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On the Spectrum of Substitution Vector-Valued Integral Operators

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Abstract. In this paper, we characterize the compact substitution vector-valued integral operators from $L^2(X)$ to $L^2(X)$ and determine their spectra.

1. Introduction and Preliminaries

Let (X, Σ, μ) be a σ-finite measure space and $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. Then $\mu \circ \varphi^{-n} \ll \mu$ where $n \geq 1$. It is assumed that the Radon-Nikodym derivative $h_n = d\mu \circ \varphi^{-n}/d\mu$ is almost everywhere finite-valued, or equivalently $\varphi^{-n}(\Sigma) \subseteq \Sigma$ is a sub-σ-finite algebra. If $n = 1$, we put $h_1 = h$. Here non-singularity of φ guarantees that the operator $f \to f \circ \varphi$ is well defined as a mapping on $L^0(\Sigma)$ where $L^0(\Sigma)$ denotes the linear space of all equivalence classes of Σ-measurable functions on X. We have the following change of variable formula:

$$\int_{\varphi^{-1}(A)} f \circ \varphi d\mu = \int_A hf d\mu \qquad A \in \Sigma, f \in L^0(\Sigma).$$

Every non-singular transformation φ from X into itself induces a linear transformation C_φ on $L^p(\mu)$ into linear space of all measurable functions on X, defined as

$$C_\varphi f = f \circ \varphi,$$

for every $f \in L^p(\mu)$. In case C_φ is continuous from $L^p(\mu)$ into itself, then it is called a composition operator on $L^p(\mu)$ induced by φ see[6]. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set.

Recall that an atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$, such that for each $B \in \Sigma$, if $B \subset A$ then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. A measure with no atoms is called non-atomic. We can easily check the following well-known facts (see[9]):

(a) Every σ-finite measure space (X, Σ, μ) can be partitioned uniquely as

$$X = (\cup_{n\in\mathbb{N}} A_n) \cup B, \tag{1}$$

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where $\{A_n\}_{n\in\mathbb{N}} \subseteq \Sigma$ is a countable collection of pairwise disjoint atoms and B, being disjoint from each A_n, is non-atomic. Since (X, Σ, μ) is σ-finite, it follows that $\mu(A_n) < \infty$ for every $n \in \mathbb{N}$.

(b) Let E be a non-atomic set with $\mu(E) > 0$. Then there exists a sequence of positive disjoint Σ-measurable subsets of E, $\{E_n\}_{n\in\mathbb{N}}$ such that $\mu(E_n) > 0$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(E_n) = 0$.

For a given complex Hilbert space $\mathcal{H}$, let $u : X \to \mathcal{H}$ be a mapping. We say that u is weakly measurable if for each $g \in \mathcal{H}$ the mapping $x \mapsto \langle u(x), g\rangle$ of X to $\mathbb{C}$ is measurable. We will denote this map by $\langle u, g\rangle$. Let $L^2(X)$ be the class of all measurable mappings $f : X \to \mathbb{C}$ such that $\|f\|_2^2 = \int_X |f(x)|^2 d\mu < \infty$.

Definition 1.1. *Let $u : X \to L^2(X)$ be a weakly measurable function. We say that u is a semi-weakly bounded function if for some $A > 0$,*

$$\|\langle u, g\rangle\|_2 \leq A\|g\|_2,$$

for each $g \in L^2(X)$

Definition 1.2. *Let $\varphi : X \to X$ be a non-singular measurable transformation and C_φ be a composition operator on $L^2(X)$. Also let $u : X \to L^2(X)$ be a weakly measurable function. Then the pair (u, φ) induces a substitution vector-valued integral operator $T_u^\varphi : L^2(X) \to L^2(X)$ defined by*

$$\langle (T_u^\varphi)^k f, g\rangle = \int_X \langle u, g\rangle\langle u \circ \varphi, g\rangle ... \langle u \circ \varphi^{k-1}, g\rangle C_{\varphi^k} f d\mu, \quad g, f \in L^2(X).$$

for every $k \in \mathbb{N}$. It is easy to see that $(T_u^\varphi)^k$ is well defined and linear and also we have $\langle T_u^\varphi f, g\rangle = \int_X \langle u, g\rangle C_\varphi f d\mu$

In [4], we have $\|T_u^\varphi f\| = \sup_{g\in\mathcal{D}} |\langle T_u^\varphi f, g\rangle|$, where $\mathcal{D}$ is the closed unit ball of L^2 and $\langle ., .\rangle$ is inner product in L^2. Some fundamental properties of the substitution vector-valued integral operator $T_u^\varphi : L^2(X) \to \mathcal{H}$ are studied by the author et al in [4].

