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The Natural Lie Algebra Brackets on Couples of Vector Fields and p-Forms

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Abstract. If $m \geq p+1 \geq 2$ or $m = p \geq 3$, all natural Lie algebra brackets on couples of vector fields and p-forms on m-manifolds are described.

1. Introduction

Let $\mathcal{M}f_m$ be the category of m-dimensional C^∞ manifolds and their embeddings.

The Courant bracket on the "doubled" tangent bundle $T \oplus T^*$ is of full interest because it is involved in the definitions of Dirac and generalized complex structures, see e.g. [1, 4, 5]. That is why, in [2], we studied "brackets" on $T \oplus T^*$ similar to the Courant one.

The Courant bracket can be extended on $T \oplus \bigwedge^p T^*$, see e.g. [5]. That is why, in [3], we described all $\mathcal{M}f_m$-natural bilinear operators

$$A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow T \oplus \bigwedge^p T^*$$

transforming pairs of couples $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^p(M)$ $(i = 1, 2)$ of vector fields and p-forms on m-manifolds M into couples $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \mathcal{X}(M) \oplus \Omega^p(M)$ of vector fields and p-forms on M.

In the present note, we extract all $\mathcal{M}f_m$-natural bilinear operators A as above satisfying the Jacobi identity in Leibniz form (or shortly, satisfying the Leibniz rule)

$$A(\rho^1, A(\rho^2, \rho^3)) = A(A(\rho^1, \rho^2), \rho^3) + A(\rho^2, A(\rho^1, \rho^3))$$

for any $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^p(M)$ $(i = 1, 2, 3)$ and $M \in obj(\mathcal{M}f_m)$.

In particular, we find all $\mathcal{M}f_m$-natural Lie algebra brackets $[-,-]$ on $\mathcal{X}(M) \oplus \Omega^p(M)$ (i.e. $\mathcal{M}f_m$-natural skew-symmetric bilinear operators $A = [-,-]$ as above satisfying the Leibniz rule).

From now on, (x^i) $(i = 1, ..., m)$ is the usual coordinates on $\mathbb{R}^m$ and $\partial_i = \frac{\partial}{\partial x^i}$ are the canonical vector fields on $\mathbb{R}^m$.

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2. On the Courant like brackets

Definition 2.1. *([3]) A bilinear $\mathcal{M}f_m$-natural operator $A : (T\oplus\bigwedge^p T^*)\times(T\oplus\bigwedge^p T^*) \rightsquigarrow T\oplus\bigwedge^p T^*$ is a $\mathcal{M}f_m$-invariant family of bilinear operators*

$$A : (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)) \to \mathcal{X}(M) \oplus \Omega^p(M)$$

for m-dimensional manifolds M, where $\mathcal{X}(M)$ is the space of vector fields on M and $\Omega^p(M)$ is the space of p-forms on M.

Remark 2.2. *In the above definition, the $\mathcal{M}f_m$-invariance of A means that if $(X^1\oplus\omega^1, X^2\oplus\omega^2) \in (\mathcal{X}(M)\oplus\Omega^p(M))\times(\mathcal{X}(M) \oplus \Omega^p(M))$ and $(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2) \in (\mathcal{X}(\overline{M}) \oplus \Omega^p(\overline{M})) \times (\mathcal{X}(\overline{M}) \oplus \Omega^p(\overline{M}))$ are φ-related by an $\mathcal{M}f_m$-map $\varphi : M \to \overline{M}$ (i.e. $\overline{X}^i \circ \varphi = T\varphi \circ X^i$ and $\overline{\omega}^i \circ \varphi = \bigwedge^p T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.*

The most important example of a bilinear $\mathcal{M}f_m$-natural operator $A : (T\oplus\bigwedge^p T^*)\times(T\oplus\bigwedge^p T^*) \rightsquigarrow T\oplus\bigwedge^p T^*$ is the generalized Courant bracket.

