On Linear Operators for Which TT^D Is Normal

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Abstract. A Drazin invertible operator $T \in \mathcal{B}(\mathcal{H})$ is called skew D-quasi-normal operator if T^* and TT^D commute or equivalently TT^D is normal. In this paper, firstly we give a list of conditions on an operator T, each of which is equivalent to T being skew D-quasi-normal. Furthermore, we obtain the matrix representation of these operators. We also develop some basic properties of such operators. Secondly we extend the Kaplansky theorem and the Fuglede-Putnam commutativity theorem for normal operators to skew D-quasi-normal matrices.

1. Introduction

The symbol $\mathcal{B}(\mathcal{H})$ stands for the algebra of bounded linear operators on Hilbert space $\mathcal{H}$, over the field $\mathbb{C}$ of complex numbers. As usual, $I = I_{\mathcal{H}}$ denotes the identity operator. For $T \in \mathcal{B}(\mathcal{H})$, let T^* denote its adjoint, $\mathcal{N}(T)$ its nullity, $\mathcal{R}(T)$ its range and $\sigma(T)$ its spectrum. Let $\mathcal{H}$ be a finite complex Hilbert space, we identify $\mathcal{B}(\mathcal{H})$ with the space $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ complex matrices in the natural way. For $T \in \mathcal{M}_n(\mathbb{C})$, denote by T^T and $\overline{T}$ the transpose and the conjugate of T, respectively. A closed subspace $\mathcal{M} \subset \mathcal{H}$ is said to be invariant for an operator $T \in \mathcal{B}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$, and in this situation we denote by $T \mid_{\mathcal{M}}$ the restriction of T to $\mathcal{M}$.

Throughout this paper, we need some notations. Let $T = U + iV$, where $U = ReT = \frac{T+T^*}{2}$ and $V = ImT = \frac{T-T^*}{2i}$ are the real and imaginary parts of T. We shall write $B^2 = TT^*$ and $C^2 = T^*T$, where B and C are non-negative definite.

The famous Fuglede-Putnam's theorem is as follows: the operator equation $SX = XT$ implies $S^*X = XT^*$ when S and T are normal operators. This theorem is a very useful tool when dealing with products (and even sums) involving normal operators. First, Fuglede [8] proved it in the case when $S = T$ and then Putnam [12] proved it in a general case. For works related to products of normal matrices and operators, the reader may consult [7, 10].

For bounded linear operators, the Drazin inverse was introduced and studied by Caradus [2]. It is shown that the Drazin inverse has proved helpful in analyzing Markov chains, difference equation, differential equations, Cauchy problems and iterative procedures [1, 3, 11, 13, 14].

Now, we recall some definitions and basic fact about the Drazin inverse. For $T \in \mathcal{B}(\mathcal{H})$, if there exists an operator $T^D \in \mathcal{B}(\mathcal{H})$ satisfying the following three operator equations:

$$TT^D = T^DT, \qquad T^DTT^D = T^D, \qquad T^{k+1}T^D = T^k$$

where $k = ind(T)$, the index of T, is the smallest nonnegative integer for which $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$ and $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$, then T^D is called a Drazin inverse of T. In particular, when $ind(T) = 1$, the operator T^D is called the group inverse of T, and is denoted by $T^\sharp$. Clearly, $ind(T) = 0$ if and only if T is invertible and in this case $T^D = T^{-1}$. For $T \in \mathcal{B}(\mathcal{H})$, it is well known that the Drazin inverse T^D of T is unique if it exists and $(T^*)^D = (T^D)^*$. If T is Drazin invertible, then the spectral idempotent T^π of T corresponding to $\{0\}$ is given by $T^\pi = I - TT^D$. We note that if T is nilpotent, then it is Drazin invertible, $T^D = 0$, and $ind(T) = r$, where r is the power of nilpotency of T.

We require a few preliminary lemmas and definitions.

