytic in the unit disk $|z| < 1$ in the complex plane, of the form $f(z) = z^p + \ldots$ such that $\Re\mathfrak{e} f^{(p)}(z)/f^{(p-1)}(z) > 0$ in the unit disc. The object of the present paper is to derive some bounds for coefficients in this class and relation with the functions satisfying condition $\Re\mathfrak{e} f^{(k)}(z)/f^{(p-k)}(z) > 0$ in the unit disc.

## 1. Introduction

We denote by $\mathcal{H}$ the class of functions $f(z)$ which are holomorphic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f$ analytic in a domain $D \in \mathbb{C}$ is called $p$-valent in $D$, if for every complex number $w$, the equation $f(z) = w$ has at most $p$ roots in $D$, so that there exists a complex number $w_0$ such that the equation $f(z) = w_0$ has exactly $p$ roots in $D$. The properties of multivalent functions under several operators were established recently in several papers, see for instance [3, 6, 8, 16]. Meromorphic multivalent functions was considered recently in [4, 5, 9]. Denote by $\mathcal{A}_p$, $p \in \mathbb{N} = \{1, 2, \ldots\}$, the class of functions $f(z) \in \mathcal{H}$ given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in \mathbb{D}). \tag{1}$$

Let $\mathcal{A} = \mathcal{A}_1$. Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent. Also let $\mathcal{S}_p^*$ and $C_p$ be the subclasses of $\mathcal{A}_p$ defined as follows

$$\mathcal{S}_p^* = \left\{ f(z) \in \mathcal{A}_p : \Re\mathfrak{e} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0,\ z \in \mathbb{D} \right\},$$

$$C_p = \left\{ f(z) \in \mathcal{A}_p : \Re\mathfrak{e} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0,\ z \in \mathbb{D} \right\}.$$

The classes $\mathcal{S}_p^*$ and $C_p$ will be called the class of $p$-valently starlike functions and the class of $p$-valently convex functions, respectively. Note that $\mathcal{S}_1^* = \mathcal{S}^*$ and $C_1 = C$, where $\mathcal{S}^*$ and $C$ are usual classes of starlike and convex functions respectively.

In this paper we need the following lemmas.

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**Lemma 1.1.** *[13, Theorem 5] If $f(z) \in \mathcal{A}_p$, then for all $z \in \mathbb{D}$, we have*

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \quad \Rightarrow \quad \forall k \in \{1,\ldots,p-1\}: \quad \Re\mathrm{e}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0. \tag{2}$$

**Corollary 1.2.** *If $f(z) \in \mathcal{A}_p$, then for $r \in (0,1]$, we have*

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad |z| < r \quad \Rightarrow \quad \forall k \in \{1,\ldots,p-1\}: \quad \Re\mathrm{e}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0, \quad |z| < r.$$

**Lemma 1.3.** *[14] Let $p$ be analytic function in $|z| < 1$, with $p(0) = 1$. If there exists a point $z_0$, $|z_0| < 1$, such that*

$$\Re\mathrm{e}\{p(z)\} > 0 \ \ for \ \ |z| < |z_0|$$

*and*

$$p(z_0) = \pm ia$$

*for some $a > 0$, then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik\arg\{p(z_0)\}}{\pi}, \tag{3}$$

*for some $k \geq (a + a^{-1})/2 \geq 1$.*

**Lemma 1.4.** *[13] If $f(z) \in \mathcal{A}_p$, and there exists a positive integer $j$, $1 \leq j \leq p$ for which*

$$\Re\mathrm{e}\left\{j + \frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right\} > 0, \quad (z \in \mathbb{D}), \tag{4}$$

*then for all $z \in \mathbb{D}$ we have*

$$\forall k \in \{1,\ldots,j\}: \quad \Re\mathrm{e}\left\{k - 1 + \frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0. \tag{5}$$

**Corollary 1.5.** *If $f(z) \in \mathcal{A}_p$, and there exists a positive integer $j$, $1 \leq j \leq p$ for which*

$$\Re\mathrm{e}\left\{j + \frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right\} > 0, \quad (|z| < r), \tag{6}$$

*then for $|z| < r$, we have*

$$\forall k \in \{1,\ldots,j\}: \quad \Re\mathrm{e}\left\{k - 1 + \frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0, \quad (|z| < r). \tag{7}$$

## 2. Main results

Coefficient bounds for $p$-valent functions was considered recently in [15] while the coefficient neighborhoods of certain $p$-valently analytic functions with negative coefficients, in [1]. Some convolution (Hadamard product) conditions for starlikeness and convexity of meromorphically multivalent functions one can find in [11].

