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Univalence of Certain Integral Operators Involving q-Bessel Functions

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Abstract. In this investigation we make a systematic study of the univalence of certain families of integral operators, which are defined by means of the normalized forms of Jackson's second and third q-Bessel functions.

1. Introduction

Bessel functions, also called cylinder functions because of their close association with cylindrical domains are widely used in electromagnetic theory and numerous other areas of physics and engineering. The classical Bessel function of the first kind of order ν is defined by the infinite series

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{n!\Gamma(n+\nu+1)}, \tag{1}$$

where Γ stands for the Euler gamma function, $z \in \mathbb{C}$ and $\nu \in \mathbb{R}$. In his research of basic numbers, F.H. Jackson [1–3] defines the oldest q-analogues of the Bessel functions, namely $J_\nu^{(1)}(z;q)$. In the following years, the Jackson's second and third q-Bessel functions [4, 5] were defined by:

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{n(n+\nu)}, \tag{2}$$

and

$$J_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n z^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{\frac{1}{2}n(n+1)}, \tag{3}$$

where $z \in \mathbb{C}$, $\nu > -1$, $q \in (0,1)$ and

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}), \quad (a;q)_\infty = \prod_{k\geq 1}(1-aq^{k-1}).$$

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It is well known that:
(i) if z is fixed, then $J_\nu^{(1)}((z(1-q);q) \to J_\nu(z)$, $J_\nu^{(2)}((1-z)q;q) \to J_\nu(z)$ and $J_\nu^{(3)}((1-z)q;q) \to J_\nu(2z)$ as $q \to 1^-$;
(ii) $(-z^2/4;q)_\infty J_\nu^{(1)}(z;q) = J_\nu^{(2)}(z;q)$, for $|z| < 2$.

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane and $\mathcal{A}$ the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U, \tag{4}$$

which are analytic in U and satisfy the condition $f(0) = f'(0) - 1 = 0$. Also, we consider S the class of functions $f \in \mathcal{A}$ which are univalent in U.

Because the functions $J_\nu^{(2)}(\cdot;q)$ and $J_\nu^{(3)}(\cdot;q)$ do not belong to the class $\mathcal{A}$ we will need to consider the following normalizations:

$$h_\nu^{(2)}(z;q) = 2^\nu \frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty} z^{1-\frac{\nu}{2}} J_\nu^{(2)}(\sqrt{z};q) \tag{5}$$

and

$$h_\nu^{(3)}(z;q) = \frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty} z^{1-\frac{\nu}{2}} J_\nu^{(3)}(\sqrt{z};q), \tag{6}$$

which implies

$$h_\nu^{(2)}(z;q) = z + \sum_{n\geq 1} \frac{(-1)^n q^{n(n+\nu)}}{4^n (q;q)_n (q^{\nu+1};q)_n} z^{n+1} \tag{7}$$

and

$$h_\nu^{(3)}(z;q) = z + \sum_{n\geq 1} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q;q)_n (q^{\nu+1};q)_n} z^{n+1}. \tag{8}$$

Recently, Bessel functions were investigated from different points of view (see [6–12] and the references therein for more details).

Finding sufficient conditions for the univalence of integral operators is an important topic of research in Geometric Function Theory. In this work, our objective is based on the following remarkable families of integral operators:

$$H(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^n \left[e^{f_i(t)} \right]^{\sigma_i} dt \right\}^{1/\lambda}, \quad z \in U, \tag{9}$$

$$I(z) = \left\{ \mu \int_0^z t^{\mu-1} \prod_{i=1}^n \left[\frac{f_i(t)}{t} \right]^{\alpha_i} dt \right\}^{1/\mu}, \quad z \in U, \tag{10}$$

$$F(z) = \left\{ (n\beta+1) \int_0^z \prod_{i=1}^n [f_i(t)]^{\beta} dt \right\}^{1/(n\beta+1)}, \quad z \in U, \tag{11}$$

$$G(z) = \left\{ \eta \int_0^z t^{\eta-1} \prod_{i=1}^n \left[f_i'(t) \right]^{\gamma_i} dt \right\}^{1/\eta}, \quad z \in U, \tag{12}$$

where the functions $f_1, \dots f_n \in \mathcal{A}$, and the parameters $\alpha_1, \dots \alpha_n, \gamma_1, \dots \gamma_n, \sigma_1, \dots \sigma_n, \mu, \beta, \eta$ and $\lambda \in \mathbb{C}$ are given such that the integrals (9)-(12) exist (see[13] and the references therein).