Lemma 1.1 ([1]). *Let $S, T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible.*

1. *If $R \in \mathcal{B}(\mathcal{H})$ is an invertible operator, then $R^{-1}TR$ is Drazin invertible and $(R^{-1}TR)^D = R^{-1}T^DR$.*
2. *ST is Drazin invertible if and only if TS is Drazin invertible, $ind(ST) \leq ind(TS) + 1$ and $(ST)^D = S[(TS)^D]^2T$.*
3. *If S is idempotent, then $S^D = S$.*
4. *If $ST = TS$, then $(ST)^D = T^DS^D = S^DT^D$, $S^DT = TS^D$ and $ST^D = T^DS$.*
5. *If $ST = TS = 0$, then $(S + T)^D = S^D + T^D$.*

Lemma 1.2 ([1]). *If $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ are Drazin invertible with $ind(A) = m$ and $ind(B) = n$. Then $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is also Drazin invertible and*

$$M^D = \begin{pmatrix} A^D & X \\ 0 & B^D \end{pmatrix},$$

where

$$X = \sum_{i=0}^{n-1} (A^D)^{i+2} C B^i B^\pi + A^\pi \sum_{i=0}^{m-1} A^i C (B^D)^{i+2} - A^D C B^D. \tag{1}$$

In [4], the authors introduced the classes of D-normal operators and D-quasi-normal operators as a generalization of the classes of normal operators and quasi-normal operators, respectively. For more details we refer the reader to [5]. The class of skew D-quasi-normal operators was defined by Dana and Yousefi [6] as a generalization of the class of D-normal operators.

In this paper, our main goal is to further study the skew D-quasi-normal operators. The paper is carried out as follows. In Section 2, first of all, we investigate bounded linear operators T on a Hilbert space $\mathcal{H}$ for which T^* and TT^D commute or equivalently T^π is normal. Secondly we give a list of conditions on an operator T, each of which is equivalent to T being skew D-quasi-normal. We obtain the matrix representation of these operators. We also develop some basic properties of this class. In Section 3, we generalize the famous result on products of normal operators, due to I. Kaplansky, to skew D-quasi-normal matrices. Also, we use the Fuglede-Putnam theorem to prove that, for matrices S and T, if ST is skew D-quasi-normal, then ST is skew D-quasi-normal if and only if $S^*(ST)(ST)^D = (TS)(TS)^DS^*$. We use this theorem and we show that both ST and TS are skew D-quasi-normal if and only if $S^*(ST)(ST)^D = (TS)(TS)^DS^*$ and $(ST)^D(ST)T^* = T^*(TS)(TS)^D$. In addition, we deduce a result relating the factors in a polar decomposition of S to the skew D-quasi-normality of ST and TS. Finally, we extend Fuglede-Putnam theorem form normal operators to skew D-quasi-normal matrices.

2. Skew D-quasi-normal operators

In this section, we give some equivalent conditions for the skew D-quasi-normality of an operator. Also, we investigate some basic properties of the class of skew D-quasi-normal operators. Furthermore, we obtain the matrix representation of these operators.

Definition 2.1. *[6] Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. T is said skew D-quasi-normal if*

$$T^*TT^D = TT^DT^*.$$

The class of all skew D-quasi-normal operators is denoted by $[\mathfrak{SD}]$.

Dana and Yousefi [6] presented eight conditions on an operator T, each of which is equivalent to T being skew D-quasi-normal. We give other equivalent conditions for the skew D-quasi-normality of T. We compile here the full list of these conditions.

Theorem 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible and $F = TT^D + T^*$, $G = TT^D - T^*$. Then the following conditions are equivalent:*

1. *T is of class $[\mathfrak{SD}]$;*
2. *$T^{*n}TT^D = TT^DT^{*n}$, for each $n \in \mathbb{N}$;*
3. *TT^D is of class $[N]$;*
4. *T^π is of class $[N]$;*
5. *$T^\pi T^* = T^* T^\pi$;*
6. *TT^D commutes with ReT;*
7. *TT^D commutes with ImT;*
8. *G commutes with F;*
9. *TT^D commutes with F;*
10. *TT^D commutes with G;*
11. *$TT^DF + FT^* = F^2$;*
12. *$TT^DG - GT^* = G^2$.*

Lemma 2.3. *Suppose $T \in [\mathfrak{SD}]$ then $\mathcal{R}(T^D)$ reduces T.*

Proof. Since $T \in [\mathfrak{SD}]$, $TT^DT^* = T^*TT^D$. $\mathcal{R}(T^D)$ is invariant under T is obvious. We shall show that $\mathcal{R}(T^D)$ is invariant under T^*. Let $x \in \mathcal{R}(T^D)$. Then $x = T^Dy$ for some $y \in \mathcal{H}$ and

$$T^*x = T^*T^Dy = T^*TT^DT^Dy = T^DTT^*T^Dy \in \mathcal{R}(T^D).$$

Thus $\mathcal{R}(T^D)$ is invariant under T^* and $\mathcal{R}(T^D)$ reduces T. $\square$

The Lemma 2.3 enables us to give the matrix representation of skew D-quasi-normal operators.