Let $(x)_n$ denote the Pochhammer symbol which is defined in term of Gamma function $\Gamma$ as:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{for} \quad n = 0, \qquad x \neq 0, \\ x(x+1)\ldots(x+n-1) & \text{for} \quad k \in \mathbb{N} = \{1,2,3,\ldots\}. \end{cases}$$

**Theorem 2.1.** *If $f(z) \in \mathcal{A}_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if*

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad z \in \mathbb{D}, \tag{8}$$

*then for $n \geq p$, we have*

$$|a_n| \leq \frac{p!(n-p+1)}{n(n-1)(n-2)\ldots(n-(p-2))} = \frac{p!(n-p+1)}{(n-p+2)_{p-1}}$$

*The result is sharp.*

*Proof.* If a function $f(z)$ satisfies (8), then $f^{(p-1)}(z)/p! = z + b_2z^2 + \cdots$ is a starlike function. Therefore, the coefficients of $f^{(p-1)}(z)/p!$ satisfy

$$|b_n| \leq n.$$

From this we can obtain the bound for $|a_n|$. We have that $b_{n-p+1} = n(n-1)(n-2)\ldots(n-(p-2))a_n/p!$, so $|a_n| \leq p!(n-p+1)/[n(n-1)(n-2)\ldots(n-(p-2))]$ for $n \geq p$. To show that the bound is sharp it suffices to prove that the function

$$f_p(z) = \sum_{n=p}^{\infty} \frac{p!(n-p+1)}{n(n-1)(n-2)\ldots(n-(p-2))} z^n, \quad z \in \mathbb{D}, \tag{9}$$

satisfies (8). We have

$$f_p^{(p-1)}(z)/p! = \frac{z}{(1-z)^2}$$

so (8) holds.
□

It is well known that if $f(z) \in \mathcal{A}_1$, then $|a_n| \leq n$. From this and from Theorem 2.1 we the following corollary for $p \geq 1$.

**Corollary 2.2.** *If $f(z) \in \mathcal{A}_p$, $p \geq 1$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if*

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad z \in \mathbb{D},$$

*then we have*

$$|a_{p+1}| \leq \frac{4}{p+1}, \quad |a_{p+2}| \leq \frac{18}{(p+1)(p+2)}, \ldots, \quad |a_{p+k}| \leq (k+1)\frac{(k+1)!}{(p+1)\ldots(p+k)}.$$

*The result is sharp.*

Corollary 2.2 implies that the function (9) may be written as

$$f_p(z) = z^p + \frac{4z^{p+1}}{p+1} + \frac{18z^{p+2}}{(p+1)(p+2)} + \sum_{k=3}^{\infty}(k+1)\frac{(k+1)!}{(p+1)\ldots(p+k)}z^{p+k}, \quad z \in \mathbb{D}. \tag{10}$$

Now we prove an inequality of type Fekete-Szegö type for functions satisfying (8). Fekete-Szegö inequalities for $p$-valent starlike and convex functions of complex order was considered recently in [2].

**Theorem 2.3.** *If $f(z) \in \mathcal{A}_p$, $p \geq 1$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if*

$$\Re e \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0, \quad z \in \mathbb{D},$$

*then for any complex number $\mu$, we have*

$$\left| a_{p+2} - \mu a_{p+1}^2 \right| \leq \frac{6}{(p+1)(p+2)} \max \{1, |2\lambda - 1|\}, \tag{11}$$

*where*

$$\lambda = \frac{4\mu(p+2)}{3(p+1)} - 1.$$

*The bound is sharp.*

*Proof.* We have

$$zf^{(p)}(z) = f^{(p-1)}(z) \left[ 1 + q_1 z + q_2 z^2 + \cdots \right],$$

where $\Re e\{1 + q_1 z + q_2 z^2 + \cdots\} > 0$ in $\mathbb{D}$. This leads us to the conclusion

$$a_{p+1} = \frac{2q_1}{p+1}, \quad a_{p+2} = \frac{3(q_1^2 + q_2)}{(p+1)(p+2)}.$$

