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Inequalities for the Eigenvalues of the Positive Definite Solutions of the Nonlinear Matrix Equation

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Abstract. In this paper, the Hermitian positive definite solutions of the matrix equation $X^s + A^*X^{-t}A = Q$ are considered, where Q is a Hermitian positive definite matrix, s and t are positive integers. Bounds for the sum of eigenvalues of the solutions to the equation are given. The equivalent conditions for solutions of the equation are obtained. The eigenvalues of the solutions of the equation with the case $AQ = QA$ are investigated.

1. Introduction

We consider the Hermitian positive definite solutions of the matrix equation

$$X^s + A^*X^{-t}A = Q, \tag{1}$$

where Q is an $n \times n$ Hermitian positive definite matrix, s and t are positive integers. Here A^* stands for the conjugate transpose of the matrix A. Nonlinear matrix equations with the form (1) often arise in control theory, dynamic programming, ladder networks, stochastic filtering, statistics and etc.(see [1] and the reference therein). The problems for eigenvalues of the solutions of the equation were studied by several authors [1–8, 10–19] .

In this manuscript, lower bounds for the sum of some eigenvalues of the solution to the equation are given. Equivalent conditions for solutions of the equation are obtained. Eigenvalues of the solutions to the equation with the case $AQ = QA$ are investigated.

Throughout this paper, we write $B > 0$ ($B \geq 0$) if the matrix B is positive definite (semidefinite). If $B - C > 0$ ($B - C \geq 0$), then we write $B > C$ ($B \geq C$). This induces a partial ordering on the Hermitian matrices. Let $\mathbb{C}^{n\times m}$ stand for the set of all $n \times m$ complex matrices. We use trA and $rank(A)$ to denote the trace and rank of A, respectively. A solution always means a Hermitian positive definite solution. The eigenvalues of a Hermitian matrix A are ordered as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

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2. Conditions for the Existence of Solutions

Lemma 2.1. *[9, P248] Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and suppose that $1 \le m \le n$. Then*

$$\lambda_1(A) + \lambda_2(A) + \cdots + \lambda_m(A) = \max_{V\in\mathbb{C}^{n\times m}, V^*V=I_m} trV^*AV,$$

$$\lambda_{n-m+1}(A) + \lambda_{n-m+2}(A) + \cdots + \lambda_n(A) = \min_{V\in\mathbb{C}^{n\times m}, V^*V=I_m} trV^*AV.$$

Theorem 2.2. *If $rank(A) = r$ and Eq.(1) has a solution X, then*

$$\lambda_1^s(X) + \lambda_2^s(X) + \cdots + \lambda_{n-k}^s(X) \ge \lambda_{k+1}(Q) + \lambda_{k+2}(Q) + \cdots + \lambda_n(Q),$$

$$\lambda_1(X) + \lambda_2(X) + \cdots + \lambda_{n-k}(X) \ge (\lambda_{k+1}(Q) + \lambda_{k+2}(Q) + \cdots + \lambda_n(Q))^{\frac{1}{s}},$$

where $k = r, r+1, \ldots, n-1$.

Proof. Since $rank(A) = r$, there exists $U_{n-r} \in \mathbb{C}^{n\times(n-r)}$ and $U_{n-r}^*U_{n-r} = I_{n-r}$ such that

$$AU_{n-r} = 0.$$

Multiplying right side of Eq.(1) by U_{n-r} and left side by U_{n-r}^*, we have

$$U_{n-r}^*X^sU_{n-r} + U_{n-r}^*A^*X^{-t}AU_{n-r} = U_{n-r}^*QU_{n-r}$$

or

$$U_{n-r}^*X^sU_{n-r} = U_{n-r}^*QU_{n-r}.$$

Let $U_{n-r} = (u_1, u_2, \ldots, u_{n-r})$ and $U_i = (u_1, u_2, \ldots, u_i)$, $i = 1, 2, \ldots, n-r$. Noticing that $\lambda_i(X^s) = \lambda_i^s(X)$, $i = 1, 2, \ldots, n$, by Lemma 2.1 we have

$$\begin{aligned}
&\lambda_1^s(X) + \lambda_2^s(X) + \cdots + \lambda_{n-k}^s(X)\\
&= \max_{V_{n-k}\in\mathbb{C}^{n\times(n-k)}, V_{n-k}^*V_{n-k}=I_{n-k}} trV_{n-k}^*X^sV_{n-k}\\
&\ge trU_{n-k}^*X^sU_{n-k}\\
&= trU_{n-k}^*QU_{n-k}\\
&\ge \min_{V_{n-k}\in\mathbb{C}^{n\times(n-k)}, V_{n-k}^*V_{n-k}=I_{n-k}} trV_{n-k}^*QV_{n-k}\\
&= \lambda_{k+1}(Q) + \lambda_{k+2}(Q) + \cdots + \lambda_n(Q),
\end{aligned}$$

