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Toughness Condition for a Graph to be All Fractional (g, f, n)-Critical Deleted

Wei Gao[a], Weifan Wang[b], Darko Dimitrov[c]

[a] *School of Information Science and Technology, Yunnan Normal University, Kunming 650500, China*
[b] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China*
[c] *Information Studies, Novo Mesto, Slovenia*

Abstract. In data transmission networks, the feasibility of data transmission can be characterized by the existence of fractional factors. If some channels and stations are not available in the transmission network at the moment, the possibility of transmission between data is characterized by whether the corresponding graph structure of the network is critical deleted. Toughness used to measure the vulnerability and robustness of a network, which is an important parameter to be considered in network designing. In this paper, we mainly study the relationship between toughness and the all fractional critical deleted graph, and a toughness condition for a graph to be all fractional (g, f, n)-critical deleted is determined.

1. Introduction

As a relaxation of the well-known cardinality matching problem, the problem of fractional factor acts as a crucial problem in operation research and computer networks, which has been widely applied in different areas such as network designing, combinatorial polyhedron, scheduling, etc. In data transmission networks, large data packets are sent to different destinations via channels, and to efficiently improve its workload, we divide the large data packets into small ones, and thus the available assignments of data packets can be modelled and described as the problem of fractional flow. Furthermore, it can be converted to a problem of the existence of fractional factor in certain network graph. In turn, theoretical analysis can help scientists make effective network designs at the beginning. Several developed tricks on graph based networks designing can be referred to Rahimi and Haghighi [1], Haenggi et al. [2], Fardad et al. [3], Pishvaee and Rabbani [4], Lanzeni [5], Possani et al. [6], Ashwin and Postlethwaite [7], de Araujo et al. [8], Rizzelli et al. [9], and Crouzeilles et al. [10].

All graphs considered in this article are loopless, finite and without multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$ (or simply $d(x)$ and $N(x)$), respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. For two vertex-disjoint subsets S and T of G, we use $e_G(S, T)$ to denote the number of edges with one end in S and the other end in T. Minimum degree of G is

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Email addresses: gaowei@ynnu.edu.cn (Wei Gao), wwf@zjnu.cn (Weifan Wang), darko.dimitrov11@gmail.com (Darko Dimitrov)

denoted by $\delta(G) = \min\{d(x)|x \in V(G)\}$. The notations and terminologies used but undefined in this paper can be referred to Bondy and Murty [11].

Suppose that g and f are two integer-valued functions on $V(G)$ such that $0 \le g(x) \le f(x)$ for all $x \in V(G)$. A *fractional* (g, f)*-factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \le d^h_G(x) \le f(x)$, where $d^h_G(x) = \sum_{e\in E(x)} h(e)$ is called the *fractional degree* of x in G. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a fractional (g, f)-factor is a fractional $[a, b]$-factor. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional (g, f)-factor is a fractional f-factor. Moreover, if $g(x) = f(x) = k$ ($k \ge 1$ is an integer) for all $x \in V(G)$, then a fractional (g, f)-factor is just a fractional k-factor.

A graph G is called a *fractional* (g, f, m)*-deleted graph* if for each edge subset $H \subseteq E(G)$ with $|H| = m$, there exists a fractional (g, f)-factor h such that $h(e) = 0$ for all $e \in H$. In other words, after removing any m edges, the resulting graph admits a fractional (g, f)-factor. Especially when $m = 1$, fractional (g, f, m)-deleted graph is called fractional (g, f)-deleted graph. A graph G is called a *fractional* (g, f, n)*-critical graph* if after delated any n vertices from G, the resulting graph still has a fractional (g, f)-factor.

We say that G has all fractional (g, f)-factors if G has a fractional p-factor for each $p : V(G) \to \mathbb{N}$ satisfying $g(x) \le p(x) \le f(x)$ for any $x \in V(G)$. If $g(x) = a$, $f(x) = b$ for each vertex x and G has all fractional (g, f)-factors, then we say that G has all fractional $[a, b]$-factors. In data transmission networks, a network graph has all fractional (g, f)-factor corresponding to the data packets within a given capacity range which can be transmitted at a moment.