Theorem 2.4. *If T is of class $[\mathfrak{SD}]$, then T has the following matrix representation,*

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

on $\mathcal{H} = \mathcal{R}(T^D) \oplus \mathcal{N}(T^D)$ where $T_1 = T|_{\mathcal{R}(T^D)}$ is also of class $[\mathfrak{SD}]$ and T_2 is a nilpotent operator with nilpotency $ind(T_2)$. Futhermore $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. By Lemma 2.3, $\mathcal{R}(T^D)$ reduces T. Hence T has the matrix representation, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ on $\mathcal{H} = \mathcal{R}(T^D) \oplus \mathcal{N}(T^{*D})$. We note that since $T \in [\mathfrak{SD}]$, then $\mathcal{N}(T^D) = \mathcal{N}(T^{*D})$. Let P be the orthogonal projection onto $\mathcal{R}(T^D)$. Then

$$\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PT = PTP.$$

Hence

$$P(TT^DT^*)P = \begin{pmatrix} T_1 T_1^D T_1^* & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$P(T^*TT^D)P = \begin{pmatrix} T_1^*T_1T_1^D & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $T \in [\mathfrak{S}\mathfrak{D}]$, $P(T^*TT^D)P = P(T^DT^*T)P$, implying $T_1^*T_1T_1^D = T_1^DT_1^*T_1$. Hence $T_1 \in [\mathfrak{S}\mathfrak{D}]$.

For any $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathcal{H}$,

$$< T_2^D z_2, z_2 > = < T^D(I - P)z, (I - P)z > = 0.$$

Therefore $T_2^D = 0$. Then T_2 is a nilpotent operator. Since $\mathcal{R}(T^D)$ reduces T, $\sigma(T) = \sigma(T_1) \cup \sigma(T_2) = \sigma(T_1) \cup \{0\}$. $\quad\square$

In the next we add without proof some important properties of the class of skew D-quasi-normal operators (for more details we refer the reader to [6]).

Theorem 2.5. *If $T \in [\mathfrak{S}\mathfrak{D}]$, then*

1. *T^* is of class $[\mathfrak{S}\mathfrak{D}]$.*
2. *T^D is of class $[\mathfrak{S}\mathfrak{D}]$.*
3. *If $S \in \mathcal{B}(\mathcal{H})$ is Drazin invertible and unitary equivalent to T, then S is of class $[\mathfrak{S}\mathfrak{D}]$.*
4. *If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{M}$ reduces T and T^D, then $S = T|_{\mathcal{M}}$ is of class $[\mathfrak{S}\mathfrak{D}]$.*
5. *The direct sum and the tensor product of two operators in $[\mathfrak{S}\mathfrak{D}]$ are in $[\mathfrak{S}\mathfrak{D}]$.*
6. *If T and S are of class $[\mathfrak{S}\mathfrak{D}]$ such that $[T, S] = 0$, then TS is of class $[\mathfrak{S}\mathfrak{D}]$.*
7. *If T is of class $[\mathfrak{S}\mathfrak{D}]$, then T^m is of class $[\mathfrak{S}\mathfrak{D}]$ for any positive integer m.*
8. *If S and T are of class $[\mathfrak{S}\mathfrak{D}]$ such that $ST = TS = 0$, then $S + T$ is of class $[\mathfrak{S}\mathfrak{D}]$.*

Remark 2.6. *All nonzero nilpotent operators are of class $[\mathfrak{S}\mathfrak{D}]$. However they are not normal.*

Theorem 2.7. *Let T is of class $[\mathfrak{S}\mathfrak{D}]$ and $C^2T^D = T^DC^2$. Then B commutes with ReT^D and ImT^D.*

