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A New Lower Bound for the Smallest Singular Value

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Abstract. The aim of this paper is to obtain new lower bounds for the smallest singular value for some special subclasses of nonsingular H-matrices. This is done in two steps: first, unifying principle for deriving new upper bounds for the norm 1 of the inverse of an arbitrary nonsingular H-matrix is presented, and then, it is combined with some well-known upper bounds for the infinity norm of the inverse. The importance and efficiency of the results are illustrated by an example from ecological modelling, as well as on a type of large-scale matrices posessing a block structure, arising in boundary value problems.

1. Introduction and motivation

For an arbitrary complex matrix $A \in \mathbb{C}^{n,n}$ by $\sigma_{min}(A)$ we denote its smallest singular value. It is well-known that $\sigma_{min}(A) = 0$ if and only if A is singular. Since the focus of our attention belongs to nonsingular, in particular H-matrices, we will use the fact that $\sigma_{min}(A) = \|A^{-1}\|_2^{-1}$. So, in order to obtain a lower bound for $\sigma_{min}(A)$, one can think of this task as bounding the Euclidean norm of the inverse from above. As a matter of fact, the relation

$$\|B\|_2^2 \leq \|B\|_\infty \|B\|_1,$$

which holds for any $B \in \mathbb{C}^{n,n}$, makes it clear that we can use upper bounds for the inverses in both, infinity and norm 1, for this purpose. For instance, this technique has been used in [10].

It is also well-known that systems, obtained by discretization of partial differential equations - either using the finite element or finite difference methods, usually have a block structure in addition to large dimensions. This fact justified the efforts to introduce and develop the concept of block matrices, as it offers many practical benefits. The most obvious is computationally cheap reduction in size and the afterward application of point-wise results.

This paper is organized as follows. A brief introduction covering the motivation is followed by Section 2, with notations, as well as definitions and known results. Next section presents an upper bound for norm 1 of the inverse of some special subclasses of H-matrices. Section 4 contains the main result of this paper,

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offering new lower bounds for the smallest singular value, and comparing them to the existing ones, as well as addresses the block generalizations and applications of introduced bounds to large-scale matrices. Finally, we end this paper with Section 5, with numerical examples and closing remarks.

2. Notation and preliminaries

Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a square complex matrix of order n. The set of row (column) indices of A is defined as $N := \{1, 2, \dots, n\}$, and deleted row and column sums as

$$r_i := \sum_{j \in N \setminus \{i\}} |a_{ij}|, \qquad c_i := \sum_{j \in N \setminus \{i\}} |a_{ji}|, \qquad i \in N.$$

Definition 2.1. *Matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called strictly diagonally dominant (SDD) matrix if and only if $|a_{ii}| > r_i$, for all $i \in N$.*

The famous result for SDD matrices, for further reference see [1, 14], we recall as the following theorem.

Theorem 2.2 (Ahlberg-Nilson-Varah bound for SDD matrices). *Let $A \in \mathbb{C}^{n,n}$ be an SDD matrix. Then, $\|A^{-1}\|_\infty \le \nu_\infty(A)$, where*

$$\nu_\infty(A) := \frac{1}{\min\limits_{i \in N} \left(|a_{ii}| - r_i \right)}. \tag{1}$$

Diagonal dominance can also be column-wise. Due to the fact that $c_i(A) = r_i(A^T)$, a matrix is called strictly diagonally dominant by columns if and only if $|a_{ii}| > c_i(A)$, for all $i \in N$, or, equivalently, provided that A^T is an SDD matrix. Obviously, if matrix A is strictly diagonally dominant by columns, then

$$\|A^{-1}\|_1 = \|(A^{-1})^T\|_\infty = \|(A^T)^{-1}\|_\infty \le \nu_\infty(A^T) = \frac{1}{\min\limits_{i \in N} \left(|a_{ii}| - c_i \right)}.$$

Next, we recall the class of nonsingular matrices that generalizes the SDD property.

Definition 2.3. *Matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called a generalized diagonally dominant (GDD) matrix if there exists a positive vector $x \in \mathbb{R}^n$ such that*

$$|a_{ii}| x_i > \sum_{j \in N \setminus \{i\}} |a_{ij}| x_j, \quad \text{for all } i \in N.$$

This definition can be interpreted in the following way: matrix A is a GDD matrix if and only if there exists a positive diagonal matrix $X = diag(x_1, x_2, \dots, x_n)$, such that AX is an SDD matrix. Such a matrix X is usually called a scaling matrix. This class is also known in the literature as the class of nonsingular H-matrices, so we will recall this equivalent definition, as well.

Definition 2.4. *Matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called an H-matrix if its comparison matrix, $\mathcal{M}(A) = [m_{ij}] \in \mathbb{R}^{n,n}$, defined by*

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \neq j \end{cases}$$

is an M-matrix, i.e., $\mathcal{M}(A)^{-1} \ge O$.

It is known that the absolute value of the inverse for any nonsingular H-matrix can be bounded from above by the inverse of its comparison M-matrix.

Theorem 2.5 ([2]). *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a nonsingular H-matrix, then*

$$|A^{-1}| \leq \mathcal{M}(A)^{-1}.$$

One particular subclass of H-matrices, i.e. GDD matrices, will play a central role in this paper. It is the S-SDD class, and here we recall some of its equivalent definitions.