Also, some sufficient conditions for univalence of certain integral operators involving Bessel functions of the first kind were given by Szász and Kupán [14], Baricz and Frasin [15], respectively. Deniz, Orhan and Srivastava [16] investigated the univalence of integral operators involving generalized Bessel functions. Moreover, univalence conditions for an integral operator associated with the q-hypergeometric functions were deduced by Aldweby and Darus [17].

Motivated by the aforementioned works, we aim to obtain new sufficient conditions for the univalence of the integral operators (9)-(12), by using some inequalities for q-Bessel functions.

The following results will be required in our investigation:

Lemma 1.1 (Pascu[18]). *Let δ be a complex number, $\Re(\delta) > 0$ and the function $f \in \mathcal{A}$. If for all $z \in U$,*

$$\frac{1 - |z|^{2\Re(\delta)}}{\Re(\delta)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{13}$$

then for any complex number μ, $\Re(\mu) \geq \Re(\delta)$, the function

$$F_\mu(z) = \left(\mu \int_0^z t^{\mu-1} f'(t) dt \right)^{\frac{1}{\mu}} \tag{14}$$

is in the class S.

Lemma 1.2 (Pescar[19]). *Let μ be a complex number, $\Re(\mu) > 0$ and c be a complex number, with $|c| \leq 1$, $c \neq -1$. If $f \in \mathcal{A}$ satisfies*

$$\left| c|z|^{2\mu} + (1 - |z|^{2\mu}) \frac{zf''(z)}{\mu f'(z)} \right| \leq 1, \tag{15}$$

for all $z \in U$, then the function F_μ, defined by (14) is in the class S.

Lemma 1.3 (Orhan and Aktaş [12]). *Let $q \in (0, 1), \nu > -1$ and $4(1 - q)(1 - q^\nu) > q^\nu$. Then the function $h_\nu^{(2)}(z; q)$ satisfies the next two inequalities for $z \in U$:*

$$|h_\nu^{(2)}(z; q)| \leq \frac{4(1 - q)(1 - q^\nu)}{4(1 - q)(1 - q^\nu) - q^\nu}, \tag{16}$$

$$|[h_\nu^{(2)}(z; q)]'| \leq \left[\frac{4(1 - q)(1 - q^\nu)}{4(1 - q)(1 - q^\nu) - q^\nu} \right]^2. \tag{17}$$

Lemma 1.4. *Let $q \in (0, 1), \nu > -1$ and $2(1 - q)(1 - q^\nu) > q^\nu$. Then the function $h_\nu^{(2)}(z; q)$ satisfies the inequalities*

$$\left| \frac{z[h_\nu^{(2)}(z; q)]'}{h_\nu^{(2)}(z; q)} - 1 \right| \leq \frac{4q^\nu(1 - q)(1 - q^\nu)}{[4(1 - q)(1 - q^\nu) - q^\nu][4(1 - q)(1 - q^\nu) - 2q^\nu]}, \tag{18}$$

$$\left| \frac{z[h_\nu^{(2)}(z; q)]''}{[h_\nu^{(2)}(z; q)]'} \right| \leq \frac{2q^\nu a^2(1 - q^\nu)^2}{[a(1 - q^\nu) - q^\nu] \cdot [2q^{2\nu} - 4q^\nu a(1 - q^\nu) + a^2(1 - q^\nu)^2]}, \tag{19}$$

where $a = 4(1 - q)$ and $z \in U$.