Proof. Since $C^2T^D = T^DC^2$ we have $T^*TT^D = T^DT^*T$. Hence $(T^*)^DT^*T = T^*T(T^*)^D$.
Now

$$\begin{aligned} B^2ReT^D &= 1/2[TT^*(T^D + T^{D^*})] \\ &= 1/2[TT^*T(T^D)^2 + TT^{D^*}T^*] \\ &= 1/2[T(T^D)^2T^*T + TT^*(T^{D^*})^2T^*] \\ &= 1/2[T^*TT^D + T^{D^*}T^*TT^{D^*}T^*] \\ &= 1/2[T^DTT^* + (T^{D^*})^2T^*TT^*] \\ &= 1/2[T^DTT^* + T^{D^*}TT^*] \\ &= ReT^DB^2. \end{aligned}$$

Hence $BReT^D = ReT^DB$. Similarly $BImT^D = ImT^DB$. $\quad\square$

Theorem 2.8. *Let T is of class $[\mathfrak{S}\mathfrak{D}]$ and $C^2T^D = B^2T^D$. Then B commutes with ReT^D and ImT^D.*

Proof. Since $C^2T^D = B^2T^D$ we have $T^*TT^D = TT^*T^D$. And since T is of class $[\mathfrak{S}\mathfrak{D}]$ we have $T^*TT^D = T^DTT^*$. Hence $TT^*T^D = T^DTT^*$. Now

$$\begin{aligned} B^2ReT^D &= 1/2[TT^*(T^D + T^{D^*})] \\ &= 1/2[TT^DT^* + T^{D^*}TT^*] \\ &= ReT^DB^2. \end{aligned}$$

Hence $BReT^D = ReT^DB$. Similarly $BImT^D = ImT^DB$. $\quad\square$

Theorem 2.9. *Suppose that T is an operator such that*

1. *B commutes with ReT^D and ImT^D*
2. $C^2T^D = B^2T^D$

then $T \in [\mathfrak{S}\mathfrak{D}]$.

Proof. Since B commutes with ReT^D and ImT^D we have

$$B^2T^D + B^2T^{*D} = T^DB^2 + T^{*D}B^2$$

$$B^2T^D - B^2T^{*D} = T^DB^2 - T^{*D}B^2.$$

So we have

$$T^DB^2 = B^2T^D = C^2T^D.$$

The proof is comlete. □

3. Skew D-quasi-normal matrices

In what follows for this section we suppose that all of operators are in $\mathcal{M}_n(\mathbb{C})$.

Proposition 3.1. *Let $T \in [\mathfrak{S}\mathfrak{D}]$. Then $TT^D\overline{T} = \overline{T}TT^D$ if and only if $TT^DT^T = T^TTT^D$.*

Proof. Suppose that $TT^D\overline{T} = \overline{T}TT^D$. Then $TT^DT^T = T^TTT^D$ by Fuglede theorem [8]. In a similar way, we see that if $TT^DT^T = T^TTT^D$, then $TT^D\overline{T} = \overline{T}TT^D$, so these two statements are equivalent when T is of class $[\mathfrak{S}\mathfrak{D}]$. □

Lemma 3.2. *Let $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{M}_n(\mathbb{C})$ (A and B are square matrices). Then M is of class $[\mathfrak{S}\mathfrak{D}]$ if and only if A and B are of class $[\mathfrak{S}\mathfrak{D}]$ and $AX + CB^D = 0$, where X is defined by (1).*

Proof. Let M be of class $[\mathfrak{S}\mathfrak{D}]$, then MM^D is normal. Using condition 8 in [9], we have AA^D and BB^D are of class $[N]$ and $AX + CB^D = 0$. In a similar way, we see that if A and B are of class $[\mathfrak{S}\mathfrak{D}]$ and $AX + CB^D = 0$, then MM^D is normal. □

Corollary 3.3. *Let $M = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ where $a, b, c \in \mathbb{C}$ and $a, b, c \neq 0$. Then M is of class $[\mathfrak{S}\mathfrak{D}]$.*

It is well known that every $T \in \mathcal{M}_n(\mathbb{C})$ has a polar decomposition as $T = UP$ where $P \in \mathcal{M}_n(\mathbb{C})$ is positive semidefinite Hermitian and $U \in \mathcal{M}_n(\mathbb{C})$ is unitary. We have the following theorem.