Definition 1.3 ([4]). *Let $u : X \to \mathcal{H}$ be a weakly measurable function. We say that $(u, \varphi, \mathcal{H})$ has absolute property, if for each $f \in L^2(X)$, there exists $g_f \in \mathcal{D}$ such that $\sup_{g\in\mathcal{D}} \int_X |\langle u, g\rangle||C_\varphi f| d\mu = \int_X |\langle u, g_f\rangle||C_\varphi f| d\mu$, and $\langle u, g_f\rangle = e^{i(-\arg C_\varphi f + \theta_f)}|\langle u, g_f\rangle|$, for a constant θ_f.*

Proposition 1.4 ([4]). *Assume that $(u, \varphi, \mathcal{H})$ has the absolute property. Then*

$$\sup_{g\in\mathcal{D}} |\int_X \langle u, g\rangle C_\varphi f d\mu| = \sup_{g\in\mathcal{D}} \int_X |\langle u, g\rangle||C_\varphi f| d\mu.$$

Throughout of this paper we assume that $(u, \varphi, \mathcal{H})$ has the absolute property.

The aim of this paper is to carry some of the results obtained for the weighted composition operators in [1, 3, 5, 7] to a substitution vector-valued integral operator on $L^2(X)$ space. In this note, we will determine under certain conditions the specrum of T_u^φ on $L^2(X)$ space.

2. The main results

In this section we give necessary conditions for the compactness of T_u^φ from $L^2(X)$ to $L^2(X)$. Then, we determine the spectrum $\sigma(T_u^\varphi)$ of a compact substitution vector-valued integral operator T_u^φ on $L^2(X)$.

Theorem 2.1. *Let $u : X \to L^2(X)$ be a semi-weakly bounded function and let $h \in L^\infty(\Sigma)$, then $T_u^\varphi : L^2(X) \to L^2(X)$ is bounded.*

Proof. Let $f \in L^2(X)$. By Holder's inequality and change of variable formula we have

$$\begin{aligned}\|T_u^{\varphi} f\| &= \sup_{g\in\mathcal{D}} |\int_X \langle u, g\rangle C_{\varphi} f d\mu| \\ &\leq \sup_{g\in\mathcal{D}} (\int_X |\langle u, g\rangle|^2 d\mu)^{\frac{1}{2}} (\int_X |f\circ\varphi|^2 d\mu)^{\frac{1}{2}} \\ &\leq \sup_{g\in\mathcal{D}} (A\|g\|_2)(\int_X h|f|^2 d\mu)^{\frac{1}{2}} \\ &\leq A\sqrt{\|h\|_\infty}\|f\|_2.\end{aligned}$$

This shows that T_u^{φ} is bounded. □

Theorem 2.2. *Let (X, Σ, μ) be partitioned as (1.1) and the substitution vector-valued integral operator T_u^{φ} be a compact operator on $L^2(X)$. Then*

(i) $\sup_{g\in\mathcal{D}} |\langle u(\varphi^{-1}(B)), g\rangle| = 0$,

(ii) for any $\epsilon > 0$, the set $\{i \in \mathbb{N}; \mu(\{y \in \varphi^{-1}(A_i); \sup_{g\in\mathcal{D}} |\langle u(y), g\rangle| > \epsilon\} > 0\}$, is finite.