Example 2.3. *([5]) The generalized Courant bracket is given by*

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_C = [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1 + \frac{1}{2}d(i_{X^2}\omega^1 - i_{X^1}\omega^2))$$

for any $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^p(M)$, $i = 1, 2$, where $\mathcal{L}$ denotes the Lie derivative, d the exterior derivative, $[-,-]$ the usual bracket on vector fields and i is the insertion derivative. For $p = 1$ we obtain the usual Courant bracket as in [1].

Remark 2.4. *If $m = p$, $\mathcal{L}_X\omega = di_X\omega + i_Xd\omega = di_X\omega$ for any vector field X and any m-form ω on a m-manifold M as $d\omega = 0$, and then $[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_C = [X^1, X^2] \oplus \frac{1}{2}(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1)$.*

Theorem 2.5. *([3]) If $m \geq p + 1 \geq 2$ (or $m = p \geq 3$), any bilinear $\mathcal{M}f_m$-natural operator $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow T \oplus \bigwedge^p T^*$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + c_1d(i_{X^2}\omega^1) + c_2d(i_{X^1}\omega^2))$$

for uniquely determined by A real numbers a, b_1, b_2, c_1, c_2 (or a, b_1, b_2, c_1, c_2 with $c_1 = c_2 = 0$).

Corollary 2.6. *([3]) If $m \geq p + 1 \geq 2$ (or $m = p \geq 3$), any skew-symmetric bilinear $\mathcal{M}f_m$-natural operator $A : (T \otimes \bigwedge^p T^*) \times (T \otimes \bigwedge^p T^*) \rightsquigarrow T \oplus \bigwedge^p T^*$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) + cd(i_{X^2}\omega^1 - i_{X^1}\omega^2))$$

for uniquely determined by A real numbers a, b, c (or a, b, c with $c = 0$), i.e. roughly speaking, any such A coincides with the generalized Courant bracket up to three (or two) real constants.

3. The main result

The main result of the present note is the following

Theorem 3.1. *If $m \geq p+1 \geq 2$ (or $m = p \geq 3$), any bilinear $\mathcal{M}f_m$-natural operator $A : (T\oplus\bigwedge^p T^*)\times(T\oplus\bigwedge^p T^*) \rightsquigarrow T \oplus \bigwedge^p T^*$ satisfying the Leibniz rule as in Introduction is the constant multiple of one of the following four (or three respectively) operators A_1, A_2, A_3, A_4 (or A_1, A_2, A_3) given by*

$$\begin{aligned}
&A_1(\rho^1, \rho^2) = [X^1, X^2] \oplus 0\,,\\
&A_2(\rho^1, \rho^2) = [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1)\,,\\
&A_3(\rho^1, \rho^2) = [X^1, X^2] \oplus \mathcal{L}_{X^1}\omega^2\,,\\
&A_4(\rho^1, \rho^2) = [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1 + d(i_{X^2}\omega^1))\,,
\end{aligned}$$

where $\rho^i = X^i \oplus \omega^i$. The operators $A_1, ..., A_4$ satisfy the Leibniz rule.

Proof. Let $A : (T \oplus \bigwedge^p T^*) \times (T \oplus \bigwedge^p T^*) \rightsquigarrow T \oplus \bigwedge^p T^*$ be a bilinear $\mathcal{M}f_m$-natural operator satisfying the Leibniz rule. By Theorem 2.5, if $m \geq p+1 \geq 2$ (or $m = p \geq 3$), A is of the form

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d(i_{X^2}\omega^1) + c_2 d(i_{X^1}\omega^2))$$

for uniquely determined by A real numbers a, b_1, b_2, c_1, c_2 (or a, b_1, b_2, c_1, c_2 with $c_1 = c_2 = 0$). Then for any $X^1, X^2, X^3 \in \mathcal{X}(M)$ and $\omega^1, \omega^2, \omega^3 \in \Omega^p(M)$ we have

$$A(X^1 \oplus \omega^1, A(X^2 \oplus \omega^2, X^3 \oplus \omega^3)) = a^2[X^1, [X^2, X^3]] \oplus \Omega \,,$$