Lu [12] presented the sufficient and necessary condition for a graph having all fractional (g, f)-factors. Zhou and Sun [13] introduced the concept of all fractional (a, b, n)-critical graph, i.e., a graph G is called an all fractional (a, b, n)-critical graph if after deleting any n vertices of G the remaining graph of G admits all fractional $[a, b]$-factors. Also, the necessary and sufficient condition for a graph to be all fractional (a, b, n)-critical is derived.

Gao et al. [14] combined two concepts of all fractional (g, f, m)-deleted graph and all fractional (g, f, n)-critical graph together. A graph G is called an all fractional (g, f, n, m)-critical deleted graph if after deleting any n vertices of G the remaining graph of G is still an all fractional (g, f, m)-deleted graph. If $g(x) = a$, $f(x) = b$ for each $x \in V(G)$, then all fractional (g, f, n, m)-critical deleted graph becomes all fractional (a, b, n, m)-critical deleted graph, which means, after deleting any n vertices of G the remaining graph of G is still an all fractional (a, b, m)-deleted graph. If $m = 1$, then all fractional (g, f, m)-deleted, all fractional (g, f, n, m)-critical deleted graph, all fractional (a, b, n, m)-critical deleted graph are all fractional (g, f)-deleted, all fractional (g, f, n)-critical deleted graph, and all fractional (a, b, n)-critical deleted graph, respectively. Since in data transmission networks, each site is modelled as a vertex and each channel is expressed as an edge, the concept of all fractional (g, f, n, m)-critical deleted graph implies that the data packets within a given capacity range can be still transmitted when certain sites and channels are damaged or blocked.

The notation of *toughness* was first introduced by Chvátal [15] to measure the vulnerable of networks: $t(G) = \infty$ if G is a complete graph; otherwise

$$t(G) = \min\{\frac{|S|}{\omega(G-S)} : \omega(G-S) \ge 2\}$$

where $\omega(G-S)$ is the number of connected components of $G-S$. It's a vital parameter in network designing: if the toughness is large, the network is strong and stable, but the constructed network is complex and difficult to control; if the toughness is small, it's convenient to set control points in the network, but the whole network is easily attacked and damaged. In the specific network designing process, we often look for a balance and take a moderate value for $t(G)$, thus we take into account the both vulnerability and complexity.

In the past 40 years, there have already been rich results on the relationship between toughness and factors as well as fractional factors. Enomoto published a series of articles to explain the relationship between toughness and existence of k-factors, see [16], [17], [18] and [19] for details. Chen [20] proposed the sharp bound of the toughness for the existence of a $[2, b]$-factor in a graph. Gao and Gao [21], Gao et al. [22], and Gao and Wang [23] discussed the toughness condition for a graph to be fractional (g, f, n)-critical deleted. Zhou et al. [24] proved that a graph G is a fractional (k,m)–deleted graph if $\delta(G) \geq k+2m$ and $t(G) \geq k+\frac{2m-1}{k}$ for $k \geq 2$. However, there is a fatal mistake in the counterexample for its sharpness. Therefore, the tight toughness bound for fractional (k,m)–deleted graphs is still open (see Gao et al. [25] for more details). Gao et al. [26] determined two tight independent set conditions for fractional (g,f,m)-deleted graphs. More results on toughness and other conditions for fractional factors can refer to Zhou et al. [27], [28], [29], [30], [31], [32] and [33], and Gao et al. [34], [35] and [36].

In this work, we focus on the all fractional (g,f,n,m)-critical deleted graph in the special case $m=1$, i.e., all fractional (g,f,n)-critical deleted graph. We study the relationship between toughness and all fractional (g,f,n)-critical deleted graph, and the main result in our paper is stated as follows.

Theorem 1.1. *Let G be a graph and let g,f be two integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) \leq b$ with $1 \leq a \leq b$, $b \geq 2$ and $(a,b) \neq (1,2)$ for all $x \in V(G)$, where a,b are positive integers. Let n be a non-negative integer and $\Delta = b-a$. Assume $|V(G)| \geq n+b+2$ if G is complete. If $t(G) \geq \frac{b^2-\Delta-1}{a}+n$, then G is an all fractional (g,f,n)-critical deleted graph.*

The rest of paper is organized as follows: first we introduce some lemmas which are used in the proof of Theorem 1.1; the detailed proof is presented in the third section; at last, we discuss the toughness bound and future prospects.

