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Generalized Jordan Triple (σ, τ)-Higher Derivation on Triangular Algebras

Mohammad Ashraf[a], Aisha Jabeen[a], Mohd Shuaib Akhtar[a]

[a] *Department of Mathematics, Aligarh Muslim University, Aligarh,202002, India*

Abstract. Let $\mathcal{R}$ be a commutative ring with unity, $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of unital algebras $\mathcal{A}, \mathcal{B}$ and $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let σ and τ be two automorphisms of $\mathfrak{A}$. A family $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ of $\mathcal{R}$-linear mappings $\delta_n : \mathfrak{A} \to \mathfrak{A}$ is said to be a generalized Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$ if there exists a Jordan triple (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$ such that $\delta_0 = I_{\mathfrak{A}}$, the identity map of $\mathfrak{A}$ and $\delta_n(XYX) = \sum\limits_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau^i(Y))d_k(\tau^{n-k}(X))$ holds for all $X, Y \in \mathfrak{A}$ and each $n \in \mathbb{N}$. In this article, we study generalized Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$ and prove that every generalized Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$ is a generalized (σ, τ)-higher derivation on $\mathfrak{A}$.

1. Introduction

Let $\mathcal{R}$ be a commutative ring with unity and $\mathcal{A}$ be an unital algebra over $\mathcal{R}$ and $Z(\mathcal{A})$ be the center of $\mathcal{A}$. Recall that an $\mathcal{R}$-linear map $d : \mathcal{A} \to \mathcal{A}$ is called a derivation (resp. Jordan derivation) on $\mathcal{A}$ if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in \mathcal{A}$. An $\mathcal{R}$-linear map $d : \mathcal{A} \to \mathcal{A}$ is said to be a Jordan triple derivation on $\mathcal{A}$ if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in \mathcal{A}$. An $\mathcal{R}$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ is called a generalized derivation (resp. generalized Jordan derivation) on $\mathcal{A}$ if there exists a derivation (resp. Jordan derivation) d on $\mathcal{A}$ such that $\delta(xy) = \delta(x)y + xd(y)$ (resp. $\delta(x^2) = \delta(x)x + xd(x)$) holds for all $x, y \in \mathcal{A}$. Clearly, every generalized derivation on $\mathcal{A}$ is a generalized Jordan derivation on $\mathcal{A}$ but the converse need not be true in general. Zhu and Xiong [16] proved that every generalized Jordan derivation from a 2-torsion free semiprime ring with identity into itself is a generalized derivation.

The concept of derivation was extended in various directions to different rings and algebras. Let σ, τ be two endomorphisms on a ring $\mathcal{R}$. An additive map $\delta : \mathcal{R} \to \mathcal{R}$ is called a generalized (σ, τ)-derivation (resp. generalized Jordan (σ, τ)-derivation) on $\mathcal{R}$ if there exists a (σ, τ)-derivation (resp. Jordan (σ, τ)-derivation) d such that $\delta(xy) = \delta(x)\tau(y) + \sigma(x)d(y)$ (resp. $\delta(x^2) = \delta(x)\tau(x) + \sigma(x)d(x)$) holds for all $x, y \in \mathcal{R}$ and δ is said to be a generalized Jordan triple (σ, τ)-derivation on $\mathcal{R}$ if there exists a Jordan triple (σ, τ)-derivation d such that $\delta(xyx) = \delta(x)\tau(y)\tau(x) + \sigma(x)d(y)\tau(x) + \sigma(x)\sigma(y)d(x)$ holds for all $x, y \in \mathcal{R}$. It is easy to observe that a generalized $(I_{\mathcal{R}}, I_{\mathcal{R}})$-derivation, generalized Jordan $(I_{\mathcal{R}}, I_{\mathcal{R}})$-derivation and generalized Jordan triple $(I_{\mathcal{R}}, I_{\mathcal{R}})$-derivation is

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Email addresses: mashraf80@hotmail.com (Mohammad Ashraf), ajabeen329@gmail.com (Aisha Jabeen), mshuaibakhtar@gmail.com (Mohd Shuaib Akhtar)

simply a generalized derivation, generalized Jordan derivation and generalized Jordan triple derivation on $\mathcal{R}$ respectively.

Let $\mathbb{N}$ be the set of all nonegative integers. Following Hasse and Schmidt [12], a family $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ of additive mappings $d_n : \mathcal{R} \to \mathcal{R}$ such that $d_0 = I_{\mathcal{R}}$, the identity map on $\mathcal{R}$, is said to be a higher derivation (resp. Jordan higher derivation) on $\mathcal{R}$ if $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ (resp. $d_n(x^2) = \sum_{i+j=n} d_i(x)d_j(x)$) holds for all $x, y \in \mathcal{R}$ and for each $n \in \mathbb{N}$. Furthermore, motivated by the concept of generalized derivation Cortes and Haetinger [8] introduced the notion of generalized higher derivation. A family $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ of additive mappings $\delta_n : \mathcal{R} \to \mathcal{R}$ such that $\delta_0 = I_{\mathcal{R}}$, the identity map on $\mathcal{R}$, is said to be a generalized higher derivation (resp. generalized Jordan higher derivation) on $\mathcal{R}$ if there exists a higher derivation (resp. Jordan higher derivation) $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathcal{R}$ such that $\delta_n(xy) = \sum_{i+j=n} \delta_i(x)d_j(y)$ (resp. $\delta_n(x^2) = \sum_{i+j=n} \delta_i(x)d_j(x)$) holds for all $x, y \in \mathcal{R}$ and for each $n \in \mathbb{N}$.