and

$$\begin{aligned}
&\lambda_1(X) + \lambda_2(X) + \cdots + \lambda_{n-k}(X)\\
&\ge \left(\lambda_1^s(X) + \lambda_2^s(X) + \cdots + \lambda_{n-k}^s(X)\right)^{\frac{1}{s}}\\
&\ge (\lambda_{k+1}(Q) + \lambda_{k+2}(Q) + \cdots + \lambda_n(Q))^{\frac{1}{s}},
\end{aligned}$$

for $k = r, r+1, \ldots, n-1$. This completes the proof. □

Corollary 2.3. *If $rank(A) = r$ and Eq.(1) has a solution X, then*

i. $trX^s \ge \lambda_{r+1}(Q) + \lambda_{r+2}(Q) + \cdots + \lambda_n(Q)$,

ii. $trX \ge (\lambda_{r+1}(Q) + \lambda_{r+2}(Q) + \cdots + \lambda_n(Q))^{\frac{1}{s}}$.

In particular, we have $trX^s \ge (n-r)\lambda_n(Q)$ and $trX \ge (n-r)^{\frac{1}{s}}(\lambda_n(Q))^{\frac{1}{s}}$. Several bounds for the traces of the solutions of the equation were presented in [16, 19].

Theorem 2.4. *Let Eq.(1) have a solution X. The following are equivalent.*

a. $rank(A) = r$;

b. $\lambda_1\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = \lambda_2\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = \cdots = \lambda_{n-r}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = 1$ *and*
$1 > \lambda_{n-r+1}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) \geq \lambda_{n-r+2}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) \geq \cdots \geq \lambda_n\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right)$;

c. $\lambda_1\left(X^sQ^{-1}\right) = \lambda_2\left(X^sQ^{-1}\right) = \cdots = \lambda_{n-r}\left(X^sQ^{-1}\right) = 1$ *and*
$1 > \lambda_{n-r+1}\left(X^sQ^{-1}\right) \geq \lambda_{n-r+2}\left(X^sQ^{-1}\right) \geq \cdots \geq \lambda_n\left(X^sQ^{-1}\right)$;

d. $\lambda_1\left(Q^{-1}X^s\right) = \lambda_2\left(Q^{-1}X^s\right) = \cdots = \lambda_{n-r}\left(Q^{-1}X^s\right) = 1$ *and*
$1 > \lambda_{n-r+1}\left(Q^{-1}X^s\right) \geq \lambda_{n-r+2}\left(Q^{-1}X^s\right) \geq \cdots \geq \lambda_n\left(Q^{-1}X^s\right)$.

Proof. Multiplying both sides of Eq.(1) by $Q^{-\frac{1}{2}}$, we have

$$Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}} + Q^{-\frac{1}{2}}A^*X^{-t}AQ^{-\frac{1}{2}} = I. \tag{2}$$

(a) $\Leftrightarrow$ (b): We show (a) $\Rightarrow$ (b). If $rank(A) = r$, then $rank\left(AQ^{-\frac{1}{2}}\right) = r$. There exist linearly independent vectors $x_1, x_2, \ldots, x_{n-r}$ such that $AQ^{-\frac{1}{2}}(x_1, x_2, \ldots, x_{n-r}) = 0$. Multiplying right side of Eq.(2) by $(x_1, x_2, \ldots, x_{n-r})$, we have

$$Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}(x_1, x_2, \ldots, x_{n-r}) = (x_1, x_2, \ldots, x_{n-r}),$$

i.e.

$$Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}x_i = x_i, i = 1, 2, \ldots, n-r.$$

Noticing that $Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}} \leq I$, we get

$$\lambda_1\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = \lambda_2\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = \cdots = \lambda_{n-r}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = 1.$$

If $\lambda_{n-r+1}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = 1$, then it implies that $rank(A)$=$rank\left(AQ^{-\frac{1}{2}}\right) < r$, which is a contradiction. Thus

$$1 > \lambda_{n-r+1}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) \geq \lambda_{n-r+2}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) \geq \cdots \geq \lambda_n\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right).$$

For (b) $\Rightarrow$ (a), since

$$\lambda_1\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = \lambda_2\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = \cdots = \lambda_{n-r}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = 1,$$

there exist linearly independent vectors $y_1, y_2, \ldots y_{n-r}$ such that

$$Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}y_i = y_i, i = 1, 2, \ldots, n-r.$$

Multiplying right side of Eq.(2) by $(y_1, y_2, \ldots, y_{n-r})$, we obtain

$$Q^{-\frac{1}{2}}A^*X^{-t}AQ^{-\frac{1}{2}}(y_1, y_2, \ldots, y_{n-r}) = 0.$$

It means that $rank(A)$=$rank\left(Q^{-\frac{1}{2}}A^*X^{-t}AQ^{-\frac{1}{2}}\right) \leq r$. If $rank(A)$=$rank\left(AQ^{-\frac{1}{2}}\right) < r$, it contradicts the assumption (b). We get $rank(A) = r$. Hence (a) and (b) are equivalent.