2. Some useful lemmas

Recall that Gao et al. [14] characterized the necessary and sufficient condition of all fractional (g,f,n,m)-critical deleted graphs which is described as follows.

Theorem 2.1. (Gao et al. [14]) *Let m and n be nonnegative integers. Let $g,f : V(G) \to \mathbb{Z}^+$ be two valued functions with $g(x) \leq f(x)$ for each $x \in V(G)$, and H be a subgraph of G with m edges. Then G is all fractional (g,f,n,m)-critical deleted if and only if*

$$g(S) - f(T) + \sum_{x\in T} d_{G-S}(x) \geq \max_{U\subseteq S, |U|=n, H\subseteq E(G-U), |H|=m} \{g(U) + \sum_{x\in T} d_H(x) - e_H(S,T)\},$$

for any non-disjoint subsets $S,T \subseteq V(G)$ with $|S| \geq n$.

Let

$$\varepsilon(S,T) = \begin{cases} 2, & T \text{ is not an independent set} \\ 1, & T \text{ is an independent set and } e_G(T, V(G)\setminus(S\cup T)) \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

By setting $m=1$ in the necessary and sufficient condition in Theorem 2.1, we deduce the necessary and sufficient condition for all fractional (g,f,n)-critical deleted graphs which is used as the criterion for proofing the main conclusion in our paper.

Lemma 2.2. (Gao et al. [14]) *Let n be nonnegative integer, and $g,f : V(G) \to \mathbb{Z}^+$ be two valued functions with $g(x) \leq f(x)$ for each $x \in V(G)$. Then G is all fractional (g,f,n)-critical deleted if and only if*

$$g(S) - f(T) + \sum_{x\in T} d_{G-S}(x) \geq \max\{g(U) : U \subseteq S, |U| = n\} + \varepsilon(S,T),$$

for any non-disjoint subsets $S,T \subseteq V(G)$ with $|S| \geq n$.

The lemma stated below manifested the relationship between toughness and minimum degree of graph.

Lemma 2.3. (Chvátal [15]) *If a graph G is not complete, then $t(G) \leq \frac{1}{2}\delta(G)$.*

In next section, we depend heavily on the following two lemmas which are determined by Liu and Zhang [37].

Lemma 2.4. (Liu and Zhang [37]) *Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \dots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1}(k-j)c_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $j = 1, \dots, k-1$.

Obviously, Lemma 2.4 is also correct for $\delta(H) \geq 0$. In terms of the proving procedure of Lemma 2.2 in [37], we obtain the following important Lemma in which the way of presentation is revised.

Lemma 2.5. (Liu and Zhang [37]) *Let G be a graph and let $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \geq 2$. Then there exists an independent set I and the covering set $C = V(H) - I$ of H satisfying*

$$|V(H)| \leq \sum_{i=1}^{k}(k-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

and

$$|C| \leq \sum_{i=1}^{k}(k-i)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

where $I^{(i)} = \{x \in I, d_H(x) = k-i\}$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k}|I^{(i)}| = |I|$.

3. Proof of Themorem 1.1

If G is complete, the result is obtained by means of $|V(G)| \geq n+b+2$. In what follows, we assume that G is not complete. Suppose that G satisfies the conditions of Theorem 1.1, but is not an all fractional (g,f,n)-critical deleted graph. By Lemma 2.2 and the fact that $\varepsilon(S,T) \leq 2$, there exist disjoint subsets S ($|S| \geq n$) and T of $V(G)$ satisfying

$$\begin{aligned} a|S| - b|T| + \sum_{x\in T} d_{G-S}(x) - an &\leq a(|S|-n) + \sum_{x\in T}(d_{G-S}(x) - b) \\ \leq \quad g(S-U) + \sum_{x\in T}(d_{G-S}(x) - f(x)) &\leq 1. \end{aligned} \tag{1}$$

We select S and T such that $|T|$ is minimum. Clearly, $T \neq \emptyset$ (due to Lemma 2.2 and $\varepsilon(S,T) = 0$ if T is empty). Clearly, $d_{G-S}(x) \leq b-1$ for any $x \in T$ since $|T|$ is minimum.