Proof. We first show that the compactness T_u^{φ} implies (i). Assuming the contrary, we can find some $\delta > 0$ and $g_1 \in \mathcal{D}$ such that $\mu(\varphi^{-1}(B)) \cap D_{\delta,g_1}(u)) > 0$, where $D_{\delta,g_1}(u) := \{x \in X : |\langle u(x), g_1\rangle| > \delta\}$. Consider ψ, the restriction of φ to $D_{\delta,g_1}(u)$. As the measure $\mu\psi^{-1}$ is absolutely continuous with respect to μ, hence there exists a non-negative Σ-measurable function v such that

$$\mu\psi^{-1}(E) = \int_E v d\mu \quad \text{for all} \quad E \in \Sigma.$$

Now, it follows from $\mu\psi^{-1}(B) = \mu(\varphi^{-1}(B)) \cap D_{\delta,g_1}(u)) > 0$ that there is some $\alpha > 0$ satisfying $\mu(\{x \in B : v(x) \geq \alpha\}) > 0$. Since the set $\{x \in B : v(x) \geq \alpha\}$ is non-atomic, we can find a sequence of pairwise disjoint Σ-measurable subsets $\{J_n\}_{n\in\mathbb{N}}$ with $0 < \mu(J_n) < \infty$ for all $n \in \mathbb{N}$. Define $f_n = \frac{\chi_{J_n}}{\mu(J_n)^{\frac{1}{2}}}$ for each $n \in \mathbb{N}$. Hence, $\|f_n\|_2 = 1$ and $\mu\psi^{-1}(J_n) = \int_{J_n} v d\mu \geq \alpha\mu(J_n) > 0$. Moreover, for any $m, n \in \mathbb{N}$ with $m \neq n$, we get that

$$\begin{aligned}\|T_u^{\varphi} f_n - T_u^{\varphi} f_m\| &= \sup_{g\in\mathcal{D}} \int_X |\langle u, g\rangle||f_n - f_m|\circ\varphi d\mu \\ &\geq \int_X |\langle u, g_1\rangle||f_n - f_m|\circ\varphi d\mu \\ &= \int_X |\langle u, g_1\rangle||\frac{\chi_{J_n}}{\mu(J_n)^{\frac{1}{2}}} - \frac{\chi_{J_m}}{\mu(J_m)^{\frac{1}{2}}}|\circ\varphi d\mu \\ &\geq \int_{\psi^{-1}(J_n)\cup\psi^{-1}(J_m)} |\langle u, g_1\rangle||\frac{\chi_{J_n}}{\mu(J_n)^{\frac{1}{2}}} - \frac{\chi_{J_m}}{\mu(J_m)^{\frac{1}{2}}}|\circ\varphi d\mu \\ &= \int_{\psi^{-1}(J_n)} |\langle u, g_1\rangle||\frac{\chi_{J_n}}{\mu(J_n)^{\frac{1}{2}}} - \frac{\chi_{J_m}}{\mu(J_m)^{\frac{1}{2}}}|\circ\varphi d\mu + \int_{\psi^{-1}(J_m)} |\langle u, g_1\rangle||\frac{\chi_{J_n}}{\mu(J_n)^{\frac{1}{2}}} - \frac{\chi_{J_m}}{\mu(J_m)^{\frac{1}{2}}}|\circ\varphi d\mu \\ &= \frac{1}{\mu(J_n)^{\frac{1}{2}}} \int_{\psi^{-1}(J_n)\cap\varphi^{-1}(J_n)} |\langle u, g_1\rangle| d\mu + \frac{1}{\mu(J_m)^{\frac{1}{2}}} \int_{\psi^{-1}(J_m)\cap\varphi^{-1}(J_m)} |\langle u, g_1\rangle| d\mu \\ &> \delta(\frac{\mu(\psi^{-1}(J_n))}{\mu(J_n)^{\frac{1}{2}}} + \frac{\mu(\psi^{-1}(J_m))}{\mu(J_m)^{\frac{1}{2}}}).\end{aligned}$$

This implies that the sequence $\{T_u^{\varphi} f_n\}_n$ does not contain a convergent subsequence, but this contradicts compactness of T_u^{φ}.

Next, we prove (ii). Suppose, on the contrary, the set

$$\{i \in \mathbb{N}; \mu(\{y \in \varphi^{-1}(A_i); \sup_{g\in\mathcal{D}} |\langle u(y), g\rangle| > \epsilon_0\} > 0\}$$

is infinite for some $\epsilon_0 > 0$. Then, there is a subsequence of disjoint atoms $\{A_k\}_{k\in\mathbb{N}}$ such that for any $k \in \mathbb{N}$, the set $\{y \in \varphi^{-1}(A_k) : \sup_{g\in\mathcal{D}} |\langle u(y), g\rangle| > \epsilon_0\}$ has positive measure. Hence we obtain for any $k \in \mathbb{N}$ there exists $g_k \in \mathbb{N}$ such that the set

$$v_k = \{y \in \varphi^{-1}(A_k) : \sup_{g\in\mathcal{D}} |\langle u(y), g\rangle| > \epsilon_0\},$$

has positive measure and

$$\mu(v_k) \le \mu(\varphi^{-1}(A_k) = \int_{A_k} h d\mu = h(A_k)\mu(A_k) < \infty.$$