Motivated by the notion of (σ, τ)-derivation the first author together with Khan and Haetinger [2] introduced the concept of (σ, τ)-derivation as follows : A family of $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ of additive mappings $d_n : \mathcal{R} \to \mathcal{R}$ is said to be a (σ, τ)-higher derivation (resp. Jordan (σ, τ)-higher derivation) on $\mathcal{R}$ if $d_0 = I_{\mathcal{R}}$, the identity map of $\mathcal{R}$ and $d_n(xy) = \sum_{i+j=n} d_i(\sigma^{n-i}(x))d_j(\tau^{n-j}(y))$ (resp. $d_n(x^2) = \sum_{i+j=n} d_i(\sigma^{n-i}(x))d_j(\tau^{n-j}(x))$) holds for all $x, y \in \mathcal{R}$ and for each $n \in \mathbb{N}$. Following [2], a family $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ of additive mappings $\delta_n : \mathcal{R} \to \mathcal{R}$ such that $\delta_0 = I_{\mathcal{R}}$, the identity map of $\mathcal{R}$, is said to be a generalized (σ, τ)-higher derivation (resp. generalized Jordan (σ, τ)-higher derivation) on $\mathcal{R}$ if there exists a (σ, τ)-higher derivation (resp. Jordan (σ, τ)-higher derivation) $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathcal{R}$ such that $\delta_n(xy) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(x))d_j(\tau^{n-j}(y))$ (resp. $\delta_n(x^2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(x))d_j(\tau^{n-j}(x))$) holds for all $x, y \in \mathcal{R}$ and for each $n \in \mathbb{N}$. Also, they obtained that under certain assumptions, if $\mathcal{R}$ is a prime ring of characteristic different from 2, then every generalized Jordan (σ, τ)-higher derivation on $\mathcal{R}$ is a generalized (σ, τ)-higher derivation on $\mathcal{R}$, where σ, τ are commuting endomorphisms of $\mathcal{R}$.

The $\mathcal{R}$-algebra $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$ under the usual matrix operations is called a triangular algebra, where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras over $\mathcal{R}$ and $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule. The notion of triangular algebra was first introduced by Chase [4] in 1960. Further, in the year 2000, Cheung [5] initiated the study of linear maps on triangular algebras. He described Lie derivations, commuting maps and automorphisms of triangular algebras [6, 7]. Recently, Han and Wei [11] studied generalized Jordan (σ, τ)-derivation on the triangular algebras $\mathfrak{A}$ and obtained that if $\mathfrak{A}$ is a triangular algebra consisting of unital algebra $\mathcal{A}, \mathcal{B}$ and $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ which is faithful as a left $\mathcal{A}$-module and also faithful as a right $\mathcal{B}$-module, then the following statements are equivalent (*i*) δ is a generalized Jordan (σ, τ)-derivation on $\mathfrak{A}$, (*ii*) δ is a generalized Jordan triple (σ, τ)-derivation on $\mathfrak{A}$, (*iii*) δ is a generalized (σ, τ)-derivation on $\mathfrak{A}$. Motivated by [2, 11], our main purpose is to study generalized (σ, τ)-higher derivations on triangular algebras. In fact, we obtain the condition on a triangular algebra $\mathfrak{A}$ under which every generalized Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$ is a generalized (σ, τ)-higher derivation on $\mathfrak{A}$.

In the last section of this article we shall give some applications of our results in a special case viz. nest algebra.

2. Preliminaries

Throughout, this paper we shall use the following notions: Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over $\mathcal{R}$ and let $\mathcal{M}$ be $(\mathcal{A}, \mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module, that is, for $A \in \mathcal{A}$, $A\mathcal{M} = 0$ implies $A = 0$ and also as a right $\mathcal{B}$-module, that is, for $B \in \mathcal{B}$, $\mathcal{M}B = 0$ implies $B = 0$. The triangular algebra $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is 2-torsion free. The center of $\mathfrak{A}$ is $Z(\mathfrak{A}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| am = mb \text{ for all } m \in \mathcal{M} \right\}$. Define two natural projections $\pi_{\mathcal{A}} : \mathfrak{A} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathfrak{A} \to \mathcal{B}$ by $\pi_{\mathcal{A}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = a$ and $\pi_{\mathcal{B}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = b$. Moreover, $\pi_{\mathcal{A}}(Z(\mathfrak{A})) \subseteq Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathfrak{A})) \subseteq Z(\mathcal{B})$ and there exists a unique algebraic isomorphism $\xi : \pi_{\mathcal{A}}(Z(\mathfrak{A})) \to \pi_{\mathcal{B}}(Z(\mathfrak{A}))$ such that $am = m\xi(a)$ for all $a \in \pi_{\mathcal{A}}(Z(\mathfrak{A})), m \in \mathcal{M}$.