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_b and let $T_0 = \{x \in V(H') | d_{G-S}(x) = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those l components isomorphic to K_b.

If $|V(H)| = 0$, then from (1) we obtain

$$a|S| \leq b|T_0| + bl + an + 1$$

or

$$|S| \leq \frac{b(|T_0|+l)+an+1}{a}.$$

If $\omega(G-S) = |T_0| + l > 1$, then $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{b(|T_0|+l)+an+1}{a(|T_0|+l)} < \frac{b+an+1}{a}$, which contradicts $t(G) \geq \frac{b^2-\Delta-1}{a} + n$ and $b \geq 2$. Suppose $\omega(G-S) = |T_0| + l = 1$. By Lemma 2.3, $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq 2t(G)$, and thus

$2t(G) \le b-1+|S| \le b-1+\frac{an+b+1}{a}$. Using $t(G) \ge \frac{b^2-\Delta-1}{a}+n$, we infer $(a,b,n)=(1,2,0)$, which contradicts to the assumption of Theorem 1.1. Therefore, we have $|V(H)|>0$.

Let $H=H_1\cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x)=b-1$ for every vertex $x\in V(H_1)$ and $H_2=H-H_1$. By Lemma 2.5, H_1 has a maximum independent set I_1 and the covering set $C_1=V(H_1)-I_1$ such that

$$|V(H_1)| \le \sum_{i=1}^{b}(b-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \tag{2}$$

and

$$|C_1| \le \sum_{i=1}^{b}(b-i)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \tag{3}$$

where $I^{(i)}=\{x\in I_1, d_{H_1}(x)=b-i\}$ for $1\le i\le b$ and $\sum_{i=1}^{b}|I^{(i)}|=|I_1|$. On the other hand, let $T_j=\{x\in V(H_2)|d_{G-S}(x)=j\}$ for $1\le j\le b-1$. By the definitions of H and H_2 we can also see that each component of H_2 has a vertex of degree at most $b-2$ in $G-S$. According to Lemma 2.4, H_2 has a maximal independent set I_2 and the covering set $C_2=V(H_2)-I_2$ such that

$$\sum_{j=1}^{b-1}(b-j)c_j \le \sum_{j=1}^{b-1}(b-2)(b-j)i_j, \tag{4}$$

where $c_j=|C_2\cap T_j|$ and $i_j=|I_2\cap T_j|$ for every $j=1,\dots,b-1$. Set $W=V(G)-S-T$ and $U=S\cup C_1\cup (N_G(I_1)\cap W))\cup C_2\cup(N_G(I_2)\cap W)$. We derive

$$\begin{aligned}
& |C_2|+|N_G(I_2)\cap W| \\
= \; & |V(H_2)|-|I_2|+|N_{G-S-T}(I_2)| \\
= \; & |V(H_2)|-|I_2|+|N_{G-S}(I_2)|-|N_T(I_2)| \\
= \; & (|V(H_2)|-|I_2|-|N_T(I_2)|)+|N_{G-S}(I_2)| \\
\le \; & (|V(H_2)|-|I_2|-|N_{H_2}(I_2)|)+|N_{G-S}(I_2)| \\
\le \; & 0+\sum_{j=1}^{b-1} ji_j = \sum_{j=1}^{b-1} ji_j.
\end{aligned}$$

Furthermore, we get

$$|U| \le |S|+|C_1|+\sum_{j=1}^{b-1} ji_j+\sum_{i=1}^{b}(i-1)|I^{(i)}| \tag{5}$$

and

$$\omega(G-U) \ge t_0+l+|I_1|+\sum_{j=1}^{b-1} i_j, \tag{6}$$

where $t_0=|T_0|$. When $\omega(G-U)>1$, we have

$$|U| \ge t\omega(G-U), \tag{7}$$

and it also holds when $\omega(G-U)=1$.