Proof. By using the triangle inequality, its consequences and the fact that the inequalities

$$q^{n(n+\nu)} \leq q^{n\nu}, \quad (1-q)^n \leq (q;q)_n \quad \text{and} \quad (1-q^\nu)^n \leq (q^{\nu+1};q)_n \tag{20}$$

take place for $q \in (0,1)$ and $\nu > -1$, we obtain

$$\left|\frac{z}{h_\nu^{(2)}(z;q)}\right| \leq \frac{4(1-q)(1-q^\nu)-q^\nu}{4(1-q)(1-q^\nu)-2q^\nu}, \tag{21}$$

$$\left|[h_\nu^{(2)}(z;q)]' - \frac{h_\nu^{(2)}(z;q)}{z}\right| \leq \frac{4q^\nu(1-q)(1-q^\nu)}{[4(1-q)(1-q^\nu)-q^\nu]^2}, \tag{22}$$

$$\left|\frac{1}{[h_\nu^{(2)}(z;q)]'}\right| \leq \frac{[a(1-q^\nu)-q^\nu]^2}{2q^{2\nu}-4q^\nu a(1-q^\nu)+a^2(1-q^\nu)^2}, \tag{23}$$

$$\left|z[h_\nu^{(2)}(z;q)]''\right| \leq \frac{2q^\nu a^2(1-q^\nu)^2}{[a(1-q^\nu)-q^\nu]^3}, \; z \in U. \tag{24}$$

By combining the inequalities (21) and (22) we find that (18) holds. Also, from (23) and (24) we deduce that (19) is valid. □

Lemma 1.5 (Orhan and Aktaş [12]). *Let $q \in (0,1), \nu > -1$ and $(1-q)(1-q^\nu) > \sqrt{q}$. Then the function $h_\nu^{(3)}(z;q)$ satisfies the next two inequalities for $z \in U$:*

$$|h_\nu^{(3)}(z;q)| \leq \frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu)-\sqrt{q}}, \tag{25}$$

and

$$|[h_\nu^{(3)}(z;q)]'| \leq \left[\frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu)-\sqrt{q}}\right]^2. \tag{26}$$

Lemma 1.6. *Let $q \in (0,1), \nu > -1$ and $(1-q)(1-q^\nu) > 2\sqrt{q}$. Then the function $h_\nu^{(3)}(z;q)$ satisfies the inequalities:*

$$\left|\frac{z[h_\nu^{(3)}(z;q)]'}{h_\nu^{(3)}(z;q)} - 1\right| \leq \frac{\sqrt{q}(1-q)(1-q^\nu)}{[(1-q)(1-q^\nu)-\sqrt{q}][(1-q)(1-q^\nu)-2\sqrt{q}]}, \tag{27}$$

$$\left|\frac{z[h_\nu^{(3)}(z;q)]''}{[h_\nu^{(3)}(z;q)]'}\right| \leq \frac{2\sqrt{q}b^2(1-q^\nu)^2}{[b(1-q^\nu)-\sqrt{q}]\cdot[2q-4\sqrt{q}b(1-q^\nu)+b^2(1-q^\nu)^2]}, \tag{28}$$

where $b = 1-q$ and $z \in U$.

Proof. Because

$$q^{\frac{1}{2}n(n+1)} \leq q^{\frac{1}{2}n}, \quad (1-q)^n \leq (q;q)_n \quad \text{and} \quad (1-q^\nu)^n \leq (q^{\nu+1};q)_n \tag{29}$$

take place for $q \in (0,1)$ and $\nu > -1$, by using the triangle inequality and its consequences we find that:

$$\left|\frac{z}{h_\nu^{(3)}(z;q)}\right| \leq \frac{(1-q)(1-q^\nu)-\sqrt{q}}{(1-q)(1-q^\nu)-2\sqrt{q}}, \tag{30}$$

$$\left|[h_\nu^{(3)}(z;q)]' - \frac{h_\nu^{(3)}(z;q)}{z}\right| \le \frac{\sqrt{q}(1-q)(1-q^\nu)}{[(1-q)(1-q^\nu)-\sqrt{q}]^2}, \tag{31}$$

$$\left|\frac{1}{[h_\nu^{(3)}(z;q)]'}\right| \le \frac{[b(1-q^\nu)-\sqrt{q}]^2}{2q-4\sqrt{q}b(1-q^\nu)+b^2(1-q^\nu)^2}, \tag{32}$$

$$\left|z[h_\nu^{(3)}(z;q)]''\right| \le \frac{2\sqrt{q}b^2(1-q^\nu)^2}{[b(1-q^\nu)-\sqrt{q}]^3},\ z \in U. \tag{33}$$