Let $1_{\mathcal{A}}$ (resp.$1_{\mathcal{B}}$) be the identity of the algebra $\mathcal{A}$ (resp.$\mathcal{B}$) and let I be the identity of triangular algebra $\mathfrak{A}$, $e = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}$, $f = I - e = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$ and $\mathfrak{A}_{11} = e\mathfrak{A}e$, $\mathfrak{A}_{12} = e\mathfrak{A}f$, $\mathfrak{A}_{22} = f\mathfrak{A}f$. Thus $\mathfrak{A} = e\mathfrak{A}e + e\mathfrak{A}f + f\mathfrak{A}f = \mathfrak{A}_{11} + \mathfrak{A}_{12} + \mathfrak{A}_{22}$, where $\mathfrak{A}_{11}$ is subalgebra of $\mathfrak{A}$ isomorphic to $\mathcal{A}$, $\mathfrak{A}_{22}$ is subalgebra of $\mathfrak{A}$ isomorphic to $\mathcal{B}$ and $\mathfrak{A}_{12}$ is $(\mathfrak{A}_{11}, \mathfrak{A}_{22})$-bimodule isomorphic to $\mathcal{M}$. Also, $\pi_{\mathcal{A}}(Z(\mathfrak{A}))$ and $\pi_{\mathcal{B}}(Z(\mathfrak{A}))$ are isomorphic to $eZ(\mathfrak{A})e$ and $fZ(\mathfrak{A})f$ respectively. Then there is an algebra isomorphisms $\xi : eZ(\mathfrak{A})e \to fZ(\mathfrak{A})f$ such that $am = m\xi(a)$ for all $m \in e\mathfrak{A}f$.

Let $\mathbb{N}$ be the set of all nonnegative integers, σ, τ be automorphisms of triangular algebra $\mathfrak{A}$ and $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ be the family of $\mathcal{R}$-linear maps $d_n : \mathfrak{A} \to \mathfrak{A}$ such that $d_0 = I_{\mathfrak{A}}$. Then $\mathfrak{D}$ is said to be a (σ, τ)-higher derivation(resp. Jordan (σ, τ)-higher derivation) on $\mathfrak{A}$ if $d_n(XY) = \sum\limits_{i+j=n} d_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(Y)))$(resp. $d_n(X^2) = \sum\limits_{i+j=n} d_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X)))$ for all $X, Y \in \mathfrak{A}$ and for each $n \in \mathbb{N}$ and $\mathfrak{D}$ is said to be a Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$ if $d_n(XYX) = \sum\limits_{i+j+k=n} d_i(\sigma^{n-i}(X))d_j(\sigma^k(\tau^i(Y)))d_k(\tau^{n-k}(X))$ for all $X, Y \in \mathfrak{A}$ and for each $n \in \mathbb{N}$. Obviously, every (σ, τ)-higher derivation is a Jordan (σ, τ)-higher derivation on $\mathfrak{A}$ and every Jordan (σ, τ)-higher derivation is a Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$ but the converse statements are not true in general. First two authors together with Parveen [3] proved the following result:

Theorem 2.1. *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents , σ, τ be automorphisms of $\mathfrak{A}$ such that $\sigma\tau = \tau\sigma$ and let $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ be the family of $\mathcal{R}$-linear maps $d_n : \mathfrak{A} \to \mathfrak{A}$ such that $d_0 = I_{\mathfrak{A}}$. Then the following statements are equivalent:*

(*i*) *$\mathfrak{D}$ is a (σ, τ)-higher derivation on $\mathfrak{A}$,*
(*ii*) *$\mathfrak{D}$ is a Jordan (σ, τ)-higher derivation on $\mathfrak{A}$,*
(*iii*) *$\mathfrak{D}$ is a Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$.*

Motivated by the notion of generalized higher derivation on triangular algebra $\mathfrak{A}$, we introduce the notion of generalized (σ, τ)-higher derivation on $\mathfrak{A}$.

Let $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ be the family of $\mathcal{R}$-linear maps $\delta_n : \mathfrak{A} \to \mathfrak{A}$ such that $\delta_0 = I_{\mathfrak{A}}$, the identity map of $\mathfrak{A}$. Then $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is said to be a

(*i*) generalized (σ, τ)-higher derivation on $\mathfrak{A}$ if there exists a (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$ and

$$\delta_n(XY) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(Y))$$

for all $X, Y \in \mathfrak{A}$ and for each $n \in \mathbb{N}$,

(*ii*) generalized Jordan (σ, τ)-higher derivation on $\mathfrak{A}$ if there exists a Jordan (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$ and

$$\delta_n(X^2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X))$$

for all $X \in \mathfrak{A}$ and for each $n \in \mathbb{N}$,

(*iii*) generalized Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$ if there exists a Jordan triple (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$ and

$$\delta_n(XYX) = \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k(\tau^i(Y)))d_k(\tau^{n-k}(X))$$

for all $X, Y \in \mathfrak{A}$ and for each $n \in \mathbb{N}$.