$$A(A(X^1 \oplus \omega^1, X^2 \oplus \omega^2), X^3 \oplus \omega^3) = a^2[[X^1, X^2], X^3] \oplus \Theta \,,$$

$$A(X^2 \oplus \omega^2, A(X^1 \oplus \omega^1, X^3 \oplus \omega^3)) = a^2[X^2, [X^1, X^3]] \oplus \mathcal{T} \,,$$

where

$$\begin{aligned}\Omega = \; & b_1 \mathcal{L}_{a[X^2,X^3]}\omega^1 + c_1 d(i_{a[X^2,X^3]}\omega^1) \\ & + b_2 \mathcal{L}_{X^1}(b_1 \mathcal{L}_{X^3}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^3 + c_1 d(i_{X^3}\omega^2) + c_2 d(i_{X^2}\omega^3)) \\ & + c_2 d(i_{X^1}(b_1 \mathcal{L}_{X^3}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^3 + c_1 d(i_{X^3}\omega^2) + c_2 d(i_{X^2}\omega^3))) \,,\end{aligned}$$

$$\begin{aligned}\Theta = \; & b_2 \mathcal{L}_{a[X^1,X^2]}\omega^3 + c_2 d(i_{a[X^1,X^2]}\omega^3) \\ & + b_1 \mathcal{L}_{X^3}(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d(i_{X^2}\omega^1) + c_2 d(i_{X^1}\omega^2)) \\ & + c_1 d(i_{X^3}(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d(i_{X^2}\omega^1) + c_2 d(i_{X^1}\omega^2))) \,,\end{aligned}$$

$$\begin{aligned}\mathcal{T} = \; & b_1 \mathcal{L}_{a[X^1,X^3]}\omega^2 + c_1 d(i_{a[X^1,X^3]}\omega^2) \\ & + b_2 \mathcal{L}_{X^2}(b_1 \mathcal{L}_{X^3}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^3 + c_1 d(i_{X^3}\omega^1) + c_2 d(i_{X^1}\omega^3)) \\ & + c_2 d(i_{X^2}(b_1 \mathcal{L}_{X^3}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^3 + c_1 d(i_{X^3}\omega^1) + c_2 d(i_{X^1}\omega^3))) \,.\end{aligned}$$

The Leibniz rule of A is equivalent to

$$\Omega = \Theta + \mathcal{T} \,. \tag{1}$$

Assume $m = p \geq 3$. Then $c_1 = c_2 = 0$ and equation (1) gives

$$\begin{aligned}& b_1 a \mathcal{L}_{[X^2,X^3]}\omega^1 + b_2 b_1 \mathcal{L}_{X^1}\mathcal{L}_{X^3}\omega^2 + b_2^2 \mathcal{L}_{X^1}\mathcal{L}_{X^2}\omega^3 \\ & = (b_2 a \mathcal{L}_{[X^1,X^2]}\omega^3 + b_1^2 \mathcal{L}_{X^3}\mathcal{L}_{X^2}\omega^1 + b_1 b_2 \mathcal{L}_{X^3}\mathcal{L}_{X^1}\omega^2) \\ & + (b_1 a \mathcal{L}_{[X^1,X^3]}\omega^2 + b_2 b_1 \mathcal{L}_{X^2}\mathcal{L}_{X^3}\omega^1 + b_2^2 \mathcal{L}_{X^2}\mathcal{L}_{X^1}\omega^3) \,.\end{aligned}$$

If we put $X^1 = \partial_1$, $X^2 = x^1\partial_1$, $X^3 = 0$ and $\omega^1 = 0$, $\omega^2 = 0$, $\omega^3 = d(x^1)^2 \wedge dx^2 \wedge \ldots \wedge dx^m$, we get

$$4b_2^2 dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m = 2b_2 a dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m + 2b_2^2 dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m \,.$$