Let S be a non-empty subset of the index set N, and let $\overline{S} := N \setminus S$ denote its complement. Then, one can define

$$r_i^S := \sum_{j \in S \setminus \{i\}} |a_{ij}| \quad \text{and} \quad r_i^{\overline{S}} := \sum_{j \in \overline{S} \setminus \{i\}} |a_{ij}|, \quad \text{for all } i \in N.$$

Obviously, it holds that $r_i = r_i^S + r_i^{\overline{S}}$, for all $i \in N$. Also, note that the class of N-SDD matrices is, in fact, the class of SDD matrices.

Definition 2.6 ([5]). *Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, and given $\emptyset \neq S \subseteq N$, we say that A is an S-SDD matrix if it has at least one strictly diagonally dominant row, and $(|a_{ii}| - r_i^S)(|a_{jj}| - r_j^{\overline{S}}) > r_i^{\overline{S}} r_j^S$, for all $i \in S$ and all $j \in \overline{S}$.*

Later on, this definition was equivalently rewritten in many forms. One of them has been used in order to obtain eigenvalue localization sets in [3]:

Definition 2.7. *Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, and given $\emptyset \neq S \subseteq N$, we say that A is an S-SDD matrix if the following conditions are fulfilled:*

1. $|a_{ii}| > r_i^S$, *for all $i \in S$ and*
2. $(|a_{ii}| - r_i^S)(|a_{jj}| - r_j^{\overline{S}}) > r_i^{\overline{S}} r_j^S$, *for all $i \in S$ and all $j \in \overline{S}$.*

In the same paper [3], the authors have derived an explicit expression for the choice of a scaling matrix $X = diag(x_1, x_2, \ldots, x_n)$ (for which AX is an SDD matrix):

$$x_i = \begin{cases} \gamma & , \quad i \in S \\ 1 & , \quad i \in \overline{S} \end{cases} \tag{2}$$

where

$$0 \leq B_1 := \max_{i \in S} \frac{r_i^{\overline{S}}}{|a_{ii}| - r_i^S} < \gamma < \min_{j \in \overline{S} : r_j^S \neq 0} \frac{|a_{jj}| - r_j^{\overline{S}}}{r_j^S} =: B_2.$$

As a matter of fact, an equivalent characterization of the S-SDD class (for S being a proper subset of N, such that $r_j^S \neq 0$ for at least one $j \in \overline{S}$), is that the interval (B_1, B_2) is non-empty.

Regarding a bound for the infinity norm of the inverse of an S-SDD matrix, in the case of S being a *proper* subset of N, we refer to [9]:

Theorem 2.8 ([9]). *Let $A \in \mathbb{C}^{n,n}$, $n \geq 2$, be an S-SDD matrix for some non-empty proper subset S of N. Then, $\|A^{-1}\|_\infty \leq \nu_\infty^S(A)$, where*

$$\nu_\infty^S(A) := \max\left\{\theta_{ij}^S(A), \theta_{ji}^{\overline{S}}(A)\right\}, \tag{3}$$

with

$$\theta_{ij}^S(A) := \max_{\substack{i \in S \\ j \in \overline{S}}} \frac{|a_{ii}| - r_i^S + r_j^S}{(|a_{ii}| - r_i^S)(|a_{jj}| - r_j^{\overline{S}}) - r_i^{\overline{S}} r_j^S}.$$

Finally, let us end this section with some known results regarding block H- matrices. Let ℓ be an arbitrary positive integer such that $\ell \leq n$. Then, define $L := \{1, 2, \ldots, \ell\}$, and by $\pi = \{p_j\}_{j=0}^{\ell}$ denote a partition of N, provided that positive numbers p_j, $j = 1, 2, \ldots, \ell$, satisfy the following criteria

$$p_0 := 0 < p_1 < p_2 < \cdots < p_\ell := n.$$

Such a partition π enables one to perform a structural subdivision of A into $\ell \times \ell$ blocks, that is

$$A = \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1\ell} \\ \hline A_{21} & A_{22} & \cdots & A_{2\ell} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A_{\ell 1} & A_{\ell 2} & \cdots & A_{\ell\ell} \end{array}\right] = [A_{ij}]_{\ell\times\ell}.$$

Upon one specific choice of partition, for which $p_i = i$, for all $i \in N$, one is dealing with the point-wise case. In order to generalize (point-wise) subclasses of nonsingular H-matrices, the notion of a comparison matrix is required. Although it can be done in an arbitrary p-norm, for the purpose of this paper, we will only focus on the Euclidean case, that is $p = 2$, and the accompanying results related to it.

Let $A \in \mathbb{C}^{n,n}$ and $\pi = \{p_j\}_{j=0}^{\ell}$ be a partition of the index set.

The comparison matrix with respect to partition π, $\langle A \rangle_\pi = [\alpha_{ij}] \in \mathbb{R}^{\ell,\ell}$, is defined in the following way:

$$\alpha_{ij} = \begin{cases} (\|A_{ii}^{-1}\|_2)^{-1}, & i = j \\ -\|A_{ij}\|_2, & i \neq j \end{cases}$$

where $(\|A_{ii}^{-1}\|_2)^{-1}$ is defined to be zero if A_{ii} is a singular matrix.

Definition 2.9. *Given a partition π, block matrix $[A_{ij}]_{\ell\times\ell}$ is called block π H-matrix ($B^\pi H$) if its comparison matrix $\langle A \rangle_\pi$ is a nonsingular H-matrix (i.e. a nonsingular M-matrix).*

Apart from the fact that every $B^\pi H$-matrix is nonsingular, this definition implies that all its diagonal blocks are nonsingular matrices.

In general, for every nonsingular (point-wise) subclass $\mathbb{K}$ of H-matrices, we say that A is $B^\pi\mathbb{K}$ provided that $\langle A \rangle_\pi$ belongs to $\mathbb{K}$.