Define $f_k := \frac{\chi_{A_k}}{(\mu(A_k))^{\frac{1}{2}}}$. Hence $\|f_k\|_2 = 1$. For any $m, n \in \mathbb{N}$ with $m \neq n$, we get that

$$\begin{aligned}
\|T_u^{\varphi} f_n - T_u^{\varphi} f_m\| &= \sup_{g\in\mathcal{D}} \int_X |\langle u, g\rangle||f_n - f_m| \circ \varphi d\mu \\
&= \sup_{g\in\mathcal{D}} \int_X |\langle u, g\rangle|| \frac{\chi_{A_n}}{\mu(A_n)^{\frac{1}{2}}} - \frac{\chi_{A_m}}{\mu(A_m)^{\frac{1}{2}}}| \circ \varphi d\mu \\
&\ge \sup_{g\in\mathcal{D}} \int_{V_n \cup V_m} |\langle u, g\rangle|| \frac{\chi_{A_n}}{\mu(A_n)^{\frac{1}{2}}} - \frac{\chi_{A_m}}{\mu(A_m)^{\frac{1}{2}}}| \circ \varphi d\mu \\
&= \sup_{g\in\mathcal{D}} \int_{V_n\cap\varphi^{-1}(A_n)} |\langle u, g\rangle| \frac{1}{\mu(A_n)^{\frac{1}{2}}} d\mu + \sup_{g\in\mathcal{D}} \int_{V_m\cap\varphi^{-1}(A_m)} |\langle u, g\rangle| \frac{1}{\mu(A_m)^{\frac{1}{2}}} d\mu \\
&\ge \int_{V_n\cap\varphi^{-1}(A_n)} |\langle u, g_n\rangle| \frac{1}{\mu(A_m)^{\frac{1}{2}}} d\mu + \int_{V_m\cap\varphi^{-1}(A_m)} |\langle u, g_m\rangle| \frac{1}{\mu(A_m)^{\frac{1}{2}}} |d\mu \\
&> \epsilon_0 \left(\frac{\mu(v_n)}{\mu(A_n)^{\frac{1}{2}}} + \frac{\mu(v_m)}{\mu(A_m)^{\frac{1}{2}}} \right).
\end{aligned}$$

But this shows that T_u^{φ} is not compact. □

The kth iterate φ^k of the non-singular measurable transformation $\varphi : X \to X$ is defined by $\varphi^0(x) = x$ and $\varphi^k(x) = \varphi(\varphi^{k-1}(x))$ for all $x \in X$ and $k \in \mathbb{N}$.

Definition 2.3. *A atom A is called an invariant atom with respect to φ, if for all $n \in \mathbb{Z}$, $\varphi^n(A)$ is an atom. An invariant atom A with respect to φ is called a fixed atom of φ of order one, if for each $g \in \mathcal{D}$, $\langle u(A), g\rangle \neq 0$ and $\varphi(A) = A$. Also, it is called of order $2 \le k \in \mathbb{N}$, if for each $g \in \mathcal{D}$, $\Pi_{i=0}^{k-1} \langle u(\varphi^i(A)), g\rangle \neq 0, \varphi^k(A) = A$ and $\varphi^i(A) \neq A$ for all $i = 1, ..., k-1$.*

Recall that a complex λ is in the spectrum $\sigma(T_u^{\varphi})$ of T_u^{φ}, if $T_u^{\varphi} - \lambda I$ is not invertible.

Theorem 2.4. *Let T_u^{φ} be a compact substitution vector-valued operator integral from $L^2(X)$ to $L^2(X)$ and also let (X, Σ, μ) be partitioned as (1.1). If we set*

$$\Lambda = \{\lambda \in \mathbb{C} : \langle \lambda^k, g\rangle = \Pi_{i=0}^{k-1} \langle u(\varphi^i(A)), g\rangle)\mu(A)\},$$

for some fixed atom A of φ of order k and for each $g \in \mathcal{D}$. Then, we have $\sigma(T_u^{\varphi}) \cup \{0\} = \Lambda \cup \{0\}$.