If we put $X^1 = 0$, $X^2 = \partial_1$, $X^3 = x^1\partial_1$ and $\omega^1 = d(x^1)^2 \wedge dx^2 \wedge \ldots \wedge dx^m$, $\omega^2 = 0$, $\omega^3 = 0$, we get

$$2b_1 a dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m = 2b_1^2 dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m + 4b_2 b_1 dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m \,.$$

If we put $X^1 = \partial_1$, $X^2 = 0$, $X^3 = x^1\partial_1$ and $\omega^1 = 0$, $\omega^2 = d(x^1)^2 \wedge dx^2 \wedge \ldots \wedge dx^m$, $\omega^3 = 0$, we get

$$4b_2 b_1 dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m = 2b_1 b_2 dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m + 2b_1 a dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m \,.$$

So,

$$b_2 a = b_2^2 \,, \;\; b_1 a = b_1^2 + 2b_1 b_2 \,, \;\; b_1 b_2 = b_1 a \,. \tag{2}$$

From the first equality we get $b_2 = 0$ or $b_2 = a$. From the third one we get $b_1 = 0$ or $b_2 = a$. Adding the first two equalities we get $(b_2 + b_1)a = (b_2 + b_1)^2$, i.e. $b_2 + b_1 = 0$ or $b_2 + b_1 = a$. Consequently

$$(b_1, b_2) = (0, 0) \text{ or } (b_1, b_2) = (0, a) \text{ or } (b_1, b_2) = (-a, a)\,. \tag{3}$$

Theorem 3.1 for $m = p \geq 3$ is complete.

So, we may assume $m \geq p + 1 \geq 2$. Applying the differentiation d to both sides of the equality (1) and using the well-known formula $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d$ we get

$$\begin{aligned}
&b_1 a\mathcal{L}_{[X^2,X^3]}d\omega^1 + b_2 b_1\mathcal{L}_{X^1}\mathcal{L}_{X^3}d\omega^2 + b_2^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}d\omega^3 \\
&= (b_2 a\mathcal{L}_{[X^1,X^2]}d\omega^3 + b_1^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}d\omega^1 + b_1 b_2\mathcal{L}_{X^3}\mathcal{L}_{X^1}d\omega^2) \\
&+(b_1 a\mathcal{L}_{[X^1,X^3]}d\omega^2 + b_2 b_1\mathcal{L}_{X^2}\mathcal{L}_{X^3}d\omega^1 + b_2^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}d\omega^3)\,.
\end{aligned}$$

If we put $X^1 = \partial_1\,,\ X^2 = x^1\partial_1\,,\ X^3 = 0$ and $\omega^1 = 0\,,\ \omega^2 = 0\,,\ \omega^3 = (x^1)^2dx^2 \wedge ... \wedge dx^{p+1}$, we get

$$4b_2^2dx^1 \wedge dx^2 \wedge ... \wedge dx^{p+1} = 2b_2 a dx^1 \wedge dx^2 \wedge ... \wedge dx^{p+1} + 2b_2^2dx^1 \wedge dx^2 \wedge ... \wedge dx^{p+1}\,.$$

If we put $X^1 = 0\,,\ X^2 = \partial_1\,,\ X^3 = x^1\partial_1$ and $\omega^1 = (x^1)^2dx^2 \wedge ... \wedge dx^{p+1}\,, \omega^2 = 0\,,\ \omega^3 = 0$, we get

$$2b_1 a dx^1 \wedge dx^2 \wedge ... \wedge dx^{p+1} = 2b_1^2dx^1 \wedge dx^2 \wedge ... \wedge dx^{p+1} + 4b_2 b_1 dx^1 \wedge dx^2 \wedge ... \wedge dx^{p+1}\,.$$

If we put $X^1 = \partial_1\,, X^2 = 0\,,\ X^3 = x^1\partial_1$ and $\omega^1 = 0\,,\ \omega^2 = (x^1)^2dx^2 \wedge ... \wedge dx^{p+1}\,,\ \omega^3 = 0$, we get

$$4b_2 b_1 dx^1 \wedge dx^2 \wedge ... \wedge dx^{p+1} = 2b_1 b_2 dx^1 \wedge dx^2 \wedge ... \wedge dx^{p+1} + 2b_1 a dx^1 \wedge dx^2 \wedge ... \wedge dx^{p+1}\,.$$

So,

$$b_2 a = b_2^2\,,\ \ b_1 a = b_1^2 + 2b_1 b_2\,,\ \ b_1 b_2 = b_1 a\,,$$

i.e. equations (2). Consequently, we have (as above) alternative (3).