Lastly, we recall the link between the Euclidean norm of the inverse of a block matrix on one hand, and the the inverse of its comparison (point-wise) matrix, on the other:

Theorem 2.10 ([4]). *Given a partition π, if matrix $A = [A_{ij}]_{\ell,\ell}$ is a $B^\pi H$-matrix, then*

$$\|A^{-1}\|_2 \leq \|(\langle A \rangle_\pi)^{-1}\|_2.$$

3. New upper bounds for the norm 1 of the inverse of *H*-matrices

Before we derive the upper bounds for the norm 1, we propose some auxiliary results.

Lemma 3.1. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be an M-matrix, $z \in \mathbb{R}^n$ be a vector with at least one component positive and $A^{-1}z =: \delta$. Denoting $A^{-1} = [b_{ij}]$, it holds that*

$$\min_{i\in N} \sum_{j\in N} b_{ij} \geq \frac{\min\limits_{i\in N} \delta_i}{\max\limits_{i\in N} z_i}. \tag{4}$$

Proof: Denote by $k \in N$ the index for which $z_k \geq z_j$, for all $j \in N$. Then, according to the assumption that vector z has at least one component positive, we infer that $z_k > 0$ and for every $i \in N$

$$\delta_i = \sum_{j\in N} b_{ij} z_j \leq \sum_{j\in N} b_{ij} z_k, \text{ i.e. } \sum_{j\in N} b_{ij} \geq \frac{\delta_i}{z_k},$$

therefore

$$\min_{i\in N} \sum_{j\in N} b_{ij} \geq \frac{\min\limits_{i\in N} \delta_i}{\max\limits_{i\in N} z_i}. \quad \square$$

We remark that the presented bound is meaningful under the condition that $\delta > 0$.

Lemma 3.2. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be an M-matrix, $z \in \mathbb{R}^n$ be a vector with at least one component positive and $(A^T)^{-1} z =: \eta$. Denoting $A^{-1} = [b_{ij}]$, it holds that*

$$\min_{j\in N} \sum_{i\in N} b_{ij} \geq \frac{\min\limits_{i\in N} \eta_i}{\max\limits_{i\in N} z_i}. \tag{5}$$

Proof: Directly follows from Lemma 3.1, since the class of M-matrices is transpose- invariant and due to the fact that $(A^{-1})^T = (A^T)^{-1}$. $\square$

In this section we will derive a unifying framework for bounding the norm 1 of the inverse of an arbitrary nonsingular H-matrix. First, we formulate the result in its general form, applicable to any nonsingular H-matrix, and then discuss applications to some special cases.

Theorem 3.3. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a nonsingular H-matrix and $\delta > 0$ be a positive vector, for which $\mathcal{M}(A)\delta > 0$ (such a vector always exists). Then,*

$$\|A^{-1}\|_1 \leq \xi + \frac{\|\delta\|_1 - \xi \|\mathcal{M}(A)\delta\|_1}{\min\limits_{k\in N} (\mathcal{M}(A)\delta)_k},$$

where

$$\xi := \frac{\min\limits_{i\in N} \delta_i}{\max\limits_{i\in N} (\mathcal{M}(A)^T \delta)_i}.$$

Proof: First, observe that $\mathcal{M}(A)$ is an M-matrix, which ensures that there exists a vector $\delta > 0$, such that $\mathcal{M}(A)\delta > 0$. Additionally, $\mathcal{M}(A)^T$ is also an M-matrix, since this class is transpose-invariant. Let us prove that vector $y = \mathcal{M}(A)^T \delta$ has at least one component positive.

In order to confirm this, we assume the opposite, i.e. $y_i \leq 0$, for all $i \in N$. From $\delta = (\mathcal{M}(A)^T)^{-1} y$, since all elements of $(\mathcal{M}(A)^T)^{-1}$ are nonnegative, it follows that, for all $i \in N$

$$\delta_i = ((\mathcal{M}(A)^T)^{-1} y)_i \leq 0,$$

which is an obvious contradiction. Therefore, choosing $\eta = \delta$, according to Lemma 3.2, it directly follows that

$$\min_{j\in N} \sum_{i\in N} b_{ij} \geq \frac{\min\limits_{i\in N} \delta_i}{\max\limits_{i\in N} (\mathcal{M}(A)^T \delta)_i} =: \xi. \tag{6}$$

On the other hand, denoting $\mathcal{M}(A)^{-1} = [b_{ij}]$ and defining $\mathcal{M}(A)\delta =: z$, we proceed by seeing that, component-wise, $\mathcal{M}(A)^{-1} z = \delta$ is equivalent to

$$\sum_{j\in N} b_{ij} z_j = \delta_i, \quad \text{for all } \ i \in N,$$

from whence, taking the sum over i, we get

$$\sum_{i\in N}\sum_{j\in N} b_{ij}z_j = \sum_{i\in N}\delta_i,$$

that is

$$\sum_{j\in N} z_j \sum_{i\in N} b_{ij} = \sum_{i\in N}\delta_i.$$

We may spot that for every $k \in N$,

$$z_k\sum_{i\in N} b_{ik} + \sum_{j\in N\setminus\{k\}} z_j \sum_{i\in N} b_{ij} = \sum_{i\in N}\delta_i,$$

from where we deduce, using the lower bound (6), that

$$\sum_{i\in N}\delta_i \geq z_k\sum_{i\in N} b_{ik} + \xi \sum_{j\in N\setminus\{k\}} z_j.$$