By (5)-(7), we yield

$$|S| + |C_1| \geq \sum_{j=1}^{b-1}(t-j)i_j + t(t_0 + l) + t|I_1| - \sum_{i=1}^{b}(i-1)|I^{(i)}|. \tag{8}$$

In view of $b|T| - d_{G-S}(T) \geq a|S| - an - 1$, we have

$$bt_0 + bl + |V(H_1)| + \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j \geq a|S| - an - 1.$$

Combining with (8), we deduce

$$\begin{aligned} & |V(H_1)| + \sum_{j=1}^{b-1}(b-j)c_j + a|C_1| \\ \geq \ & \sum_{j=1}^{b-1}(at - aj - b + j)i_j + (at-b)(t_0 + l) + at|I_1| - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - an - 1. \end{aligned} \tag{9}$$

By (2) and (3), we get

$$|V(H_1)| + a|C_1| \leq \sum_{i=1}^{b}(ab - ai + b - (i-1))|I^{(i)}| - \frac{(a+1)|I^{(1)}|}{2}. \tag{10}$$

Using (4), (9) and (10), we yield

$$\begin{aligned} & \sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b - (i-1))|I^{(i)}| \\ \geq \ & \sum_{j=1}^{b-1}(at - aj - b + j)i_j + (at-b)(t_0 + l) + at|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - an - 1. \end{aligned} \tag{11}$$

Now, we discuss the following cases according to the value of $t_0 + l$.

Case 1. $t_0 + l \geq 1$. In this case, by $at \geq b^2 + an - \Delta - 1$ and $(a, b) \neq (1, 2)$, we have $(at - b)(t_0 + l) - an - 1 \geq b^2 - 2b + a - 2 \geq 0$. Thus (11) becomes

$$\begin{aligned} & \sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b - (i-1))|I^{(i)}| \\ \geq \ & \sum_{j=1}^{b-1}(at - aj - b + j)i_j + at|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|. \end{aligned} \tag{12}$$

And then, at least one of the following two subcases must hold.

Subcase 1.1. $\sum_{j=1}^{b-1}(b-2)(b-j)i_j \geq \sum_{j=1}^{b-1}(at - aj - b + j)i_j$.

There is at least one j such that

$$(b-2)(b-j) \geq at - aj - b + j,$$

which implies

$$at \leq (b-2)(b-j) + aj + b - j = b(b-2) + (a-b+1)j + b.$$

If $a=b$, then $at \le a(a-2)+j+a \le a^2-1$. By $t(G) \ge \frac{b^2-1}{a}+n$, we get $n=0$ and $\sum_{j=1}^{b-2} i_j = 0$, which contradicts the definition of H_2 and the choice of I_2 (see the proof of Lemma 2.3 in [37] such that $\sum_{j=1}^{b-2} i_j \neq 0$).

If $a<b$, then $at \le b(b-2)+(a-b+1)+b = (b^2-1)+(a-b)+(2-b) < b^2-\Delta-1$, which contradicts $t(G) \ge \frac{b^2-\Delta-1}{a}+n$.

Subcase 1.2. $\sum_{i=1}^{b}(ab-ai+b-(i-1))|I^{(i)}| \ge at|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|$.

If $t_0+l \ge 2$ or $b \ge 3$, then by $(bt-a)(t_0+l)-an-1 \ge 1$, we have

$$\begin{aligned}
&\sum_{i=1}^{b}(ab-ai+b-(i-1))|I^{(i)}| \\
\ge\ & at|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| + 1 \\
\ge\ & (b^2+an-\Delta-1)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| + 1 \\
\ge\ & (b^2-\Delta-1)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| + 1.
\end{aligned}$$

That is to say,

$$|I^{(1)}|(ab+2b-\frac{5}{2}a-b^2+\frac{1}{2}) + \sum_{i=2}^{b}(ab+2b-2a-i+2-b^2)|I^{(i)}| \ge 1.$$

Let

$$h_1(b) = -b^2+(a+2)b-\frac{5}{2}a+\frac{1}{2}.$$

In light of $b \ge a$ and $h_1(b)<0$ if $a=1$, we infer

$$\max\{h_1(b)\} = h_1(a) = -\frac{a}{2}+\frac{1}{2} < 0.$$

Furthermore, $ab+b-2a-i+2-b^2 \le -b^2+(a+2)b-2a$ due to $i \ge 2$. Let

$$h_2(b) = -b^2+(a+2)b-2a.$$

Using $b \ge a$, we deduce

$$\max h_2(b) = h_2(a) = 0,$$

which leads to a contradiction.