Then, using the formula $\mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega = \mathcal{L}_{[X,Y]}\omega$ and alternative (3), one can easily verify that

$$\begin{aligned}
&b_1 a\mathcal{L}_{[X^2,X^3]}\omega^1 + b_2 b_1\mathcal{L}_{X^1}\mathcal{L}_{X^3}\omega^2 + b_2^2\mathcal{L}_{X^1}\mathcal{L}_{X^2}\omega^3 \\
&= (b_2 a\mathcal{L}_{[X^1,X^2]}\omega^3 + b_1^2\mathcal{L}_{X^3}\mathcal{L}_{X^2}\omega^1 + b_1 b_2\mathcal{L}_{X^3}\mathcal{L}_{X^1}\omega^2) \\
&+(b_1 a\mathcal{L}_{[X^1,X^3]}\omega^2 + b_2 b_1\mathcal{L}_{X^2}\mathcal{L}_{X^3}\omega^1 + b_2^2\mathcal{L}_{X^2}\mathcal{L}_{X^1}\omega^3)\,.
\end{aligned}$$

The last formula for $(a, b_1, b_2) = (1, 0, 0)$ or $(a, b_1, b_2) = (1, 0, 1)$ or $(a, b_1, b_2) = (1, -1, 1)$ means that the operators A_1, A_2, A_3 satisfy the Leibniz rule, as well.

More, the last formula implies that the Leibniz rule (1) is equivalent to alternative (3) and the following condition

$$\begin{aligned}
&c_1 ad(i_{[X^2,X^3]}\omega^1) + b_2\mathcal{L}_{X^1}(c_1 d(i_{X^3}\omega^2) + c_2 d(i_{X^2}\omega^3)) \\
&+c_2 d(i_{X^1}(b_1\mathcal{L}_{X^3}\omega^2 + b_2\mathcal{L}_{X^2}\omega^3 + c_1 d(i_{X^3}\omega^2) + c_2 d(i_{X^2}\omega^3))) \\
&= c_2 ad(i_{[X^1,X^2]}\omega^3) + b_1\mathcal{L}_{X^3}(c_1 d(i_{X^2}\omega^1) + c_2 d(i_{X^1}\omega^2)) \\
&+c_1 d(i_{X^3}(b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + c_1 d(i_{X^2}\omega^1) + c_2 d(i_{X^1}\omega^2))) \\
&+c_1 ad(i_{[X^1,X^3]}\omega^2) + b_2\mathcal{L}_{X^2}(c_1 d(i_{X^3}\omega^1) + c_2 d(i_{X^1}\omega^3)) \\
&+c_2 d(i_{X^2}(b_1\mathcal{L}_{X^3}\omega^1 + b_2\mathcal{L}_{X^1}\omega^3 + c_1 d(i_{X^3}\omega^1) + c_2 d(i_{X^1}\omega^3)))\,.
\end{aligned} \tag{4}$$

If we put (in (4)) $X^1 = \partial_1$, $X^2 = \partial_2$, $X^3 = 0$, $\omega^1 = \omega^2 = 0$ and $\omega^3 = (x^2)^2dx^1 \wedge dx^3 \wedge ... \wedge dx^{p+1}$, we get

$$2c_2 b_2 dx^2 \wedge dx^3 \wedge ... \wedge dx^{p+1} = 2c_2 b_2 dx^2 \wedge dx^3 \wedge ... \wedge dx^{p+1} + 2c_2^2dx^2 \wedge dx^3 \wedge ... \wedge dx^{p+1}\,.$$

Then $c_2 = 0$.