The positivity of z ensures that, after dividing both sides of the latter inequality by z_k, we have

$$\sum_{i\in N} b_{ik} \leq \frac{1}{z_k}\Big(\sum_{i\in N}\delta_i - \xi \sum_{j\in N\setminus\{k\}} z_j\Big),$$

which is equivalent to

$$\sum_{i\in N} b_{ik} \leq \frac{1}{z_k}\Big(\sum_{i\in N}\delta_i - \xi \sum_{j\in N} z_j + \xi z_k\Big) = \xi + \frac{\|\delta\|_1 - \xi\|z\|_1}{z_k}.$$

Finally, using this estimation,

$$\|\mathcal{M}(A)^{-1}\|_1 = \max_{k\in N}\sum_{i\in N} b_{ik} \leq \max_{k\in N}\Big(\xi + \frac{\|\delta\|_1 - \xi\|z\|_1}{z_k}\Big) =$$

$$= \xi + \frac{\|\delta\|_1 - \xi\|\mathcal{M}(A)\delta\|_1}{\min_k(\mathcal{M}(A)\delta)_k}.$$

According to Theorem 2.5, $|A^{-1}| \leq \mathcal{M}(A)^{-1}$. Finally, the monotonicity of norm 1 ensures that

$$\|A^{-1}\|_1 \leq \|\mathcal{M}(A)^{-1}\|_1,$$

which completes the proof. □

From the application point of view, in order to obtain results which can be practically evaluated, it is convenient to know vector δ. As it turns out, this vector is exactly the one from Definition 2.3, and it is known for some subclasses of H-matrices. For instance, as far as SDD matrices are concerned, it is known that $\delta = e$, where e is a vector with all components equal to 1. If we switch to SDD matrices, and take $\delta = e$, our Theorem 3.3 becomes Theorem 1 from [10]:

Corollary 3.4 (Theorem 1 from [10]). *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, be an SDD matrix. Then, $\|A^{-1}\|_1 \leq \nu_1(A)$, where*

$$\nu_1(A) := \frac{n - \dfrac{\sum_{i\in N}(|a_{ii}| - r_i) - \min_{i\in N}(|a_{ii}| - r_i)}{\max_{j\in N}(|a_{jj}| - c_j)}}{\min_{i\in N}(|a_{ii}| - r_i)}. \tag{7}$$

Proof: Using the result of Theorem 3.3 for a particular choice $\delta = e$, one can readily check that

$$\mathcal{M}(A)\delta = |a_{ii}| - r_i, \quad \mathcal{M}(A)^T\delta = |a_{ii}| - c_i, \quad \|\delta\|_1 = n, \quad \xi = \frac{1}{\max\limits_{j\in N}(|a_{jj}| - c_j)},$$

hence,

$$\xi + \frac{\|\delta\|_1 - \xi\|\mathcal{M}(A)\delta\|_1}{\min\limits_{k\in N}(\mathcal{M}(A)\delta)_k} =$$

$$= \frac{1}{\max\limits_{j\in N}(|a_{jj}| - c_j)} + \frac{n - \frac{1}{\max\limits_{j\in N}(|a_{jj}| - c_j)}\sum\limits_{i\in N}(|a_{ii}| - r_i)}{\min\limits_{k\in N}(|a_{kk}| - r_k)},$$

which is the same as (7). $\square$

For all the classes for which the positive diagonal matrix that scales a given nonsingular $H-$matrix to an SDD one is known, it is possible to derive upper bounds for the norm 1 of their inverse, with the aid of the aforementioned lemmas.

Here, we will restrict ourselves to the class of S-SDD matrices, since they arise in practical applications, including the optimization of wireless sensor networks, neural networks and empirical food webs. The upper bound for the norm 1 of their inverse is our following result.

Theorem 3.5. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, be an S-SDD matrix, for some nonempty subset of the index set. Then, $\|A^{-1}\|_1 \leq \nu_1^S(A)$, where*

$$\nu_1^S(A) := \xi + \frac{mg + n - m - \xi\left(g\sum\limits_{i\in S}|a_{ii}| + \sum\limits_{i\in\overline{S}}|a_{ii}| - g\sum\limits_{i\in N} r_i^S - \sum\limits_{i\in N} r_i^{\overline{S}}\right)}{\min\left\{\min\limits_{i\in S}\left(g\left(|a_{ii}| - r_i^S\right) - r_i^{\overline{S}}\right), \min\limits_{i\in\overline{S}}\left(|a_{ii}| - gr_i^S - r_i^{\overline{S}}\right)\right\}}, \tag{8}$$

with

$$\xi = \frac{\min\left\{1, g\right\}}{\max\left\{\max\limits_{i\in S}\left(g\left(|a_{ii}| - c_i^S\right) - c_i^{\overline{S}}\right), \max\limits_{i\in\overline{S}}\left(|a_{ii}| - gc_i^S - c_i^{\overline{S}}\right)\right\}}, \tag{9}$$

$$g := \frac{B_1 + B_2}{2}, \qquad B_1 := \max_{i\in S}\frac{r_i^{\overline{S}}}{|a_{ii}| - r_i^S},$$

$$B_2 := \begin{cases} 2 - B_1 & \text{if} \quad \overline{S} = \emptyset \\ B_1 & \text{if} \quad \overline{S} \neq \emptyset \text{ and for all } j \in \overline{S},\ r_j^S = 0 \\ \min\limits_{j\in\overline{S}: r_j^S\neq 0}\dfrac{|a_{jj}| - r_j^{\overline{S}}}{r_j^S} & \text{if} \quad \overline{S} \neq \emptyset \text{ and there exists } j \in \overline{S} \text{ such that } r_j^S \neq 0 \end{cases},$$

and with the usual convention that sum, min and max are set to be zero if they are taken over an empty set.