By combining the inequalities (30) and (31) we get (27). Also, from (32) and (33) we deduce that (28) holds. □

2. Main results

Theorem 2.1. *Let n be a natural number,* $\nu_1, \dots, \nu_n > -1$, $\nu = \min\{\nu_1, \dots, \nu_n\}$, $q \in (0,1)$ *and* $\sigma_1, \dots \sigma_n, \delta \in \mathbb{C}$, *with* $\Re(\delta) > 0$.

(i) Consider the functions

$$h_{\nu_i}^{(2)}(z;q) = 2^{\nu_i}\frac{(q;q)_\infty}{(q^{\nu_i+1};q)_\infty} z^{1-\frac{\nu_i}{2}} J_{\nu_i}^{(2)}(\sqrt{z};q),\ i \in \{1,\dots,n\}. \tag{34}$$

If

$$4(1-q)(1-q^{\nu_i}) > q^{\nu_i},\ i \in \{1,\dots,n\}, \tag{35}$$

$$(2\Re(\delta)+1)^{\frac{2\Re(\delta)+1}{2\Re(\delta)}} \cdot \frac{[4(1-q)(1-q^\nu)-q^\nu]^2}{32(1-q)^2(1-q^\nu)^2} \ge \sum_{i=1}^n |\sigma_i|, \tag{36}$$

then for any complex number λ, *with* $\Re(\lambda) \ge \Re(\delta)$, *the function*

$$H_{\nu_1,\dots,\nu_n}^{(2)}(z) = \left\{\lambda \int_0^z t^{\lambda-1} \prod_{i=1}^n \left[e^{h_{\nu_i}^{(2)}(t;q)}\right]^{\sigma_i} dt\right\}^{1/\lambda}, \quad z \in U, \tag{37}$$

is in the class S.

(ii) Consider the functions

$$h_{\nu_i}^{(3)}(z;q) = \frac{(q;q)_\infty}{(q^{\nu_i+1};q)_\infty} z^{1-\frac{\nu_i}{2}} J_{\nu_i}^{(3)}(\sqrt{z};q),\ i \in \{1,\dots,n\}. \tag{38}$$

If

$$(1-q)(1-q^{\nu_i}) > \sqrt{q},\ i \in \{1,\dots,n\}, \tag{39}$$

and

$$(2\Re(\delta)+1)^{\frac{2\Re(\delta)+1}{2\Re(\delta)}} \cdot \frac{[(1-q)(1-q^\nu)-\sqrt{q}]^2}{2(1-q)^2(1-q^\nu)^2} \ge \sum_{i=1}^n |\sigma_i|, \tag{40}$$

then for any complex number λ, *with* $\Re(\lambda) \ge \Re(\delta)$, *the function*

$$H_{\nu_1,\dots,\nu_n}^{(3)}(z) = \left\{\lambda \int_0^z t^{\lambda-1} \prod_{i=1}^n \left[e^{h_{\nu_i}^{(3)}(t;q)}\right]^{\sigma_i} dt\right\}^{1/\lambda}, \quad z \in U, \tag{41}$$

is in the class S.