Theorem 3.4. *If $T = UP$ is the polar decomposition of T, then the following conditions are equivalent:*

1. *T is of class $[\mathfrak{S}\mathfrak{D}]$.*
2. $(UP)(UP)^D = (PU)(PU)^D$.
3. $TT^DU = UTT^D$.

Proof. $(1 \Leftrightarrow 2)$: If $(UP)(UP)^D = (PU)(PU)^D$ then

$$\begin{aligned} T^DTT^* &= (UP)^D(UP)P^*U^* \\ &= (PU)(PU)^DP^*U^* & \text{(By hypotheses)} \\ &= PUP(UP)^DU^* & \text{(by Lemma 1.1)} \\ &= P(PU)(PU)^DU^* & \text{(By hypotheses)} \\ &= P^2UP((UP)^D)^2 & \text{(by Lemma 1.1)} \\ &= P^2(UP)^D. & (2) \end{aligned}$$

That is

$$T^D T^* = P^2 (UP)^D. \tag{3}$$

On the other hand

$$
\begin{aligned}
T^* T T^D &= P^* U^* (UP)(UP)^D \\
&= P^* U^* (UP) U ((PU)^D)^2 P && \text{(by Lemma 1.1)} \\
&= P^* (PU)^D P \\
&= P^* P (UP)^D && \text{(by Lemma 1.1)} \\
&= P^2 (UP)^D.
\end{aligned}
$$

That is

$$T^* T^D = P^2 ((UP)^D)^2. \tag{4}$$

From (3) and (4), we conclude $T \in [\mathfrak{S}\mathfrak{D}]$.

Conversely, if T is skew D-quasi-normal, then $TT^D P^2 = P^2 TT^D$ $(P^2 = T^* T)$. Thus $TT^D P = P TT^D$. So

$$((UP)(UP)^D - (PU)(PU)^D)P = TT^D P - PTT^D = 0.$$

Thus $((UP)(UP)^D - (PU)(PU)^D) = 0$ on $\mathcal{R}(P)$. But if $f \in \mathcal{R}(P)^\perp = \mathcal{N}(P)$ then since $\mathcal{N}(P) = \mathcal{N}(U)$, we have $Uf = 0$. Therefore $(UP)(UP)^D = (PU)(PU)^D$.

$(2 \Leftrightarrow 3)$: If $(UP)(UP)^D = (PU)(PU)^D$, then

$$
\begin{aligned}
TT^D U &= (UP)(UP)^D U \\
&= (UP) U (PU)^D \\
&= U (UP)(UP)^D \\
&= UTT^D.
\end{aligned}
$$

Conversely, if $TT^D U = UTT^D$ then $(UP)(UP)^D U = U(UP)(UP)^D$. So we have $U(PU)(PU)^D = U(UP)(UP)^D$. Then multiplying by U^* we see that $(UP)(UP)^D = (PU)(PU)^D$. $\square$

We are mainly interested in generalizing the following famous result on products of normal operators, due to I. Kaplansky, to skew D-quasi-normal matrices:

Theorem 3.5 ([10]). *Let S and T be two bounded operators on a Hilbert space such that ST and S are normal. Then T commutes with SS^* iff TS is normal.*

We have the following Kaplansky-like theorem:

Proposition 3.6. *Let $S, T \in \mathcal{M}_n(\mathbb{C})$ such that S is normal and ST is skew D-quasi-normal. Then*

$$SS^* T = TSS^* \implies TS \text{ is skew D-quasi-normal.}$$

Proof. Since S is normal, we know that

$$S = UP = PU$$

where P is positive and U is unitary. Hence

$$SS^* T = TSS^* \implies P^2 T = TP^2 \implies PT = TP$$

so that

$$U^* STU = U^* UPTU = PTU = TS.$$

Hence TS is unitary equivalent to a skew D-quasi-normal matrix and thus by Theorem 2.5, is skew D-quasi-normal itself. $\square$

The reverse implication does not hold in the previous result as shown in the following example:

Example 3.7. *Consider the two matrices* $S, T \in \mathcal{M}_3(\mathbb{C})$, *where*

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then $(ST)^D = (TS)^D = 0$. *Hence* $ST, TS \in [\mathfrak{SD}]$. *Also* $S \in [N]$. *But* $SS^*T \neq TSS^*$.