Proof: A is an S-SDD matrix, hence there exists a positive diagonal scaling matrix X, for which AX is an SDD matrix.

($\star$) If $S = N$, meaning that A is an SDD matrix, we can obviously choose X to be the identity matrix.

If S is a *proper* subset of N, a scaling matrix $X = diag(x_1, x_2, \dots, x_n)$ (for which AX is an SDD matrix) has the following form:

$$x_i = \begin{cases} \gamma & , \quad i \in S \\ 1 & , \quad i \in \overline{S} \end{cases},$$

where

($\star\star$) if for all $j \in \overline{S}$, $r_j^S = 0$, then

$$0 \le B_1 = \max_{i \in S} \frac{r_i^{\overline{S}}}{|a_{ii}| - r_i^S} < \gamma.$$

($\star\star\star$) if there exists $j \in \overline{S}$ such that $r_j^S \neq 0$, then

$$0 \le B_1 = \max_{i \in S} \frac{r_i^{\overline{S}}}{|a_{ii}| - r_i^S} < \gamma < \min_{j \in \overline{S}: r_j^S \neq 0} \frac{|a_{jj}| - r_j^{\overline{S}}}{r_j^S} = B_2.$$

Note that the direct implications of the fact that A is an S-SDD matrix are:

- the value B_1 is always well defined,
- in the case ($\star\star\star$), the interval (B_1, B_2) is non-empty.

In the case ($\star\star\star$), we will choose γ to be the middle point of the given interval, $\frac{B_1 + B_2}{2}$, while in the case ($\star\star$), we will choose γ to be equal to B_1. Obviously, the case ($\star$), when $S = N$, can be considered as a case for which γ is chosen to be equal to 1.

Generally speaking, in all three cases, ($\star$), ($\star\star$) and ($\star\star\star$), we can say that there exists a scaling matrix $X = diag(x_1, x_2, \dots, x_n)$ (for which AX is an SDD matrix) of the form:

$$x_i = \begin{cases} g & , \quad i \in S \\ 1 & , \quad i \in \overline{S} \end{cases} \qquad \text{where} \qquad g = \frac{B_1 + B_2}{2},$$

and B_1, B_2 are defined in Theorem. For such a choice of $\gamma = g$, the condition for AX to be an SDD matrix is equivalent to $\mathcal{M}(A)\delta > 0$, where

$$\delta_i = \begin{cases} g & , \quad i \in S \\ 1 & , \quad i \in \overline{S} \end{cases}.$$

Since

$$(\mathcal{M}(A)\delta)_i = \begin{cases} g|a_{ii}| - g r_i^S - r_i^{\overline{S}} & , \quad i \in S \\ |a_{ii}| - g r_i^S - r_i^{\overline{S}} & , \quad i \in \overline{S} \end{cases}$$

and

$$(\mathcal{M}(A)^T\delta)_i = \begin{cases} g|a_{ii}| - g c_i^S - c_i^{\overline{S}} & , \quad i \in S \\ |a_{ii}| - g c_i^S - c_i^{\overline{S}} & , \quad i \in \overline{S} \end{cases}$$

on the grounds of Theorem 3.3, we conclude that

$$||A^{-1}||_1 \le \xi + \frac{||\delta||_1 - \xi ||\mathcal{M}(A)\delta||_1}{\min_{k \in N} (\mathcal{M}(A)\delta)_k},$$

where

$$\xi = \frac{\min_{i \in N} \delta_i}{\max_{i \in N} (\mathcal{M}(A)^T \delta)_i}. \tag{10}$$

First, we will confirm that (8) holds.

To that end, let us assume that $card(S) = m$. Then,

$$\|\delta\|_1 = mg + n - m = mg + n - m,$$

$$\|\mathcal{M}(A)\delta\|_1 = \sum_{i\in N}(\mathcal{M}(A)\delta)_i = \sum_{i\in S}\left(g\left(|a_{ii}| - r_i^S\right) - r_i^{\overline{S}}\right) + \sum_{j\in\overline{S}}\left(|a_{jj}| - gr_j^S - r_j^{\overline{S}}\right) =$$

$$= g\sum_{i\in S}|a_{ii}| + \sum_{i\in\overline{S}}|a_{ii}| - g\sum_{i\in N} r_i^S - \sum_{i\in N} r_i^{\overline{S}},$$

and

$$\min_{k\in N}(\mathcal{M}(A)\delta)_k =$$

$$= \min\left\{\min_{i\in S}\left(g\left(|a_{ii}| - r_i^S\right) - r_i^{\overline{S}}\right), \min_{i\in\overline{S}}\left(|a_{ii}| - r_i^S - r_i^{\overline{S}}\right)\right\}.$$

This minimum is always positive. It is obvious for cases $(\star)$ and $(\star\star)$, while in the case $(\star\star\star)$ it is also true, because for every $i \in S$ and every $j \in \overline{S}$, for which $r_j^S \neq 0$,

$$\frac{r_i^{\overline{S}}}{|a_{ii}| - r_i^S} \leq B_1 < g < B_2 \leq \frac{|a_{jj}| - r_j^{\overline{S}}}{r_j^S},$$

which implies that for every $i \in S$

$$g\left(|a_{ii}| - r_i^S\right) - r_i^{\overline{S}} > \frac{r_i^{\overline{S}}}{|a_{ii}| - r_i^S}\left(|a_{ii}| - r_i^S\right) - r_i^{\overline{S}} = 0,$$

and, similarly, for every $j \in \overline{S}$, for which $r_j^S \neq 0$,

$$|a_{jj}| - gr_j^S - r_j^{\overline{S}} > |a_{jj}| - \frac{|a_{jj}| - r_j^{\overline{S}}}{r_j^S} r_j^S - r_j^{\overline{S}} = 0.$$