Proof. (i) We consider the function

$$\varphi(z) = \int_0^z \prod_{i=1}^n \left[e^{h_{\nu_i}^{(2)}(t;q)}\right]^{\sigma_i} dt. \tag{42}$$

Therefore, we obtain

$$\frac{\varphi''(z)}{\varphi'(z)} = \sum_{i=1}^n \sigma_i \cdot (h_{\nu_i}^{(2)}(z;q))'. \tag{43}$$

Next, using the inequality (17) for all the functions $h_{\nu_1}^{(2)}, \ldots, h_{\nu_n}^{(2)}$, we get

$$\left|\frac{\varphi''(z)}{\varphi'(z)}\right| \le \sum_{i=1}^n |\sigma_i| \cdot \left[\frac{4(1-q)(1-q^{\nu_i})}{4(1-q)(1-q^{\nu_i}) - q^{\nu_i}}\right]^2. \tag{44}$$

Since the function

$$T(x) = \left[\frac{4(1-q)(1-q^x)}{4(1-q)(1-q^x) - q^x}\right]^2, \; x > 0 \tag{45}$$

is decreasing, we find that

$$\left[\frac{4(1-q)(1-q^{\nu_i})}{4(1-q)(1-q^{\nu_i}) - q^{\nu_i}}\right]^2 \le \left[\frac{4(1-q)(1-q^{\nu})}{4(1-q)(1-q^{\nu}) - q^{\nu}}\right]^2, \tag{46}$$

which implies

$$\frac{1-|z|^{2\Re(\delta)}}{\Re(\delta)} \left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \le \frac{1-|z|^{2\Re(\delta)}}{\Re(\delta)} \cdot |z| \cdot \left[\frac{4(1-q)(1-q^{\nu})}{4(1-q)(1-q^{\nu}) - q^{\nu}}\right]^2 \sum_{i=1}^n |\sigma_i|. \tag{47}$$

Because

$$\max_{|z|\le 1} \frac{1-|z|^{2\Re(\delta)}}{\Re(\delta)} \cdot |z| = \frac{2}{(2\Re(\delta)+1)^{\frac{2\Re(\delta)+1}{2\Re(\delta)}}}, \tag{48}$$

we have

$$\frac{1-|z|^{2\Re(\delta)}}{\Re(\delta)} \left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \le \frac{2}{(2\Re(\delta)+1)^{\frac{2\Re(\delta)+1}{2\Re(\delta)}}} \left[\frac{4(1-q)(1-q^{\nu})}{4(1-q)(1-q^{\nu}) - q^{\nu}}\right]^2 \sum_{i=1}^n |\sigma_i|. \tag{49}$$

Finally, using (36) in the last inequality and applying Pascu univalence criteria, we obtain that $H_{\nu_1,\ldots,\nu_n}^{(2)} \in S$.
(ii) Similarly, employing (26) we find that $H_{\nu_1,\ldots,\nu_n}^{(3)} \in S$. □

Theorem 2.2. *Let $q \in (0,1)$, $\nu_1, \ldots, \nu_n > -1$, $\nu = \min\{\nu_1, \ldots, \nu_n\}$, $\alpha_1, \ldots, \alpha_n, c, \mu \in \mathbb{C}$, with $c \neq -1$ and $\Re(\mu) > 0$.*
(i) If the functions $h_{\nu_i}^{(2)}$ defined in (34) satisfy

$$2(1-q)(1-q^{\nu_i}) > q^{\nu_i}, \; i \in \{1, \ldots, n\}, \tag{50}$$

and

$$|c| + \frac{1}{|\mu|} \frac{4q^{\nu}(1-q)(1-q^{\nu})}{[4(1-q)(1-q^{\nu}) - q^{\nu}][4(1-q)(1-q^{\nu}) - 2q^{\nu}]} \sum_{i=1}^n |\alpha_i| \le 1, \tag{51}$$

then the function

$$I^{(2)}_{\nu_1,\dots,\nu_n}(z) = \left\{\mu \int_0^z t^{\mu-1} \prod_{i=1}^{n} \left[\frac{h^{(2)}_{\nu_i}(t;q)}{t}\right]^{\alpha_i} dt\right\}^{1/\mu}, \quad z \in U, \tag{52}$$

is in the class S.