Proof. To prove the theorem, we adopt the method by Kamowitz [5] and Takagi [7]. Firstly, we show the inclusion $\Lambda \cup \{0\} \subseteq \sigma(T_u^{\varphi}) \cup \{0\}$. Let λ be a non-zero number in Λ such that for each $g \in \mathcal{D}$,

$$\langle \lambda^k, g\rangle = \langle u(A), g\rangle\langle u(\varphi(A)), g\rangle ... \langle u(\varphi^{k-1}(A)), g\rangle\mu(A)$$

for some fixed atom A of φ of order k.
If $k = 1$, then $\langle \lambda, g\rangle = \langle u(A), g\rangle\mu(A)$ and $\varphi(A) = A$. We claim that there exists no $f \in \mathcal{D}$ such that $T_u^{\varphi} f - \lambda f = \chi_A$ μ-a.e. on X. Indeed, since $\varphi(A) = A$, for each $g \in L^2$, we get that $\langle (T_u^{\varphi} f - \lambda f)(A), g\rangle = \int_A \langle u, g\rangle (f \circ \varphi) d\mu - \langle u(A), g\rangle\mu(A) f(A) = 0$, whereas $\chi_A(A) = 1$. This shows that $T_u^{\varphi} f - \lambda f$ is not surjective. Hence, $\lambda \in \sigma(T_u^{\varphi})$.
When $k \geq 2$, again there exists no $f \in L^2(X)$ which satisfies $T_u^{\varphi} f - \lambda f = \chi_A$ μ-a.e. on X. For, if such a function f exists, then by induction we have

$$\lambda^k f(A) - ((T_u^{\varphi})^k(f))(A) = \lambda^{k-1} + \sum_{j=1}^{k-1} \lambda^{k-j-1}((T_u^{\varphi})^j(\chi_A))(A). \tag{2}$$

Therefore for each $g \in \mathcal{D}$, we have

$$\langle \lambda^k f(A) - ((T_u^{\varphi})^k(f))(A), g\rangle = \langle \lambda^{k-1} + \sum_{j=1}^{k-1} \lambda^{k-j-1}((T_u^{\varphi})^j(\chi_A))(A), g\rangle.$$

Moreover

$$\langle (T_u^{\varphi})^k f(A), g\rangle = \langle u(A), g\rangle ... \langle u(\varphi^{k-1}(A)), g\rangle f \circ \varphi^k(A)\mu(A),$$

Since $\varphi^k(A) = A$ and $\varphi^j(A) \neq A$ for $1 \leq j \leq k-1$, the left hand side of (2.1) equals 0, while the right hand side of (2.1) equals $\langle \lambda^{k-1}, g\rangle$, which is non-zero. This contradiction shows that $\lambda \in \sigma(T_u^{\varphi})$. Therefore $\Lambda \cup \{0\} \subseteq \sigma(T_u^{\varphi}) \cup \{0\}$.

Now, we show the opposite inclusion. Let $\lambda \notin \Lambda \cup \{0\}$, and suppose that an L^2 function f satisfies $\lambda f = T_u^{\varphi} f$. All that we have to show is that f is zero μ-almost everywhere on X. For, if this holds, λ is not an eigenvalue of T_u^{φ}, and by Fredholm alternative for compact operators, λ is not in $\sigma(T_u^{\varphi})$, and thus we get $\sigma(T_u^{\varphi}) \cup \{0\} \subseteq \Lambda \cup \{0\}$. We first show that f vanishes μ-almost everywhere on $\cup_{n\in\mathbb{N}} A_n$, or equivalently, $f(A) = 0$ for every invariant atom A. Let A be a fixed atom of φ of order k. Since $T_u^{\varphi} f = \lambda f$, by induction, we get $(T_u^{\varphi})^k f = \lambda^k f$. Hence for each $g \in \mathcal{D}$, $\langle (T_u^{\varphi})^k f(A), g\rangle = \langle \lambda^k f(A), g\rangle$. Since $\varphi^k(A) = A$ and $\langle \lambda^k, g\rangle \neq \langle u(A), g\rangle\langle u(\varphi(A)), g\rangle ... \langle u(\varphi^{k-1}(A)), g\rangle\mu(A)$, we can easily deduce that $f(A) = 0$.