If $n \ge 1$, we get

$$\begin{aligned}
&\sum_{i=1}^{b}(ab-ai+b-(i-1))|I^{(i)}| \\
\ge\ & (b^2-\Delta-1+an)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| \\
\ge\ & (b^2-\Delta-1)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| + 2.
\end{aligned}$$

Hence,

$$|I^{(1)}|(ab+2b-\frac{5}{2}a-b^2+\frac{1}{2}) + \sum_{i=2}^{b}(ab+2b-2a-i+2-b^2)|I^{(i)}| \ge 2,$$

a contradiction.

In conclusion, we have $n = 0$, $t_0 + l = 1$ and $(a, b) = (2, 2)$. Then the result comes from the main result in Yu et al. [38] which determined that G is fractional 2-deleted graph if $t(G) \geq \frac{3}{2}$.

Case 2. $t_0 + l = 0$**.** In this case, by (11) we have,

$$\begin{aligned}
&\sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b - (i-1))|I^{(i)}| \\
\geq\ &\sum_{j=1}^{b-1}(at - aj - b + j)i_j + at|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - an - 1.
\end{aligned} \tag{13}$$

The following discussion is divided into three subcases relying on whether I_1 or I_2 is empty.

Subcase 2.1. $|I_1| = 0$.

In this case, (13) becomes

$$\sum_{j=1}^{b-1}((b-2)(b-j) - (at - aj - b + j))i_j + an + 1 \geq 0. \tag{14}$$

Let

$$\begin{aligned}
h_j &= (b-2)(b-j) - (at - aj - b + j) = b^2 + (a - b + 1)j - b - at \\
&\leq b^2 + (a - b + 1)j - b - a \cdot \frac{b^2 - \Delta - 1 + an}{a} \\
&= (a - b + 1)j - a + 1 - an.
\end{aligned}$$

Subcase 2.1.1. If $b \geq a + 1$, then by $b \geq 3$, we have

$$(a - b + 1)j - a + 1 - an \leq -b + 2 - an \leq -an - 1.$$

The equation holds if and only if $b = 3$ and $a = b - 1 = 2$. It implies $(a, b) = (2, 3)$, $\sum_{j=2}^{b-1} i_j = 0$, $|C_2| \leq |I_2|$ and $|T| \leq 2|I_2|$. Thus,

$$|S| \leq \frac{2|I_2|(b-1) + an + 1}{a}.$$

If $|I_2| = 1$, then $|S| \leq \frac{2(b-1)+an+1}{a} = \frac{5+2n}{2}$ and $\delta(G) \leq |S| + 1 \leq \frac{7+2n}{2}$. By $\delta(G) \geq 2t(G) \geq \frac{7+2n}{2}$, we get $|S| = \frac{5+2n}{2}$, $t(G) = \frac{7+2n}{4}$, $S \cup T = K_{\frac{9+2n}{2}}$. Since G is not a complete graph, $G - S - T \neq \emptyset$ and $\omega(G - S) \geq 2$. We obtain

$$\frac{7+2n}{4} \leq \frac{|S|}{\omega(G-S)} \leq \frac{5+2n}{4},$$

a contradiction. Hence, we deduce $|I_2| \geq 2$.