We improve Proposition 3.6 by omitting the requirement that S be normal.

Proposition 3.8. *Let* $S, T \in \mathcal{M}_n(\mathbb{C})$ *such that* ST *is skew D-quasi-normal. Then*

$$S^*ST = TSS^* \implies TS \text{ is skew D-quasi-normal}.$$

Proof. Let $S = UP$, where P is positive and U is unitary. Note that there exists a positive semidefinite $K \in \mathcal{M}_n(\mathbb{C})$ such that $S = KU$. We obtain

$$
\begin{aligned}
P^2 T &= S^*ST \\
&= (TS)S^* \\
&= TK^2.
\end{aligned}
$$

Hence, since P and K are positive semidefinite, $PT = TK$. Then $PTU = TKU$. So $PTU = TUP$. Thus

$$U^*STU = U^*UPTU = PTU = TS.$$

Hence TS is unitary equivalent to a skew D-quasi-normal operator and thus by Theorem 2.5, is skew D-quasi-normal itself. □

Proposition 3.9. *Let* $S, T \in \mathcal{M}_n(\mathbb{C})$ *such that* ST *is skew D-quasi-normal. Then*

$$S^*(ST)(ST)^D = (TS)(TS)^D S^* \iff TS \text{ is skew D-quasi-normal}.$$

Proof. Let ST and TS be skew D-quasi-normal matrices. Hence, since

$$S(TS)(TS)^D = (ST)(ST)^D S$$

by Fuglede-Putnam Theorem,

$$S^*(ST)(ST)^D = (TS)(TS)^D S^*.$$

Conversely, if $S^*(ST)(ST)^D = (TS)(TS)^D S^*$, then $S^*S(TS)^D T = T(ST)^D SS^*$. Let $S = UP$ where P is positive and U is unitary. Note that there exists a positive semidefinite $K \in \mathcal{M}_n(\mathbb{C})$ such that $S = KU$. So, $P^2(TS)^D T = T(ST)^D K^2$. Hence, since P and K are positive semidefinite, $P(TS)^D T = T(ST)^D K$. So, we have

$$P(TS)^D TU = T(ST)^D KU. \tag{5}$$

Now,

$$
\begin{aligned}
U^*(ST)(ST)^D U &= U^*S(TS)^D TU && \text{(by Lemma 1.1)} \\
&= U^*UP(TS)^D TU \\
&= T(ST)^D KU && \text{(By (5))} \\
&= T(ST)^D S \\
&= (TS)(TS)^D.
\end{aligned}
$$

Hence $(TS)(TS)^D$ is unitary equivalent to a normal operator and thus is normal itself. □

Theorem 3.10. *Let $S, T \in \mathcal{M}_n(\mathbb{C})$. Then ST and TS are of class $[\mathfrak{S}\mathfrak{D}]$ if and only if $S^*(ST)(ST)^D = (TS)(TS)^D S^*$ and $(ST)^D(ST)T^* = T^*(TS)(TS)^D$.*

Proof. Let ST and TS be skew D-quasi-normal matrices. Hence, since

$$S(TS)^D(TS) = (ST)(ST)^D S$$

by Fuglede-Putnam Theorem,

$$(TS)(TS)^D S^* = S^*(ST)(ST)^D.$$

Similarly, from

$$(TS)(TS)^D T = T(ST)(ST)^D$$

we get that

$$T^*(TS)(TS)^D = (ST)(ST)^D T^*.$$

Conversely, if $S^*(ST)(ST)^D = (TS)(TS)^D S^*$ and $(ST)(ST)^D T^* = T^*(TS)(TS)^D$, then multiplying the first equation by T^* and the second one by S^* we see that ST and TS are of class $[\mathfrak{S}\mathfrak{D}]$. □

By applying Theorem 3.10, we prove the following theorem:

Theorem 3.11. *Let $S = UP$, where P is positive semidefinite and U is unitary, and let $T \in \mathcal{M}_n(\mathbb{C})$.*