(ii) If the functions $h^{(3)}_{\nu_i}$ defined in (38) satisfy

$$(1-q)(1-q^{\nu_i}) > 2\sqrt{q},\ i \in \{1,\dots,n\}, \tag{53}$$

and

$$|c| + \frac{1}{|\mu|}\frac{\sqrt{q}(1-q)(1-q^{\nu})}{[(1-q)(1-q^{\nu}) - \sqrt{q}][(1-q)(1-q^{\nu}) - 2\sqrt{q}]}\sum_{i=1}^{n}|\alpha_i| \le 1, \tag{54}$$

then the function

$$I^{(3)}_{\nu_1,\dots,\nu_n}(z) = \left\{\mu \int_0^z t^{\mu-1} \prod_{i=1}^{n} \left[\frac{h^{(3)}_{\nu_i}(t;q)}{t}\right]^{\alpha_i} dt\right\}^{1/\mu}, \quad z \in U, \tag{55}$$

is in the class S.

Proof. (i) Let the function

$$\phi(z) = \int_0^z \prod_{i=1}^{n} \left[\frac{h^{(2)}_{\nu_i}(t;q)}{t}\right]^{\alpha_i} dt. \tag{56}$$

Thus, we have

$$\frac{z\phi''(z)}{\phi'(z)} = \sum_{i=1}^{n}\alpha_i\left[\frac{z[h^{(2)}_{\nu_i}(z;q)]'}{h^{(2)}_{\nu_i}(z;q)} - 1\right]. \tag{57}$$

Next, using the inequality (18) for all the functions $h^{(2)}_{\nu_1},\dots,h^{(2)}_{\nu_n}$, we get

$$\left|\frac{z\phi''(z)}{\phi'(z)}\right| \le \sum_{i=1}^{n}|\alpha_i|\,\frac{4q^{\nu_i}(1-q)(1-q^{\nu_i})}{[4(1-q)(1-q^{\nu_i}) - q^{\nu_i}][4(1-q)(1-q^{\nu_i}) - 2q^{\nu_i}]}. \tag{58}$$

Because the function

$$g(x) = \frac{4q^{x}(1-q)(1-q^{x})}{[4(1-q)(1-q^{x}) - q^{x}][4(1-q)(1-q^{x}) - 2q^{x}]}, \tag{59}$$

is decreasing, by using the triangle inequality and the assertion of Theorem 2, we find that

$$\begin{aligned} &\left|c|z|^{2\mu} + \left(1 - |z|^{2\mu}\right)\frac{z\phi''(z)}{\mu\phi'(z)}\right| \le \\ &|c| + \frac{1}{|\mu|}\frac{4q^{\nu}(1-q)(1-q^{\nu})}{[4(1-q)(1-q^{\nu}) - q^{\nu}][4(1-q)(1-q^{\nu}) - 2q^{\nu}]}\sum_{i=1}^{n}|\alpha_i| \le 1. \end{aligned} \tag{60}$$

Finally, applying Pescar univalence criteria, we obtain that $I^{(2)}_{\nu_1,\dots,\nu_n} \in S$.
(ii) Similarly, employing (27) we find that $I^{(3)}_{\nu_1,\dots,\nu_n} \in S$. □

Theorem 2.3. *Let n be a natural number, $\nu_1, \ldots, \nu_n > -1$, $\nu = \min\{\nu_1, \ldots, \nu_n\}$, $q \in (0,1)$ and let $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$.*

(i) If the functions $h^{(2)}_{\nu_i}$ defined in (34) satisfy

$$2(1-q)(1-q^{\nu_i}) > q^{\nu_i},\ i \in \{1, \ldots, n\}, \tag{61}$$

and

$$\frac{4q^{\nu}(1-q)(1-q^{\nu})}{[4(1-q)(1-q^{\nu}) - q^{\nu}][4(1-q)(1-q^{\nu}) - 2q^{\nu}]} \le \frac{\Re(\beta)}{n|\beta|}, \tag{62}$$

then the function

$$F^{(2)}_{\nu_1,\ldots,\nu_n}(z) = \left\{(n\beta+1)\int_0^z \prod_{i=1}^{n} \left[h^{(2)}_{\nu_i}(t;q)\right]^{\beta} dt\right\}^{1/(n\beta+1)}, \quad z \in U, \tag{63}$$

is in the class S.