Let $Z = \{x | x \in C_2, d_{G-S}(x) = 1\}$ and $z = |Z|$. Thus, $0 \leq z \leq |I_2|$, and $N_{G-S}(v) \in I_2$ if $v \in Z$. We obtain

$$|S| \leq \frac{(|I_2| + z)(b-1) + (|C_2 - z|)(b-2) + an + 1}{a}.$$

Let $Z' = \{x | x \in N_G(I_2) \cap W, d_{G-S}(x) = 1\}$, we infer

$$\begin{aligned}
&\frac{b^2 - \Delta - 1 + an}{a} \leq t(G) \leq \frac{|U - Z \cup Z'|}{\omega(G - (U - Z \cup Z'))} \\
\leq\ &\frac{\frac{(|I_2|+z)(b-1)+(|C_2-z|)(b-2)+an+1}{a} + (|C_2| - z) + (|I_2| - |C_2|)}{|I_2|}.
\end{aligned}$$

By $(a, b) = (2, 3)$ and $|C_2| \leq |I_2|$, we get $3n(1 - \frac{1}{|I_2|}) \leq \frac{1}{|I_2|} - 2$. Hence, the contradiction is derived according to $|I_2| \geq 2$.

Subcase 2.1.2. If $a = b$, then $\max\{h_j\} = h_{b-1} = -an$ and the second largest value of h_j is $h_{b-2} = -an - 1$. Analyzing the proof of Lemma 2.3 in Liu and Zhang [37], we confirm that H_2 is connected, each vertex in I_2 has degree $b - 1$ in $G - S$ except one vertex has degree $b - 2$ in $G - S$. This fact implies

$$|C_2| \leq (b-2) + (|I_2| - 1)(b - 1 - 1) = |I_2|(b-2),$$

$$|T| \leq |I_2|(b-1),$$

and

$$|S| \leq \frac{|T| + 1 + an}{a} \leq |I_2| + \frac{1 - |I_2|}{b} + n.$$

If $|I_2| = 1$, then $|S| \leq 1 + n$, $\delta(G) \leq |S| + (b - 1) \leq b + n$, which contradicts $\delta(G) \geq 2t(G) > b + n$. Hence, $|I_2| \geq 2$ and

$$\begin{aligned} b - \frac{1}{b} + n \leq t(G) &\leq \frac{|U|}{\omega(G-U)} \leq \frac{\frac{1-|I_2|}{b} + |I_2| + |I_2|(b-2) + n}{|I_2|} \\ &= (b - 1 - \frac{1}{b}) + \frac{1}{b|I_2|} + \frac{n}{|I_2|}. \end{aligned}$$

This reveals $n(1 - \frac{1}{|I_2|}) \leq \frac{1}{b|I_2|} - 1$, which contradicts $b \geq 2$ and $|I_2| \geq 2$.

Subcase 2.2. $|I_2| = 0$.

In this case, (13) becomes

$$\sum_{i=1}^{b}(ab - ai + b - (i-1))|I^{(i)}| - at|I_1| - \frac{(a+1)|I^{(1)}|}{2} + a\sum_{i=1}^{b}(i-1)|I^{(i)}| + an + 1 \geq 0.$$

This implies

$$\sum_{i=2}^{b}(ab + 2b - 2a - i + 2 - b^2)|I^{(i)}| + (ab + 2b - \frac{5}{2}a - b^2 + \frac{1}{2})|I^{(1)}| + 1 \geq 0.$$

Then by $h_1 < 0$, we get $\sum_{i=4}^{b}|I^{(i)}| = 0$, $|I^{(3)}| \leq 1$ and $|I^{(1)}| \leq 2$. Now, we consider the following three subcases.

Subcase 2.2.1. $|I^{(1)}| = 1$. In this subcase, we have $\sum_{i=3}^{b}|I^{(i)}| = 0$. By analyzing the proof process of Lemma 2.2 in Liu and Zhang [37], we obtain $|I_1| \geq 2$,

$$|T| \leq (b-1) + (|I_1| - 1)(b-1) = |I_1|(b-1),$$

$$|S| \leq \frac{|T| + 1 + an}{a} \leq \frac{|I_1|(b-1) + 1 + an}{a},$$

and

$$\begin{aligned} |U| &\leq |S| + |C_1| + \sum_{i=1}^{b}(i-1)|I^{(i)}| \\ &\leq \frac{|I_1|(b-1) + 1 + an}{a} + |I_1|(b-1) - |I_1| + (|I_1| - 1) \\ &= \frac{|I_1|(b-1) + 1 + an}{a} + |I_1|(b-1) - 1. \end{aligned}$$

Thus,

$$\frac{b^2 - \Delta - 1 + an}{a} \leq t(G) \leq \frac{|U|}{\omega(G-U)} \leq \frac{\frac{|I_1|(b-1)+1+an}{a} + |I_1|(b-1) - 1}{|I_1|}.$$

This implies $an(1-\frac{1}{|I_1|}) \le (b-a)(2-b)+\frac{1-a}{|I_1|}$, a contradiction.