By the fist part of the poof, we can assume that for all $k \in \mathbb{N} \cup 0$ and for all $g \in \mathcal{D}$, $\langle u(\varphi^k(A)), g\rangle \neq 0$. Put $\mathcal{K}(A) = \{\varphi^i(A) : i \in \mathbb{N} \cup \{0\}\}$. If $\mathcal{K}(A)$ is finite, In this case for some $n, m \in \mathbb{N}, \varphi^n(A)$ is a fixed atom of φ of order m. By a preceding discussion, we have $f(\varphi^n(A)) = 0$. On the other hand, since $\lambda^n f = (T_u^{\varphi})^n f$ and

$$\langle (T_u^{\varphi})^n f(A), g\rangle = \langle u(A), g\rangle ... \langle u(\varphi^{n-1}(A)), g\rangle\mu(A) f(\varphi^n(A)).$$

Then $f(A) = 0$.

Now, suppose that $\mathcal{K}(A)$ is infinite. We claim that the set $\{j \geq 0 : \sup_{g\in\mathcal{D}} |\langle u(\varphi^j(A)), g\rangle| > \epsilon\}$ is finite for every $\epsilon > 0$. Suppose this does not hold. Then the set $\{j \geq 0 : \mu(\{x \in \varphi^{-1}(\varphi^{j+1})(A)) : \sup_{g\in\mathcal{D}} |\langle u(x), g\rangle| \geq \epsilon\}) > 0\}$ is infinite. But this contradicts the compactness of T_u^{φ}. Hence, for any $\epsilon > 0$, there exists a M such that

$\sup_{g\in\mathcal{D}} |\langle u(\varphi^m(A), g\rangle| < \epsilon$ for all $m \geq M$. Therefore there exists $g_1 \in \mathcal{D}$ such that for each $m \geq M$,

$$\begin{aligned}
|\langle \lambda^m f(A), g_1\rangle|^2 &= |\langle (T_u^{\varphi})^m f(A), g_1\rangle|^2 = |\langle u(A), g_1\rangle|^2 ... |\langle u(\varphi^{M-1}(A), g_1\rangle)|^2 \\
&\quad ... \langle u(\varphi^{m-1}(A)), g_1\rangle|^2 |f(\varphi^m(A))|^2 \mu(A) \\
&= h_m |\langle u(A), g_1\rangle|^2 ... |\langle u(\varphi^{M-1}(A), g_1\rangle)|^2 \\
&\quad ... \langle u(\varphi^{m-1}(A)), g_1\rangle|^2 \int_{\varphi^{-m}(A)} |f|^2 d\mu \\
&\leq h_m \|f\|_2^2 \|u(A)\|_2^{2M} \|g_1\|_2^{2M} \epsilon^{2(m-M)}.
\end{aligned}$$

so

$$|f(A)|^2 < \frac{h_m \|f\|_2^2 \|u(A)\|_2^{2M} \epsilon^{2(m-M)}}{|\langle \lambda^m, g_1\rangle|^2}.$$

As $\epsilon = \frac{|\langle \lambda^m, g_1\rangle|}{2}$ and $m \to \infty$, we obtain $f(A) = 0$. Therefore we conclude that f is zero on $\cup_{n\in\mathbb{N}} A_n$.

It remains to show that f is zero μ-almost everywhere on B. Since $L^2(X) = L^2(\cup_{n\in\mathbb{N}} A_n) \oplus L^2(B)$. hence it suffices to show that f is zero as an element of $L^2(B)$. Since $\sup_{g\in\mathcal{D}} |\langle u(\varphi^{-1}(B), g\rangle| = 0$, so

$$\|T_u^{\varphi} f\|_{L^2(B)} = \sup_{g\in\mathcal{D}} \int_B |\langle u, g\rangle| |f| \circ \varphi d\mu \geq \int_B |\langle u, g_1\rangle| |f| \circ \varphi d\mu = 0.$$

Consequently $\lambda f = T_u^{\varphi} f = 0$ and hence f is zero μ-almost everywhere on B. This completes the proof of the theorem. □

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