It only remains to show (9). Apparently, $\min_{i\in N}\delta_i = \min\{1, g\}$. As far as the denominator in (10) is concerned, its positivity is implied by $\mathcal{M}(A)^T\delta$ having at least one component positive. Finally,

$$\max_{i\in N}(\mathcal{M}(A)^T\delta)_i = \max\left\{\max_{i\in S}\left(g\left(|a_{ii}| - c_i^S\right) - c_i^{\overline{S}}\right), \max_{i\in\overline{S}}\left(|a_{ii}| - gc_i^S - c_i^{\overline{S}}\right)\right\},$$

which completes the proof. □

4. Lower bounds for σ_{min}

It is well-known that

$$(\sigma_{min}(A)^{-1})^2 = \|A^{-1}\|_2^2 \leq \|A^{-1}\|_\infty \|A^{-1}\|_1,$$

so we can use this fact to immediately obtain the lower bound for the minimal singular value of an S-SDD matrix.

Theorem 4.1. *Let $A \in \mathbb{C}^{n,n}$ be an S-SDD matrix, for some nonempty proper subset S of the index set. Then,*

$$\sigma_{min}(A) \geq \frac{1}{\sqrt{\nu_\infty^S(A)\nu_1^S(A)}}.$$

However, the approach based on S-SDD matrices can help one to improve some already known lower bounds for the minimal singular value of an SDD matrix, as well. Before we formulate this new lower bound as a theorem, let us prove the following Lemma.

Lemma 4.2. *Let $A \in \mathbb{C}^{n,n}$ be an SDD matrix. Then, for every nonempty proper subset S of the index set N, it holds that*

$$\nu_\infty^S(A) \leq \nu_\infty(A). \tag{11}$$

Proof: Observe that A, being an SDD matrix, is also an S-SDD matrix for every $S \subseteq N$. In order to prove (11), it is sufficient to show that, for every $i \in S$ and for every $j \in \overline{S}$, both

$$\theta_{ij}^S(A) \leq \max_{i\in S} \frac{1}{|a_{ii}| - r_i} \quad \text{and} \quad \theta_{ji}^{\overline{S}}(A) \leq \max_{j\in \overline{S}} \frac{1}{|a_{jj}| - r_j}$$

holds. Let $i \in S$ and $j \in \overline{S}$ be chosen arbitrarily. First, if $|a_{ii}| - r_i \leq |a_{jj}| - r_j$, then this is equivalent to $|a_{ii}| - r_i^S + r_j^S - r_i^{\overline{S}} \leq |a_{jj}| - r_j^{\overline{S}}$. Multiplying both sides by $|a_{ii}| - r_i^S$ which is positive and subtracting $r_i^{\overline{S}} r_j^S$ from both sides gives

$$(|a_{ii}| - r_i^S + r_j^S)(|a_{ii}| - r_i^S - r_i^{\overline{S}}) \leq (|a_{ii}| - r_i^S)(|a_{jj}| - r_j^{\overline{S}}) - r_i^{\overline{S}} r_j^S,$$

with the right-hand side being positive by assumption. Hence,

$$\frac{|a_{ii}| - r_i^S + r_j^S}{(|a_{ii}| - r_i^S)(|a_{jj}| - r_j^{\overline{S}}) - r_i^{\overline{S}} r_j^S} \leq \frac{1}{|a_{ii}| - r_i}.$$

Taking the maximum over $i \in S$ and $j \in \overline{S}$ leads to

$$\max_{\substack{i\in S\\ j\in \overline{S}}} \frac{|a_{ii}| - r_i^S + r_j^S}{(|a_{ii}| - r_i^S)(|a_{jj}| - r_j^{\overline{S}}) - r_i^{\overline{S}} r_j^S} \leq \max_{i\in S} \frac{1}{|a_{ii}| - r_i}. \tag{12}$$

However, if $|a_{jj}| - r_j < |a_{ii}| - r_i$, then multiplying both sides with $|a_{jj}| - r_j^{\overline{S}}$, which is again positive, and subtracting $r_i^{\overline{S}} r_j^S$ from both sides produces

$$(|a_{jj}| - r_j^{\overline{S}} + r_i^{\overline{S}})(|a_{jj}| - r_j^S - r_j^{\overline{S}}) < (|a_{ii}| - r_i^S)(|a_{jj}| - r_j^{\overline{S}}) - r_i^{\overline{S}} r_j^S,$$

which implies

$$\frac{|a_{jj}| - r_j^{\overline{S}} + r_i^{\overline{S}}}{(|a_{ii}| - r_i^S)(|a_{jj}| - r_j^{\overline{S}}) - r_i^{\overline{S}} r_j^S} < \frac{1}{|a_{jj}| - r_j}.$$

From there,

$$\max_{\substack{i\in S\\ j\in \overline{S}}} \frac{|a_{jj}| - r_j^{\overline{S}} + r_i^{\overline{S}}}{(|a_{ii}| - r_i^S)(|a_{jj}| - r_j^{\overline{S}}) - r_i^{\overline{S}} r_j^S} < \max_{j\in \overline{S}} \frac{1}{|a_{jj}| - r_j}. \tag{13}$$