It can be easily seen that every generalized (σ,τ)-higher derivation is a generalized Jordan (σ,τ)-higher derivation on $\mathfrak{A}$ and every generalized Jordan (σ,τ)-higher derivation is a generalized Jordan triple (σ,τ)-higher derivation on $\mathfrak{A}$. But the converse need not be true in general. In fact, if $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a generalized (σ,τ)-higher derivation associated with (σ,τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$, then $\delta_n(XY) = \sum\limits_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(Y))$ for all $X, Y \in \mathfrak{A}$. Replacing Y by X, we obtain $\delta_n(X^2) = \sum\limits_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X))$ for all $X \in \mathfrak{A}$ and for each $n \in \mathbb{N}$. That is $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a generalized Jordan (σ,τ)-higher derivation on $\mathfrak{A}$. Again, replacing Y by YX, we obtain
$\delta_n(XYX) = \sum\limits_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k(\tau^i(Y)))d_k(\tau^{n-k}(X))$ for all $X, Y \in \mathfrak{A}$ and for each $n \in \mathbb{N}$. That is $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a generalized Jordan triple (σ,τ)-higher derivation on $\mathfrak{A}$.

In the present paper, our objective is to prove every generalized Jordan (triple) (σ,τ)-higher derivation is a generalized (σ,τ)-higher derivation on $\mathfrak{A}$. In fact, our results generalize [3, Theorem 3.7, Theorem 3.8] and [11, Proposition 4.1, Theorem 4.3].

3. Main Results

The main result of the present paper states as follows:

Theorem 3.1. *Let $\mathfrak{A} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra consisting of $\mathcal{A},\mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents , σ,τ be automorphisms of $\mathfrak{A}$ such that $\sigma\tau = \tau\sigma$ and let $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ be the family of $\mathcal{R}$-linear maps $\delta_n : \mathfrak{A} \to \mathfrak{A}$ on $\mathfrak{A}$ such that $\delta_0 = I_{\mathfrak{A}}$. Then the following statements are equivalent:*

(*i*) *Δ is a generalized (σ,τ)-higher derivation on $\mathfrak{A}$,*
(*ii*) *Δ is a generalized Jordan (σ,τ)-higher derivation on $\mathfrak{A}$,*
(*iii*) *Δ is a generalized Jordan triple (σ,τ)-higher derivation on $\mathfrak{A}$.*

In order to prove our main results, we begin with the following sequence of lemmas:

Lemma 3.2. *[14, Theorem 1] Let $\mathfrak{A} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra consisting of $\mathcal{A},\mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents. Then an $\mathcal{R}$-linear map $\sigma : \mathfrak{A} \to \mathfrak{A}$ is an automorphism of $\mathfrak{A}$ if and only if it has the form*

$$\sigma\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} \theta(a) & \theta(a)m' - m'\eta(b) + \nu(m) \\ 0 & \eta(b) \end{pmatrix},$$

where $\theta : \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{B} \to \mathcal{B}$ are automorphisms, m' is a fixed element in $\mathcal{M}$ and $\nu : \mathcal{M} \to \mathcal{M}$ is an $\mathcal{R}$-linear bijective mapping such that $\nu(am) = \theta(a)\nu(m), \nu(mb) = \nu(m)\eta(b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$ and $m \in \mathcal{M}$.

Obviously, for any $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \mathfrak{A}$, we have

$$\sigma(e) = \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} = e_m \quad , \quad \sigma(f) = \begin{pmatrix} 0 & -m \\ 0 & 1 \end{pmatrix} = f_{-m}. \tag{1}$$

Lemma 3.3. *[3, Lemma 3.3] Let $\mathfrak{A} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra consisting of $\mathcal{A},\mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and σ,τ be automorphisms of $\mathfrak{A}$. Let $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ be a Jordan (σ,τ)-higher derivation on $\mathfrak{A}$. Then for all $m \in \mathcal{M}$ and for each fixed $n \in \mathbb{N}$*

(*i*) *$\sigma^n(e) = e_m$ and $\sigma^n(f) = f_{-m}$,*
(*ii*) *$d_n(I) = 0$, where I is the identity element of $\mathfrak{A}$,*
(*iii*) *$d_n(e), d_n(f) \in \mathcal{M}$.*

Lemma 3.4. *[1, Theorem 3.2] Suppose that $\mathfrak{A} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ is a triangular algebra having algebras $\mathcal{A},\mathcal{B}$ with only trivial idempotents and an $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$. Let σ and τ be two automorphisms of $\mathfrak{A}$ and a multiplicative map $\delta : \mathfrak{A} \to \mathfrak{A}$ be a generalized Jordan (σ,τ)- derivation (not necessarily linear) on $\mathfrak{A}$ associated with a multiplicative Jordan (σ,τ)-derivation d on $\mathfrak{A}$. Then δ is an additive generalized (σ,τ)-derivation on $\mathfrak{A}$.*

Lemma 3.5. *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents ; σ, τ be endomorphisms of $\mathfrak{A}$ such that $\sigma\tau = \tau\sigma$ and $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ be a generalized Jordan (σ, τ)-higher derivation associated with a Jordan (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$. Then for all $X, Y, Z \in \mathfrak{A}$ and for each fixed $n \in \mathbb{N}$*

(*i*) $\delta_n(XY + YX) = \sum_{i+j=n} \{\delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(Y)) + \delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(X))\}$,

(*ii*) $\delta_n(XYX) = \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau^i(Y))d_k(\tau^{n-k}(X))$,

(*iii*) $\delta_n(XYZ + ZYX) = \sum_{i+j+k=n} \{\delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau^i(Y))d_k(\tau^{n-k}(Z)) + \delta_i(\sigma^{n-i}(Z))d_j(\sigma^k\tau^i(Y))d_k(\tau^{n-k}(X))\}$.