Notice that if $C, D \in \mathbb{C}^{n\times n}$, then CD and DC have exactly the same eigenvalues. It implies that (b), (c) and (d) are equivalent. This completes the proof. □

Theorem 2.4 shows that if A is singular and Eq.(1) has a solution X, then $\lambda_{max}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = 1$, which is Theorem 2.1 of [17]. Theorem 2.4 also generalizes Theorem 3.2 of [2].

Corollary 2.5. *Let $Q = I$ and Eq.(1) has a solution X, The following are equivalent.*

a. *$rank(A) = r$;*

b. $\lambda_1(X) = \lambda_2(X) = \cdots = \lambda_{n-r}(X) = 1$ *and* $1 > \lambda_{n-r+1}(X) \geq \lambda_{n-r+2}(X) \geq \cdots \geq \lambda_n(X)$;

Corollary 2.6. **a.** *If $rank(A) = r$ and Eq.(1) has a solution X, then*

$$tr\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = tr\left(X^sQ^{-1}\right) = tr\left(Q^{-1}X^s\right) \geq n - r.$$

b. *If $rank(A) = r$, $Q = I$, and Eq.(1) has a solution X, then*

$$trX \geq n - r.$$

Corollary 2.5 implies that Eq.(1) with $Q = I$ has a solution X, then (a) when A is nonsingular, $\lambda_{max}(X) < 1$; (b) when A is singular, $\lambda_{max}(X) = 1$. This is Theorem 2.1 of [12]. The conclusion that if A is singular and Eq.(1) has a solution X, then $\lambda_{max}(X) = 1$ also appear in Theorem 2 of [14].

Lemma 2.7. *[9] Let $B, C \in \mathbb{C}^{n\times n}$ and $BC = CB$. There is a unitary matrix U such that both U^*BU and U^*CU are upper triangular, respectively.*

Lemma 2.8. *Let $rank(A) = r$ and $AQ = QA$. There exist linearly independent vectors $u_1, u_2, \ldots, u_{n-r}$ and $\lambda_{i_1}(Q), \lambda_{i_2}(Q), \ldots, \lambda_{i_{(n-r)}}(Q)$ such that*

$$Au_k = 0,\ \ Qu_k = \lambda_{i_k}(Q)u_k,\ \ k = 1, 2, \ldots, n - r.$$

Proof. By Lemma 2.7, there is a unitary matrix U such that both U^*AU and U^*QU are upper triangular, respectively. Without loss of generality, we assume that $U = (u_1, u_2, \ldots, u_n)$ and $Au_k = 0$, for $k = 1, 2, \ldots, n-r$. It implies that

$$U^*QU = \begin{pmatrix} \lambda_{i_1}(Q) & & * \\ & \ddots & \\ 0 & & \lambda_{i_n}(Q) \end{pmatrix}.$$

Since Q is Hermitian positive definite matrix, then $(U^*QU)^* = U^*QU$. We get

$$U^*QU = \begin{pmatrix} \lambda_{i_1}(Q) & & 0 \\ & \ddots & \\ 0 & & \lambda_{i_n}(Q) \end{pmatrix}.$$

It implies that $Au_k = 0$, $Qu_k = \lambda_{i_k}(Q)u_k$, $k = 1, 2, \ldots, n - r$. □

Theorem 2.9. *Let $rank(A) = r$, $AQ = QA$, and Eq.(1) have a solution X. There exist $\lambda_{i_1}(Q), \lambda_{i_2}(Q), \ldots, \lambda_{i_{(n-r)}}(Q)$ and linearly independent vectors $u_1, u_2, \ldots, u_{n-r}$ such that $X^su_k = Qu_k = \lambda_{i_k}(Q)u_k$, $k = 1, 2, \ldots, n - r$.*

Proof. By Lemma 2.8, there exist linearly independent vectors $u_1, u_2, \ldots, u_{n-r}$ and $\lambda_{i_1}(Q), \lambda_{i_2}(Q), \ldots, \lambda_{i_{(n-r)}}(Q)$ such that

$$Au_k = 0,\ \ Qu_k = \lambda_{i_k}(Q)u_k,\ \ k = 1, 2, \ldots, n - r.$$

Multiplying right side of Eq.(1) by $(u_1, u_2, \ldots, u_{n-r})$, we have

$$X^s(u_1, u_2, \ldots, u_{n-r}) = Q(u_1, u_2, \ldots, u_{n-r}).$$

Thus

$$X^su_k = Qu_k = \lambda_{i_k}(Q)u_k,\ \ k = 1, 2, \ldots, n - r.$$

□

If $rank(A) = r$, $AQ = QA$ and Eq.(1) has a solution X, Theorem 2.9 shows that $\lambda_{i_k}(Q)$ $(k = 1, 2, \ldots, n - r)$ are eigenvalues of X^s, which implies that $\lambda_{i_k}^{\frac{1}{s}}(Q)$ $(k = 1, 2, \ldots, n - r)$ are eigenvalues of X.

3. Conclusions

In this paper, we give lower bounds for the sum of eigenvalues of a solution to the equation. We evaluate the eigenvalues for a solution of the equation under some conditions. We present some lower bounds for the traces of a solution of the equation. The equivalent conditions for a solution of the equation are obtained.

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