Finally, (12) and (13) imply that $\max\{\theta_{ij}^S(A), \theta_{ji}^{\overline{S}}(A)\} \leq \max_{i\in N} \frac{1}{|a_{ii}| - r_i}$, which completes the proof. $\square$

Therefore, when dealing with SDD matrices, with some investment in the calculations, we can *improve* the Varah bound (1), by optimizing the upper bound (3) over all nonempty proper subsets S. In terms of norm 1, we can expect the similar behaviour, if we optimize (8) over all nonempty $S \subseteq N$. Then, a better lower bound for the minimal singular value of an SDD matrix can be obtained:

Theorem 4.3. *Let $A \in \mathbb{C}^{n,n}$ be an SDD matrix. Then,*

$$\sigma_{min}(A) \geq \frac{1}{\sqrt{\alpha\beta}} := \mathcal{E},$$

where

$$\alpha := \min_{\emptyset \neq S \subsetneq N} \nu_\infty^S(A) \qquad \textit{and} \qquad \beta := \min_{\emptyset \neq S \subseteq N} \nu_1^S(A).$$

In the next section, we will compare this bound with the existing ones on some relevant numerical examples. Before that, let us comment how we can obtain block generalizations of the previous results.

In order to bound the smallest singular value of a block matrix $A \in \mathbb{C}^{n,n}$, provided that it is a $B_2^\pi H$-matrix, we will use Theorem 2.10, from whence we have

$$\sigma_{min}(A) = \|A^{-1}\|_2^{-1} \geq \|(\langle A\rangle_\pi^{(2)})^{-1}\|_2^{-1} = \sigma_{min}(\langle A\rangle_\pi^{(2)}).$$

In other words, the smallest singular value of a (large-scale) block matrix can be bounded from below by the smallest singular value of the comparison matrix $\langle A\rangle_\pi^{(2)} \in \mathbb{R}^{\ell,\ell}$, playing the role of a point-wise matrix, for which we can use previous results.

New lower bounds for the minimal singular value of a B^π S-SDD and B^π SDD matrices are given in the following two theorems.

Theorem 4.4. *Let $A \in \mathbb{C}^{n,n}$ be a B^π S-SDD matrix, for some nonempty proper subset S of the index set. Then,*

$$\sigma_{min}(A) \geq \frac{1}{\sqrt{\nu_\infty^S(\langle A\rangle_\pi)\nu_1^S(\langle A\rangle_\pi)}}.$$

Theorem 4.5. *Let $A \in \mathbb{C}^{n,n}$ be a B^π SDD matrix. Then,*

$$\sigma_{min}(A) \geq \frac{1}{\sqrt{\widehat{\alpha}\widehat{\beta}}} := \widehat{\mathcal{E}},$$

where

$$\widehat{\alpha} := \min_{\emptyset \neq S \subsetneq N} \nu_\infty^S(\langle A\rangle_\pi) \qquad \textit{and} \qquad \widehat{\beta} := \min_{\emptyset \neq S \subseteq N} \nu_1^S(\langle A\rangle_\pi).$$

5. Numerical examples

Example 1 (point-wise case): Consider an example from the modelling of dynamical processes occurring in ecology. When analyzing the stability of a dynamical system that models the energy flow in the soil food web, the matrix of interest is the same as in [8], where it was referred to as the community matrix, obtained as the Jacobian of the dynamical system presented in [11–13],

$$A = \begin{bmatrix} -6.0000 & 0 & 0 & 0 & 0.0200 & 0 & 0.0200 & 0 \\ 0 & -1.8400 & 0 & 0 & 0 & 0.0015 & 0 & 0 \\ 0 & 0 & -1.9200 & 0 & 0 & 0.0016 & 0 & 0 \\ 0 & 0 & 0 & -2.6800 & 0 & 0.0045 & 0 & 0 \\ -0.0263 & 0 & 0 & 0 & -6.0000 & 0 & 0.0101 & 0 \\ 0 & -10.5143 & -13.6558 & -12.0721 & 0 & -1.2000 & 0 & 0.9302 \\ -15.7632 & 0 & 0 & 0 & -15.7618 & 0 & -1.2000 & 0.9591 \\ 6.7895 & 7.0971 & 10.3866 & 7.5088 & 6.6331 & -2.9147 & -3.0590 & -6.2977 \end{bmatrix}.$$

This matrix is obviously not an SDD one, but one can confirm that it is an S-SDD for $S = \{1, 2, 3, 4, 5\}$. The results are summarized in the following table.

$\|A^{-1}\|_\infty$	ν_∞^S	$\|A^{-1}\|_1$	ν_1^S	σ_{min}	$\mathcal{E}$
12.2139	86.1454	8.3171	359.2074	0.1398	0.0057

ε_1	ε_2	ε_3	ε_4
-18.8473	-3.0402	-17.9392	-18.8473

Bound (28) from [10] is not applicable, since A is not SDD. Some other well-known lower bounds for the smallest singular value, introduced in [6, 7] and reviewed in [4], cannot be used either, since all of them are negative. Namely,

Example 2 (block case): Consider the following block tridiagonal matrix arising in boundary value problems,

$$A = \begin{bmatrix} T & -I & O & \cdots & O \\ -I & T & -I & \cdots & O \\ O & \ddots & \ddots & \ddots & O \\ \vdots & & -I & T & -I \\ O & \cdots & O & -I & T \end{bmatrix} \in \mathbb{R}^{n^2,n^2}, \text{ where } T = \begin{bmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -1 & 4 & -1 \\ 0 & \cdots & 0 & -1 & 4 \end{bmatrix} \in \mathbb{R}^{n,n},$$

with I being the identity matrix of order n. We will consider two cases, $n = 4$ and $n = 10$, while the partition will always be $\pi = \{0, n, 2n, \ldots, n^2\}$ and the norm of choice is the Euclidean norm.