Proof. (*i*) By our hypothesis for $X \in \mathfrak{A}, n \in \mathbb{N}$, we have

$$\delta_n(X^2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X)).$$

Now replace X by $X + Y$ in the above relation to get

$$\begin{aligned}\delta_n((X+Y)^2) &= \sum_{i+j=n} \delta_i(\sigma^{n-i}(X+Y))d_j(\tau^{n-j}(X+Y)) \\ &= \sum_{i+j=n} \delta_i\{\sigma^{n-i}(X) + \sigma^{n-i}(Y)\}d_j\{\tau^{n-j}(X) + \tau^{n-j}(Y)\} \\ &= \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X)) + \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(Y)) \\ &\quad +\delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(X)) + \delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(Y)). \end{aligned} \tag{2}$$

Also,

$$\begin{aligned}&\delta_n((X+Y)^2) \\ &= \delta_n(X^2) + \delta_n(XY+YX) + \delta_n(Y^2) \\ &= \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X)) + \delta_n(XY+YX) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(Y)) \end{aligned} \tag{3}$$

On comparing (2) and (3), we obtain the required result.
(*ii*) Now, replacing Y by $XY + YX$ in (*i*), we find that

$$\begin{aligned}&\delta_n(X(XY+YX) + (XY+YX)X) \\ &= \sum_{i+j=n} \{\delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(XY+YX)) + \delta_i(\sigma^{n-i}(XY+YX))d_j(\tau^{n-j}(X))\} \\ &= \sum_{i+r+s=n} \{\delta_i(\sigma^{n-i}(X))d_r(\sigma^s\tau^i(X))d_s(\tau^{n-s}(Y))\} \\ &\quad +2\sum_{i+j+k=n} \{\delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau^i(Y))d_k(\tau^{n-k}(X))\} \\ &\quad + \sum_{r+s+j=n} \{\delta_r(\sigma^{n-r}(Y))d_s(\sigma^j\tau^r(X))d_j(\tau^{n-j}(X))\}. \end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned}
&\delta_n(X(XY+YX)+(XY+YX)X)\\
&= \sum_{i+j=n}\{\delta_i(\sigma^{n-i}(X^2))d_j(\tau^{n-j}(Y))+\delta_i(\sigma^{n-i}(Y))d_j(\tau^{n-j}(X^2))\}+2\delta_n(XYX)\\
&= \sum_{r+s+j=n}\delta_r(\sigma^{n-r}(X))d_s(\sigma^j\tau^r(X))d_j(\tau^{n-j}(Y))\\
&\quad+\sum_{i+r+s=n}\{\delta_i(\sigma^{n-i}(Y))d_r(\sigma^s\tau^i(X))d_s(\tau^{n-s}(X))\}+2\delta_n(XYX). \qquad (5)
\end{aligned}$$

Combining (4) , (5) and using Theorem 2.1, we get the required result.
(*iii*) Linearizing X in (*ii*), we have

$$\begin{aligned}
\delta_n(XYZ+ZYX) &= \sum_{i+j+k=n}\{\delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau^i(Y))d_k(\tau^{n-k}(Z))\\
&\quad+\delta_i(\sigma^{n-i}(Z))d_j(\sigma^k\tau^i(Y))d_k(\tau^{n-k}(X))\}
\end{aligned}$$

for all $X, Y, Z \in \mathfrak{A}$ and for each fixed $n \in \mathbb{N}$. □

Following the above notations we prove that:

Lemma 3.6. *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and σ, τ be automorphisms of $\mathfrak{A}$. Let $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ be a generalized Jordan (σ, τ)-higher derivation associated with a Jordan (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$. Then $\delta_n(e) \in \mathcal{A} + \mathcal{M}$ and $\delta_n(f) \in \mathcal{B} + \mathcal{M}$ for each fixed $n \in \mathbb{N}$.*

Proof. By Lemma 3.4, we have $\delta(e) \in \mathcal{A} + \mathcal{M}$. Now suppose that $\delta_r(e) \in \mathcal{A} + \mathcal{M}$ for all $1 < r < n$. Using method of induction and by the definition of generalized Jordan (σ, τ)-higher derivation, we have

$$\begin{aligned}
\delta_n(e) &= \delta_n(e^2)\\
&= \sum_{i+j=n}\delta_i(\sigma^{n-i}(e))d_j(\tau^{n-j}(e))\\
&= \delta_n(e)\tau^n(e)+\delta_{n-1}(\sigma(e))d_1(\tau^{n-1}(e))+\delta_{n-2}(\sigma^2(e))d_2(\tau^{n-2}(e))\\
&\quad+\cdots+\delta_1(\sigma^{n-1}(e))d_{n-1}(\tau^1(e))+\sigma^n(e)d_n(e)\\
&= \delta_n(e)\tau^n(e)+\sigma^n(e)d_n(e).
\end{aligned}$$

Put $\delta_n(e) = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ where $a \in \mathcal{A}, m \in \mathcal{M}$ and $b \in \mathcal{B}$ and using Lemma 3.3 in the above expression, we obtain that $b = 0$. This implies that $\delta_n(e) \in \mathcal{A} + \mathcal{M}$. Similarly, we can prove that $\delta_n(f) \in \mathcal{B} + \mathcal{M}$. □

Lemma 3.7. *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and σ, τ be automorphisms of $\mathfrak{A}$ such that $\sigma\tau = \tau\sigma$. If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a generalized Jordan (σ, τ)-higher derivation associated with a Jordan (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$, then for each $n \in \mathbb{N}$*

(*i*) $\delta_n(\mathcal{M}) \subseteq \mathcal{M}$,
(*ii*) $\delta_n(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M}$,
(*iii*) $\delta_n(\mathcal{B}) \subseteq \mathcal{B} + \mathcal{M}$.