(ii) If the functions $h^{(3)}_{\nu_i}$ defined in (38) satisfy

$$(1-q)(1-q^{\nu_i}) > 2\sqrt{q},\ i \in \{1, \ldots, n\}, \tag{64}$$

and

$$\frac{\sqrt{q}(1-q)(1-q^{\nu})}{[(1-q)(1-q^{\nu}) - \sqrt{q}][(1-q)(1-q^{\nu}) - 2\sqrt{q}]} \le \frac{\Re(\beta)}{n|\beta|}, \tag{65}$$

then the function

$$F^{(3)}_{\nu_1,\ldots,\nu_n}(z) = \left\{(n\beta+1)\int_0^z \prod_{i=1}^{n} \left[h^{(3)}_{\nu_i}(t;q)\right]^{\beta} dt\right\}^{1/(n\beta+1)}, \quad z \in U, \tag{66}$$

is in the class S.

Proof. *(i) We define a regular function*

$$I_\beta(z) = \int_0^z \prod_{i=1}^{n} \left[\frac{h^{(2)}_{\nu_i}(t;q)}{t}\right]^{\beta} dt. \tag{67}$$

Therefore, we get

$$I'_\beta(z) = \prod_{i=1}^{n} \left[\frac{h^{(2)}_{\nu_i}(z;q)}{z}\right]^{\beta}. \tag{68}$$

Differentiating both sides of (68) logarithmically, we obtain

$$\frac{zI''_\beta(z)}{I'_\beta(z)} = \beta \sum_{i=1}^{n} \left[\frac{z[h^{(2)}_{\nu_i}(z;q)]'}{h^{(2)}_{\nu_i}(z;q)} - 1\right]. \tag{69}$$

Hence, using (18) for each function $h^{(2)}_{\nu_1}, h^{(2)}_{\nu_2}, \ldots, h^{(2)}_{\nu_n}$, we find that

$$\left|\frac{zI''_\beta(z)}{I'_\beta(z)}\right| \le |\beta| \sum_{i=1}^{n} \left|\frac{z[h^{(2)}_{\nu_i}(z;q)]'}{h^{(2)}_{\nu_i}(z;q)} - 1\right|$$
$$\le |\beta| \sum_{i=1}^{n} \frac{4q^{\nu_i}(1-q)(1-q^{\nu_i})}{[4(1-q)(1-q^{\nu_i}) - q^{\nu_i}][4(1-q)(1-q^{\nu_i}) - 2q^{\nu_i}]}. \tag{70}$$

Since the function

$$r(x) = \frac{4q^x(1-q)(1-q^x)}{[4(1-q)(1-q^x)-q^x][4(1-q)(1-q^x)-2q^x]} \tag{71}$$

is decreasing, taking into account the hypothesis of the theorem, we obtain

$$\frac{1-|z|^{2\Re(\beta)}}{\Re(\beta)}\left|\frac{zI''_\beta(z)}{I'_\beta(z)}\right| \leq \frac{n|\beta|}{\Re(\beta)} \cdot \frac{4q^\nu(1-q)(1-q^\nu)}{[4(1-q)(1-q^\nu)-q^\nu][4(1-q)(1-q^\nu)-2q^\nu]} \leq 1. \tag{72}$$

Because $\Re(n\beta+1) > \Re(\beta)$ *and the fact that* $F^{(2)}_{\nu_1,\ldots,\nu_n}$ *can be rewritten*

$$F^{(2)}_{\nu_1,\ldots,\nu_n}(z) = \left\{(n\beta+1)\int_0^z t^{n\beta}\prod_{i=1}^n\left[\frac{h^{(2)}_{\nu_i}(t;q)}{t}\right]^\beta dt\right\}^{1/(n\beta+1)}, \quad z \in U, \tag{73}$$

applying Pascu univalence criterion, we obtain that $F^{(2)}_{\nu_1,\ldots,\nu_n}(z) \in S$.
(ii) Similarly, employing (27) we find that $F^{(3)}_{\nu_1,\ldots,\nu_n}(z) \in S$. □

Theorem 2.4. *Let n be a natural number,* $\nu_1,\ldots,\nu_n > -1$, $\nu = \min\{\nu_1,\ldots,\nu_n\}$, $q \in (0,1)$ *and let* $\gamma_1,\ldots,\gamma_n,\delta \in \mathbb{C}$ *with* $\Re(\delta) > 0$.