If we put $X^1 = 0$, $X^2 = \partial_1$, $X^3 = \partial_2$, $\omega^1 = (x^2)^2dx^1 \wedge dx^3 \wedge ... \wedge dx^{p+1}$, and $\omega^2 = \omega^3 = 0$, we get

$$0 = 2b_1 c_1 dx^2 \wedge dx^3 \wedge ... \wedge dx^{p+1} + 2c_1^2dx^2 \wedge dx^3 \wedge ... \wedge dx^{p+1} + 2c_2 b_1 dx^2 \wedge dx^3 \wedge ... \wedge dx^{p+1}\,.$$

Then (as $c_2 = 0$) we get $c_1 = 0$ or $c_1 = -b_1$.

Consequently we obtain that $(b_1, b_2, c_1, c_2) = (0, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (0, a, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (-a, a, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (-a, a, a, 0)$.

On the other hand, $A_1, ..., A_4$ from Theorem 3.1 satisfy the Leibniz rule, see above for A_1, A_2, A_3 and see Lemma 3.2 below for A_4. Theorem 3.1 is complete. □

Lemma 3.2. *The operator A_4 from Theorem 3.1 satisfies the Leibniz rule.*

Proof. It is sufficient to prove (4) for $(a, b_1, b_2, c_1, c_2) = (1, -1, 1, 1, 0)$, i.e that

$$\begin{aligned} &di_{[X^2,X^3]}\omega^1 + \mathcal{L}_{X^1} di_{X^3}\omega^2 = -\mathcal{L}_{X^3} di_{X^2}\omega^1 - di_{X^3}\mathcal{L}_{X^2}\omega^1 \\ &+di_{X^3}\mathcal{L}_{X^1}\omega^2 + di_{X^3} di_{X^2}\omega^1 + di_{[X^1,X^3]}\omega^2 + \mathcal{L}_{X^2} di_{X^3}\omega^1 \ . \end{aligned} \tag{5}$$

The above formula (5) is the sum of the following two formulas

$$di_{[X^2,X^3]}\omega^1 = -\mathcal{L}_{X^3} di_{X^2}\omega^1 - di_{X^3}\mathcal{L}_{X^2}\omega^1 + di_{X^3} di_{X^2}\omega^1 + \mathcal{L}_{X^2} di_{X^3}\omega^1 \ , \tag{6}$$

$$\mathcal{L}_{X^1} di_{X^3}\omega^2 = di_{X^3}\mathcal{L}_{X^1}\omega^2 + di_{[X^1,X^3]}\omega^2 \ . \tag{7}$$

The formula (7) follows immediately from $\mathcal{L}_{X^1} d = d\mathcal{L}_{X^1}$ and $i_{[X^1,X^3]} = \mathcal{L}_{X^1} i_{X^3} - i_{X^3}\mathcal{L}_{X^1}$. The proof of (6) is following. Applying $\mathcal{L}_{X^3} = di_{X^3} + i_{X^3} d$ and $dd = 0$, we easily get $di_{X^3} di_{X^2}\omega^1 = \mathcal{L}_{X^3} di_{X^2}\omega^1$. We have also $\mathcal{L}_{X^2} d = d\mathcal{L}_{X^2}$ and then $\mathcal{L}_{X^2} di_{X^3}\omega^1 = d\mathcal{L}_{X^2} i_{X^3}\omega^1$. Then (6) follows from $i_{[X^2,X^3]} = \mathcal{L}_{X^2} i_{X^3} - i_{X^3}\mathcal{L}_{X^2}$. □

From Theorem 3.1 and Corollary 2.6 it follows immediately

Corollary 3.3. *If $m \geq p + 1 \geq 2$ or $m = p \geq 3$, any $\mathcal{M}f_m$-natural Lie algebra bracket on $\mathcal{X}(M) \oplus \Omega^p(M)$ is the constant multiple of one of the following two ones*

$$\begin{aligned} &[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_1 = [X^1, X^2] \oplus 0 \ , \\ &[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_2 = [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) \ . \end{aligned}$$

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