Proof. (*i*) In order to prove our lemma we follow induction method. From Lemma 3.4, we have

$$\delta_1(\mathcal{M}) \subseteq \mathcal{M}, \delta_1(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M}, \delta_1(\mathcal{B}) \subseteq \mathcal{B} + \mathcal{M}.$$

Assume that our lemma holds for component index i, where $1 < i < n$, i.e.,

$$\delta_i(\mathcal{M}) \subseteq \mathcal{M}, \delta_i(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M}, \delta_i(\mathcal{B}) \subseteq \mathcal{B} + \mathcal{M} \qquad (6)$$

Now

$$\begin{aligned}
\delta_n(m) &= \delta_n(me+em)\\
&= \sum_{i+j=n}\delta_i(\sigma^{n-i}(m))d_j(\tau^{n-j}(e)) + \sum_{i+j=n}\delta_i(\sigma^{n-i}(e))d_j(\tau^{n-j}(m))\\
&= \delta_n(m)\tau^n(e) + \delta_{n-1}(\sigma(m))d_1(\tau^{n-1}(e)) + \delta_{n-2}(\sigma^2(m))d_2(\tau^{n-2}(e))\\
&\quad + \cdots + \delta_1(\sigma^{n-1}(m))d_{n-1}(\tau^1(e)) + \sigma^n(m)d_n(e) + \delta_n(e)\tau^n(m)\\
&\quad +\delta_{n-1}(\sigma(e))d_1(\tau^{n-1}(m)) + \delta_{n-2}(\sigma^2(e))d_2(\tau^{n-2}(m))\\
&\quad + \cdots + \delta_1(\sigma^{n-1}(e))d_{n-1}(\tau^1(m)) + \sigma^n(e)d_n(m).
\end{aligned}$$

Put $\delta_n(m) = \begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix}$ where $a_1 \in \mathcal{A}, m_1 \in \mathcal{M}$ and $b_1 \in \mathcal{B}$. Now using Lemmas 3.3 and 3.6 in the above expression, we obtain that $b_1 = 0$. On the other way using $m = mf + fm$, we have $a_1 = 0$. This implies that $\delta_n(\mathcal{M}) \subseteq \mathcal{M}$.
(*ii*) For any $a \in \mathcal{A}$, we have

$$\begin{aligned}
0 &= \delta_n(af+fa)\\
&= \sum_{i+j=n}\delta_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(f)) + \sum_{i+j=n}\delta_i(\sigma^{n-i}(f))d_j(\tau^{n-j}(a))\\
&= \delta_n(a)\tau^n(f) + \delta_{n-1}(\sigma(a))d_1(\tau^{n-1}(f)) + \delta_{n-2}(\sigma^2(a))d_2(\tau^{n-2}(f))\\
&\quad + \cdots + \delta_1(\sigma^{n-1}(a))d_{n-1}(\tau^1(f)) + \sigma^n(a)d_n(f) + \delta_n(f)\tau^n(a)\\
&\quad +\delta_{n-1}(\sigma(f))d_1(\tau^{n-1}(a)) + \delta_{n-2}(\sigma^2(f))d_2(\tau^{n-2}(a))\\
&\quad + \cdots + \delta_1(\sigma^{n-1}(f))d_{n-1}(\tau^1(a)) + \sigma^n(f)d_n(a).
\end{aligned}$$

Put $\delta_n(a) = \begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix}$ where $a_2 \in \mathcal{A}, m_2 \in \mathcal{M}$ and $b_2 \in \mathcal{B}$ and using Lemmas 3.3 and 3.6 in the above expression, we obtain that $b_2 = 0$. This implies that $\delta_n(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{M}$.
(*iii*) Similar to (*ii*). □

Lemma 3.8. *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and σ, τ be automorphisms of $\mathfrak{A}$ such that $\sigma\tau = \tau\sigma$. If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a generalized Jordan (σ, τ)-higher derivation associated with a Jordan (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$, then for all $a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}$ and for each $n \in \mathbb{N}$*

(*i*) $\delta_n(am) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(m))$,

(*ii*) $\delta_n(mb) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(m))d_j(\tau^{n-j}(b))$.