(i) If the functions $h^{(2)}_{\nu_i}$ *defined in (34) satisfy*

$$2(1-q)(1-q^{\nu_i}) > q^{\nu_i},\ i \in \{1,\ldots,n\}, \tag{74}$$

and

$$\Re(\delta) \geq \frac{2q^\nu a^2(1-q^\nu)^2}{[a(1-q^\nu)-q^\nu]\cdot[2q^{2\nu}-4q^\nu a(1-q^\nu)+a^2(1-q^\nu)^2]}\sum_{i=1}^n|\gamma_i|, \tag{75}$$

where $a = 4(1-q)$*, then for any complex number* η*, with* $\Re(\eta) \geq \Re(\delta)$*, the function*

$$G^{(2)}_{\nu_1,\ldots,\nu_n}(z) = \left\{\eta\int_0^z t^{\eta-1}\prod_{i=1}^n\left[(h^{(2)}_{\nu_i}(t;q))'\right]^{\gamma_i} dt\right\}^{1/\eta}, \quad z \in U, \tag{76}$$

is in the class S.

(ii) If the functions $h^{(3)}_{\nu_i}$ *defined in (38) satisfy*

$$(1-q)(1-q^{\nu_i}) > 2\sqrt{q},\ i \in \{1,\ldots,n\}, \tag{77}$$

and

$$\Re(\delta) \geq \frac{2\sqrt{q}b^2(1-q^\nu)^2}{[b(1-q^\nu)-\sqrt{q}]\cdot[2q-4\sqrt{q}b(1-q^\nu)+b^2(1-q^\nu)^2]}\sum_{i=1}^n|\gamma_i|, \tag{78}$$

where $b = 1-q$*, then for any complex number* η*, with* $\Re(\eta) \geq \Re(\delta)$*, the function*

$$G^{(3)}_{\nu_1,\ldots,\nu_n}(z) = \left\{\eta\int_0^z t^{\eta-1}\prod_{i=1}^n\left[(h^{(3)}_{\nu_i}(t;q))'\right]^{\gamma_i} dt\right\}^{1/\eta}, \quad z \in U, \tag{79}$$

is in the class S.

Proof. (i) We define the regular function

$$p(z) = \int_0^z \prod_{i=1}^{n} \left[h_{\nu_i}^{(2)}(t;q))\right]^{\gamma_i} dt. \tag{80}$$

and we find that

$$\left|\frac{zp''(z)}{p'(z)}\right| \leq \sum_{i=1}^{n} |\gamma_i| \cdot \left|\frac{z[h_{\nu_i}^{(2)}(z;q)]''}{[h_{\nu_i}^{(2)]}(z;q)]'}\right|. \tag{81}$$

In view of the fact that

$$s(x) = \frac{2q^x a^2(1-q^x)^2}{[a(1-q^x)-q^x]\cdot[2q^{2x}-4q^x a(1-q^x)+a^2(1-q^x)^2]} \tag{82}$$

is decreasing, multiplying both sides of (81) with $\frac{1-|z|^{2\Re(\delta)}}{\Re(\delta)}$ and taking into account the relation (19), we get

$$\frac{1-|z|^{2\Re(\delta)}}{\Re(\delta)}\left|\frac{zp''(z)}{p'(z)}\right| \leq 1. \tag{83}$$

Hence, applying Pascu uivalence criterion we deduce that $G_{\nu_1,\ldots,\nu_n}^{(2)}(z) \in S$.
(ii) Similarly, employing (28) we find that $G_{\nu_1,\ldots,\nu_n}^{(3)}(z) \in S$. □

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