1. *If TU is normal and $PT(ST)^D U = T(ST)^D UP$, then ST and TS are of class $[\mathfrak{S}\mathfrak{D}]$.*
2. *If ST and TS are of class $[\mathfrak{S}\mathfrak{D}]$, then $PT(ST)^D U = T(ST)^D UP$.*

Proof. Suppose that TU is normal and $PT(ST)^D U = T(ST)^D UP$. Then

$$\begin{aligned}
(TS)^D(TS)S^* &= T((ST)^D)^2 S(TS)S^* \\
&= T(ST)^D(UP)(UP)^* \\
&= T(ST)^D UP^2 U^* \\
&= P^2 T(ST)^D UU^* \qquad\qquad \text{(By hypotheses)} \\
&= (UP)^*(UP)T(ST)^D \\
&= S^* ST(ST)^D
\end{aligned}$$

i.e.

$$(TS)(TS)^D S^* = S^* ST(ST)^D. \tag{6}$$

On the other hand

$$\begin{aligned}
P(TS)^D TU &= PT(ST)^D U \\
&= T(ST)^D UP \\
&= T(ST)^D S \\
&= (TS)(TS)^D \tag{7}
\end{aligned}$$

and

$$TUP(TS)^D = (TS)(TS)^D \tag{8}$$

From (7) and (8)

$$P(TS)^D TU = TUP(TS)^D.$$

Now, since TU is normal then by Fuglede-Putnam theorem,

$$P(TS)^D (TU)^* = (TU)^* P(TS)^D. \tag{9}$$

Now, we obtain that

$$
\begin{aligned}
(ST)^D (ST) T^* &= S((TS)^D)^2 T(ST)T^* \\
&= S(TS)^D TT^* \\
&= (UP)(TS)^D (TU)(TU)^* \\
&= (UP)(TS)^D (TU)^*(TU) &&(TU \text{ is normal}) \\
&= U(TU)^* P(TS)^D (TU) &&(\text{by } (9)) \\
&= T^* P(TS)^D (TU) \\
&= T^*(TS)(TS)^D.
\end{aligned}
$$

i.e.

$$(ST)(ST)^D T^* = T^*(TS)(TS)^D. \tag{10}$$

Therefore, by (6), (10) and Theorem 3.10, we get that ST and TS are of class $[\mathfrak{SD}]$. This proves (1). To prove (2), let ST and TS are of class $[\mathfrak{SD}]$ and note that there exists a positive semidefinite $K \in \mathcal{M}_n(\mathbb{C})$ such that $S = KU$. Using Theorem 3.10, we obtain

$$
\begin{aligned}
P^2(TS)^D T &= S^* S(TS)^D T \\
&= S^*(ST)(ST)^D \\
&= (TS)(TS)^D S^* \\
&= TST((ST)^D)^2 SS^* \\
&= T(ST)^D K^2.
\end{aligned}
$$

Hence, since P and K are positive semidefinite, $P(TS)^D T = T(ST)^D K$. Then $PT(ST)^D U = T(ST)^D UP$. □

We present an analogue of Fuglede-Putnam's theorem for skew D-quasi-normal matrices. To prove Theorem 3.13 the following lemma is needful.

Lemma 3.12. *(see [5, Lemma 2.6]). Let $S, T, X \in \mathcal{M}_n(\mathbb{C})$. If $SX = XT$, then $S^D X = XT^D$.*

Using the above Lemma and the Fuglede-Putnam's theorem, we can get the following result.

Theorem 3.13. *Let $S, T, X \in \mathcal{M}_n(\mathbb{C})$. If S and T are of class $[\mathfrak{SD}]$ and $SX = XT$, then $(SS^D)^* X = X(TT^D)^*$.*

Proof. Since $SX = XT$, by Lemma 3.12, $SS^D X = XTT^D$. Now, since S and T are of class $[\mathfrak{SD}]$, then SS^D and TT^D are normal. So by Fuglede-Putnam theorem, $(SS^D)^* X = X(TT^D)^*$. □

From Theorem 3.13 we obtain the following corollary.

Corollary 3.14. *Let $S, X \in \mathcal{M}_n(\mathbb{C})$. If S is of class $[\mathfrak{SD}]$ and $SX = XS$, then $(SS^D)^* X = X(SS^D)^*$.*

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