Thus we have

$$\left| a_{p+2} - \mu a_{p+1}^2 \right| = \frac{3}{(p+1)(p+2)} \left| q_2 - \frac{4\mu(p+2) - 3(p+1)}{3(p+1)} q_1^2 \right|. \tag{12}$$

In [10] it was proved that for any complex number $\lambda$ the following sharp estimate holds

$$|q_2 - \lambda q_1^2| \leq 2 \max \{1, |2\lambda - 1|\}. \tag{13}$$

Therefore, applying (13) in (12) gives sharp bound (11). □

**Corollary 2.4.** *If $p = 1$, then* (11) *becomes the known sharp result [10]*

$$|a_3 - \mu a_2^2| \leq \max \{1, |4\mu - 3|\},$$

*for starlike functions, i.e. the solution of Fekete-Szegö problem in the class of starlike functions.*

If $\mu = 1$, then (11) becomes the following sharp result.

**Corollary 2.5.** *If $f(z) \in \mathcal{A}_p$, $p \geq 1$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if*

$$\Re e \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0, \quad z \in \mathbb{D},$$

*then we have*

$$\left| a_{p+2} - a_{p+1}^2 \right| \leq \frac{6}{(p+1)(p+2)} \max \left\{ 1, \frac{|7-p|}{3(p+1)} \right\}.$$

*The bound is sharp which show the coefficients of* (10).

**Theorem 2.6.** *If $f(z) \in \mathcal{A}_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if*

$$\Re e \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0, \quad z \in \mathbb{D}, \tag{14}$$

*then for $|z| = r < 1$, we have*

$$\frac{1}{(1+r)^{2p}} \leq \left| \frac{f(z)}{z^p} \right| \leq \frac{1}{(1-r)^{2p}}. \tag{15}$$

*The bounds are sharp.*

*Proof.* From (14) and from Lemma 1.1, we have

$$\Re e \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in \mathbb{D}, \quad \left. \frac{zf'(z)}{f(z)} \right|_{z=0} = p,$$

and so we have for $|z| = r < 1$

$$\frac{1-r}{1+r} \leq \Re e \left\{ \frac{zf'(z)}{pf(z)} \right\} \leq \frac{1+r}{1-r}.$$

Then it follows that

$$\begin{aligned}
\log \left| \frac{f(z)}{z^p} \right| &= \Re e \int_0^z \left( \frac{f'(t)}{f(t)} - \frac{p}{t} \right) dt \\
&= \Re e \int_0^z \frac{p}{t} \left( \frac{tf'(t)}{pf(t)} - 1 \right) dt \\
&= \Re e \int_0^r \frac{p}{\rho e^{i\theta}} \left( \frac{tf'(t)}{pf(t)} - 1 \right) e^{i\theta} d\rho \\
&= \int_0^r \Re e \left\{ \frac{p}{\rho} \left( \frac{tf'(t)}{pf(t)} - 1 \right) \right\} d\rho \\
&\leq \int_0^r \frac{p}{\rho} \left( \frac{1+\rho}{1-\rho} - 1 \right) d\rho \\
&= \int_0^r \frac{2p}{1-\rho} d\rho = \log \frac{1}{(1-r)^{2p}}.
\end{aligned}$$

This shows that for $|z| = r < 1$

$$\left| \frac{f(z)}{z^p} \right| \leq \frac{1}{(1-r)^{2p}}.$$

Applying the same method as the above, we can obtain for $|z| = r < 1$

$$\frac{1}{(1+r)^{2p}} \leq \left| \frac{f(z)}{z^p} \right|.$$

The sharpness of (18) shows the function

$$g(z) = \left[ \frac{z}{(1-z)^2} \right]^p = z^p + \cdots. \tag{16}$$

This completes the proof of Theorem 2.6.