For the first case, where block matrix A is of order 16, $\sigma_{min}(A) = 0.7639$, and this value coincides with the smallest singular value of the comparison matrix, $\langle A\rangle_\pi^{(2)}$. We also note that A is not an SDD, whereas $\langle A\rangle_\pi^{(2)}$ is, so this makes it possible to compare $\mathcal{E}$ to the bound denoted by (28) in [10]. The following table presents results for the comparison matrix, for which $\varepsilon_i = 0.382$, for all $i = 1, 2, 3, 4$.

$\|(\langle A\rangle_\pi^{(2)})^{-1}\|_\infty$	α	$\|(\langle A\rangle_\pi^{(2)})^{-1}\|_1$	β	(28) from [10]	$\mathcal{E}$
1.4757	1.4757	1.4757	2.8727	0.2909	0.4857

As we can conclude, not only has bound $\mathcal{E}$ outperfomed the bound from [10], but also each of the standard bounds, ε_i, $i = 1, 2, 3, 4$, as well. The optimal choice of S for α is $S = \{1, 4\}$ and for β it is $S = \{2, 3\}$.

Now, we turn to the case $n = 10$. Matrix A is now of order 100, with $\sigma_{min}(A) = 0.162$, again the same as $\sigma_{min}(\langle A\rangle_\pi^{(2)})$. However, as before, A is not SDD, while $\langle A\rangle_\pi^{(2)}$ is, so we are able to perform our analysis in a similar way. Computations reveal that $\varepsilon_i = 0.081$, for all $i = 1, 2, 3, 4$.

This time, $\mathcal{E}$ only outperformed the bound from [10]. The optimal choice for α this time is $S = \{3, 4, 7, 8\}$, where the one for β is $S = \{2, 3, \ldots, 9\}$.

As a conclusion, let us remark that in the case of a block H-matrix, the lower bound for the smallest singular value is obtained by bounding the smallest singular value of its comparison matrix, which, in the Euclidean norm, has a very specific structure. Namely, $\langle A\rangle_\pi^{(2)} = [\alpha_{ij}]$, where

$$\alpha_{ij} = \begin{cases} \sigma_{min}(A_{ii}), & i = j, \\ -\sigma_{max}(A_{ij}), & i \neq j. \end{cases}$$

Therefore, the comparison matrix itself requires a calculation of extreme singular values of each of the blocks from the partitioned matrix. However, in our example, as well as in most examples arising from boundary value problems, these matrices are low-dimensional ones in addition to being the same along the diagonal as well as outside it, so practically it is only required to calculate two singular values, one smallest of the diagonal block, and one largest of the off-diagonal block.

$\|(\langle A\rangle_\pi^{(2)})^{-1}\|_\infty$	α	$\|(\langle A\rangle_\pi^{(2)})^{-1}\|_1$	β	(28) from [10]	$\mathcal{E}$
7.3263	12.3435	7.3263	75.5519	0.0296	0.0327

References

[1] Ahlberg, J.H., Nilson, E.N.: Convergence properties of the spline fit. J.SIAM (1963), 95-104.

[2] Berman, A., Plemmons, R.J.: Nonnegative Matrices in the Mathematical Sciences. Classics in Applied Mathematics 9, SIAM, Philadelphia, 1994

[3] Cvetković, Lj., Kostić, V., Varga, R.: A new Geršgorin-type eigenvalue inclusion area. ETNA 18 (2004), 73-80.

[4] Cvetković, Lj., Kostić, V., Doroslovački, K., Cvetković D.: Euclidean norm estimates of the inverse of some special block matrices. Appl. Math. Comput. 284 (2016), 12-23.

[5] Gao, Y.M., Wang, X.H.: Criteria for generalized diagonally dominant and *M*-matrices. Linear Algebra and Appl. 268 (1992), 257-268.

[6] Johnson, C.R.: A Gersgorin-type Lower Bound for the Smallest Singular Value. Linear Algebra and its Applications 112 (1989), 1-7.

[7] Johnson, C.R., Szulc, T.: Further Lower for the Smallest Singular Value. Linear Algebra and its Applications 272 (1998), 169-179.

[8] Kostić, V., Cvetković, Lj., Cvetković, D.: Pseudospectra localizations and their applications. Numerical Linear Algebra with Applications 23(2) (2016), 356-372.

[9] Morača, N.: Upper bounds for the infinity norm of the inverse of SDD and *S*-SDD matrices. Journal of Computational and Applied Mathematics 206 (2007), 666-678.

[10] Morača, N.: Bounds for norms of the matrix inverse and the smallest singular value. Linear Algebra and its Applications 429 (2008), 2589-2601.

[11] Moore, J.C., de Ruiter, P.C.: Energetic Food Webs. Oxford University Press, Oxford (2012).

[12] Neutel, A.M., Heesterbeek, J.A.P., van de Koppel, J., Hoenderboom, G., Vos, A., Kaldeway, C., Berendse, F., de Ruiter, P.C.: Reconciling complexity with stability in naturally assembling food webs (Letter). Nature 449 (2008), 599-602.

[13] Neutel, A.M., Thorne, M.A.S.: Interaction strengths in balanced carbon cycles and the absence of a relation between ecosystem complexity and stability. Ecology Letters 17(6) (2014), 651-661.

[14] Varah J. M.: A lower bound for the smallest value of a matrix. Linear Algebra Appl. 11(1975), 3-5.