Proof. (*i*) For any $a \in \mathcal{A}$ and $m \in \mathcal{M}$ using Lemma 3.7, we have

$$\begin{aligned}
\delta_n(am) &= \delta_n(am+ma)\\
&= \sum_{i+j=n}\delta_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(m)) + \sum_{i+j=n}\delta_i(\sigma^{n-i}(m))d_j(\tau^{n-j}(a))\\
&= \delta_n(a)\tau^n(m) + \delta_{n-1}(\sigma(a))d_1(\tau^{n-1}(m)) + \delta_{n-2}(\sigma^2(a))d_2(\tau^{n-2}(m))\\
&\quad + \cdots + \delta_1(\sigma^{n-1}(a))d_{n-1}(\tau^1(m)) + \sigma^n(a)d_n(m) + \delta_n(m)\tau^n(a)\\
&\quad +\delta_{n-1}(\sigma(m))d_1(\tau^{n-1}(a)) + \delta_{n-2}(\sigma^2(m))d_2(\tau^{n-2}(a))\\
&\quad + \cdots + \delta_1(\sigma^{n-1}(m))d_{n-1}(\tau^1(a)) + \sigma^n(m)d_n(a)\\
&= \sum_{i+j=n}\delta_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(m)).
\end{aligned}$$

(*ii*) Similar to (*i*). □

Lemma 3.9. *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and σ, τ be automorphisms of $\mathfrak{A}$ such that $\sigma\tau = \tau\sigma$. If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a generalized Jordan (σ, τ)-higher derivation associated with a Jordan (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$, then for all $a_1, a_2 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}$ and for each $n \in \mathbb{N}$*

(*i*) $\delta_n(a_1a_2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2))$,

(*ii*) $\delta_n(b_1b_2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(b_1))d_j(\tau^{n-j}(b_2))$.

Proof. From Lemma 3.8 for any $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$, we obtain that

$$\delta_n(a_1a_2m) = \sigma^n(a_1a_2)d_n(m) + \delta_n(a_1a_2)\tau^n(m) + \sum_{\substack{i+j=n\\ i,j<n}} \delta_i(\sigma^{n-i}(a_1a_2))d_j(\tau^{n-j}(m)).$$

On the other hand,

$$\begin{aligned}\delta_n(a_1a_2m) &= \sigma^n(a_1)d_n(a_2m) + \delta_n(a_1)\tau^n(a_2m) + \sum_{\substack{i+j=n\\ i,j<n}} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2m))\\ &= \sigma^n(a_1a_2)d_n(m) + \sigma^n(a_1)d_n(a_2)\tau^n(m) + \delta_n(a_1)\tau^n(a_2m)\\ &\quad +\sigma^n(a_1)\sum_{\substack{i+j=n\\ i,j<n}} \delta_i(\sigma^{n-i}(a_2))d_j(\tau^{n-j}(m)) + \sum_{\substack{i+j=n\\ i,j<n}} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2m)).\end{aligned}$$

From last two expressions, it follows that

$$\{\delta_n(a_1a_2) - \delta_n(a_1)\tau^n(a_2) - \sigma^n(a_1)d_n(a_2) - \sum_{\substack{i+j=n\\ i,j<n}} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2))\}\tau^n(m). \tag{7}$$

Since $\mathcal{M}$ is faithful left $\mathcal{A}$-module, using (7) we find that

$$\delta_n(a_1a_2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2))$$

for all $a_1, a_2 \in \mathcal{A}$.
(*ii*) Similar to (*i*). □

Theorem 3.10. *Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and σ, τ be automorphisms of $\mathfrak{A}$ such that $\sigma\tau = \tau\sigma$. If $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a generalized Jordan (σ, τ)-higher derivation associated with a Jordan (σ, τ)-higher derivation $\mathfrak{D} = \{d_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$, then Δ is a generalized (σ, τ)-higher derivations on $\mathfrak{A}$.*

Proof. For any $X, Y \in \mathfrak{A}$. Suppose that $X = a_1 + m_1 + b_1$ and $Y = a_2 + m_2 + b_2$, where $a_1, a_2 \in \mathcal{A}, m_1, m_2 \in \mathcal{M}$ and $b_1, b_2 \in \mathcal{B}$. Using Lemmas 3.8 and 3.9, we have

$$\begin{aligned}\delta_n(XY) &= \delta_n((a_1 + m_1 + b_1)(a_2 + m_2 + b_2))\\ &= \delta_n(a_1a_2 + a_1m_2 + m_1b_2 + b_1b_2)\\ &= \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2)) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(m_2))\\ &\quad + \sum_{i+j=n} \delta_i(\sigma^{n-i}(m_1))d_j(\tau^{n-j}(b_2)) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(b_1))d_j(\tau^{n-j}(b_2))\end{aligned} \tag{8}$$

On the other hand, using Lemmas 3.3, 3.5 and 3.6, we arrive at

$$\begin{aligned}
&\sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(Y)) \\
&= \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1 + m_1 + b_1))d_j(\tau^{n-j}(a_2 + m_2 + b_2)) \\
&= \sum_{i+j=n} \{\delta_i(\sigma^{n-i}(a_1)) + \delta_i(\sigma^{n-i}(m_1)) + \delta_i(\sigma^{n-i}(b_1))\} \\
&\quad \{d_j(\tau^{n-j}(a_2)) + d_j(\tau^{n-j}(m_2)) + d_j(\tau^{n-j}(b_2))\} \\
&= \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(a_2)) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(a_1))d_j(\tau^{n-j}(m_2)) \\
&\quad + \sum_{i+j=n} \delta_i(\sigma^{n-i}(m_1))d_j(\tau^{n-j}(b_2)) + \sum_{i+j=n} \delta_i(\sigma^{n-i}(b_1))d_j(\tau^{n-j}(b_2)). \qquad (9)
\end{aligned}$$