□

**Theorem 2.7.** *If $f(z) \in \mathcal{A}_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if*

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad z \in \mathbb{D}, \tag{17}$$

*then for $|z| = r < 1$, we have*

$$\frac{pr^{p-1}(1-r)}{(1+r)^{2p+1}} \leq \left|f'(z)\right| \leq \frac{pr^{p-1}(1+r)}{(1-r)^{2p}}. \tag{18}$$

*The bounds are sharp.*

*Proof.* By the same reason as in the proof of Theorem 2.6, we have for $|z| = r < 1$

$$\frac{1-r}{1+r} \leq \Re\mathrm{e}\left\{\frac{zf'(z)}{pf(z)}\right\} \leq \frac{1+r}{1-r}.$$

Applying Theorem 2.6 we easily have the proof of Theorem 2.7. The sharpness of (20) shows the function (16). □

**Theorem 2.8.** *If $f(z) \in \mathcal{A}_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if*

$$\Re\mathrm{e}\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 0, \quad z \in \mathbb{D}, \tag{19}$$

*then, we have*

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad |z| < \sqrt{2}-1. \tag{20}$$

*The bound is sharp.*

*Proof.* Let us put

$$q(z) = \frac{f^{(p-1)}(z)}{zp!}, \quad q(0) = 1.$$

From the hypothesis (19), we have

$$\Re\mathrm{e}\{q(z)\} > 0, \quad z \in \mathbb{D}.$$

Applying [7, p.186], [12, Th.2], we have

$$\left|\frac{zq'(z)}{q(z)}\right| = \left|\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - 1\right| \leq \frac{2|z|}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Therefore, we have

$$\left|\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - 1\right| < 1 \quad \text{for} \quad |z| < \sqrt{2}-1$$

and so

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \quad \text{for} \quad |z| < \sqrt{2}-1.$$

It is easy to check that the function $f_1(z)$ such that

$$f_1^{(p-1)}(z) = \frac{z(1+z)}{1-z}$$

gives

$$\left.\frac{zf_1^{(p)}(z)}{f_1^{(p-1)}(z)}\right|_{z=1-\sqrt{2}} = \left.\frac{1+2z-z^2}{1-z^2}\right|_{z=1-\sqrt{2}} = 0$$

which shows the sharpness of (20).
□

**Corollary 2.9.** *If $f(z) \in \mathcal{A}_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if*

$$\Re\mathrm{e}\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 0, \quad z \in \mathbb{D}, \tag{21}$$

*then, $f(z)$ is p-valently starlike in $|z| < \sqrt{2}-1$. The bound is sharp.*

*Proof.* From Theorem 2.8, we have (19). Then from Corollary 1.2 we have

$$\Re\mathrm{e}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad |z| < \sqrt{2}-1.$$

□

**Theorem 2.10.** *Let $f(z) \in \mathcal{A}_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and let*

$$\Re\mathrm{e}\left\{\frac{f^{(k)}(z)}{z^{p-k}}\right\} > 0, \quad z \in \mathbb{D} \tag{22}$$

*for some integer $k \in [0, p]$. Then $f(z)$ is p-valently convex in $(\sqrt{1+p^2}-1)/p$ i.e.*

$$1 + \Re\mathrm{e}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0, \quad |z| < (\sqrt{1+p^2}-1)/p. \tag{23}$$

*The result is sharp.*

*Proof.* Let us put

$$q(z) = \frac{f^{(k)}(z)}{(p)_k z^{p-k}}, \quad q(0) = 1.$$

Then it follows that

$$\frac{zq'(z)}{q(z)} = \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} - (p-k) = k + \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} - p, \quad z \in \mathbb{D}.$$

And so from the hypothesis (22), applying [7, p.186], [12, Th.2], we have

$$\left|\frac{zq'(z)}{q(z)}\right| = \left|k + \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} - p\right| \leq \frac{2|z|}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Therefore, we have

$$\Re\mathfrak{e}\left\{k + \frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\right\} > 0, \quad |z| < (\sqrt{1+p^2} - 1)/p.$$

Applying Corollary 1.5, we have

$$\Re\mathfrak{e}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad |z| < (\sqrt{1+p^2} - 1)/p.$$

Further, taking the function $f(z)$ given by

$$f(z) = \left(\frac{1+z}{1-z}\right)z^p, \quad zin\mathbb{D},$$

we see that the result is sharp.

□

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