Since from Theorem 2.1 every Jordan (σ, τ)-higher derivation on $\mathfrak{A}$ is a (σ, τ)-higher derivation on $\mathfrak{A}$. So that (8) and (9) implies that Δ is a generalized (σ, τ)-higher derivation with associated (σ, τ)-higher derivation on $\mathfrak{A}$. □

Now we are in position to prove our main result:

Proof. [Proof of Theorem 3.1] (*i*) ⇔ (*ii*) It is obvious by Theorem 3.10.
(*ii*) ⇔ (*iii*) It can be easily seen that every generalized Jordan (σ, τ)-higher derivation on $\mathfrak{A}$ is a generalized Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$ by Lemma 3.5(*ii*). Conversely, by the definition of generalized Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$, we have

$$\delta_n(XYX) = \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau^i(Y))d_k(\tau^{n-k}(X))$$

for all $X, Y \in \mathfrak{A}$. Replace Y by I, the identity map of $\mathfrak{A}$, in the above expression, we arrive at

$$\begin{aligned}
\delta_n(XIX) &= \sum_{i+j+k=n} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau^i(I))d_k(\tau^{n-k}(X)) \\
&= \sum_{\substack{i+j+k=n \\ j\neq 0}} \delta_i(\sigma^{n-i}(X))d_j(\sigma^k\tau^i(I))d_k(\tau^{n-k}(X)) \\
&\quad + \sum_{i+k=n} \delta_i(\sigma^{n-i}(X))(\sigma^k\tau^i(I))d_k(\tau^{n-k}(X)).
\end{aligned}$$

This implies that

$$\delta_n(X^2) = \sum_{i+j=n} \delta_i(\sigma^{n-i}(X))d_j(\tau^{n-j}(X))$$

for all $X \in \mathfrak{A}$. Since from Theorem 2.1 every Jordan triple (σ, τ)-higher derivation on $\mathfrak{A}$ is a Jordan (σ, τ)-higher derivation on $\mathfrak{A}$. Therefore, Δ is a generalized Jordan (σ, τ)-higher derivation with associated Jordan (σ, τ)-higher derivation on $\mathfrak{A}$. □

In particular, for $n = 1$, we find that the following result due to Han and Wei [11].

Corollary 3.11. *[11, Theorem 4.3] Let $\mathfrak{A} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, where $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents, σ, τ be automorphisms of $\mathfrak{A}$ such that $\sigma\tau = \tau\sigma$ and let δ be an $\mathcal{R}$-linear map on $\mathfrak{A}$. Then the following statements are equivalent:*

(*i*) *δ is a generalized (σ, τ)-derivation on $\mathfrak{A}$,*
(*ii*) *δ is a generalized Jordan (σ, τ)-derivation on $\mathfrak{A}$,*
(*iii*) *δ is a generalized Jordan triple (σ, τ)-derivation on $\mathfrak{A}$.*

4. Applications

As an immediate consequence we will apply Theorem 3.1 to a classical example of triangular algebra viz. nest algebras. By Theorem 3.1, we have the following results:
If we choose the identity map in place of σ and τ in Theorem 3.1, we obtain the following corollary. Note that similar result still holds if the condition that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents is deleted.

Corollary 4.1. *[15, Theorem 4.7] Let* $\mathfrak{A} = (\mathcal{A}, \mathcal{M}, \mathcal{B})$ *be a triangular algebra and* $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ *be a family of* $\mathcal{R}$*-linear maps* $\delta_n : \mathfrak{A} \to \mathfrak{A}$ *such that* $\delta_0 = I_{\mathfrak{A}}$*. Then the following statements are equivalent:*

(i) Δ *is a generalized higher derivation on* $\mathfrak{A}$,
(ii) Δ *is a generalized Jordan higher derivation on* $\mathfrak{A}$,
(iii) Δ *is a generalized Jordan triple higher derivation on* $\mathfrak{A}$.

Corollary 4.2. *[15, Corollary 4.8] For any one of the following two cases:*

(a) *Assume,* $\mathcal{N}$ *is a nest on a Banach space X,* $Alg(\mathcal{N})$ *is the nest algebra associated with* $\mathcal{N}$ *and* $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ *is a sequence of linear mappings of* $Alg(\mathcal{N})$*. Suppose that there exists a non-trivial element in* $\mathcal{N}$ *which is complemented in X,*

(b) *Let* $\mathcal{N}$ *be a nest on a complex Hilbert space H,* $Alg(\mathcal{N})$ *be the nest algebra associated with* $\mathcal{N}$ *and* $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ *be a sequence of linear mappings of* $Alg(\mathcal{N})$.

Then the following statements are equivalent:

(i) Δ *is a generalized higher derivation on* $Alg(\mathcal{N})$,
(ii) Δ *is a generalized Jordan higher derivation on* $Alg(\mathcal{N})$,
(iii) Δ *is a generalized Jordan triple higher derivation on* $Alg(\mathcal{N})$.

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