Subcase 2.2.2. $|I^{(1)}|=2$. In this subcase, we yield $\sum_{i=3}^{b}|I^{(i)}|=0$. We can get a contradiction via the discussion similar to Subcase 2.2.1.

Subcase 2.2.3. $|I^{(1)}|=0$. In this subcase, we have $\sum_{i=4}^{b}|I^{(i)}|=0$ and $|I^{(3)}|\le 1$. If $|I_1|=1$, then $|S|\le \frac{(b-1)+an+1}{a}$. Thus, we get

$$\frac{(b-1)+an+1}{a}+b-1\ge b-1+|S|\ge \delta(G)\ge 2t(G)\ge \frac{2(b^2-\Delta-1+an)}{a},$$

a contradiction. Hence, $|I_1|\ge 2$. Let $Y=N_G(I_1)\cap W$.

If there is a vertex $y\in Y$ such that y is only adjacent to one vertex in I_1. Reset

$$U=S\cup C_1\cup(N_G(I_1)\cap(W-\{y\})).$$

Then, we derive

$$|U|\le |S|+|I_1|(b-1)-1\le \frac{|I_1|(b-1)+an+1}{a}+|I_1|(b-1)-1.$$

In light of $|I_1|\ge 2$, we obtain

$$\frac{b^2-\Delta-1+an}{a}\le t(G)\le \frac{|U|}{\omega(G-U)}\le \frac{\frac{|I_1|(b-1)+an+1}{a}+|I_1|(b-1)-1}{|I_1|}.$$

This implies $an(1-\frac{1}{|I_1|}) \le (b-a)(2-b)+\frac{1-a}{|I_1|}$, a contradiction.

If each vertex in Y is adjacent to at least two vertices in I_1. We get

$$|U|\le |S|+|I_1|(b-2)+\frac{|I_1|}{2}\le \frac{|I_1|(b-1)+an+1}{a}+|I_1|(b-2)+\frac{|I_1|}{2},$$

where $U=S\cup C_1\cup(N_G(I_1)\cap W)$. Due to $|I_1|\ge 2$, we deduce

$$\frac{b^2-\Delta-1+an}{a}\le t(G)\le \frac{|U|}{\omega(G-U)}\le \frac{\frac{|I_1|(b-1)+an+1}{a}+|I_1|(b-2)+\frac{|I_1|}{2}}{|I_1|}.$$

That is to say, $an(1-\frac{1}{|I_1|}) \le (b-a)(2-b)+(\frac{1}{|I_1|}-\frac{a}{2})$, which contradicts $a\ge 1$, $(a,b)\ne(1,2)$ and $|I_1|\ge 2$.

Subcase 2.3. $|I_1|\ne 0$ and $|I_2|\ne 0$. From what we have discussed in Subcase 2.1, we get $\sum_{j=1}^{b-1}(b-2)(b-j)i_j \le \sum_{j=1}^{b-1}(at-aj-b+j)i_j+an+1$. Then, we yield

$$\sum_{i=1}^{b}(ab-ai+b-(i-1))|I^{(i)}|\ge at|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|.$$

This implies

$$\sum_{i=2}^{b}(ab+2b-2a-i+2-b^2)|I^{(i)}|+(ab+2b-\frac{5}{2}a-b^2+\frac{1}{2})|I^{(1)}|\ge 0.$$

Thus, we have $\sum_{i=4}^{b}|I^{(i)}|=0$, $|I^{(3)}|\le 1$, $|I^{(1)}|\le 2$ and $n=0$ by what we have discussed in Subsection 1.2. We only discuss the situation of $|I^{(1)}|=0$, and other two cases for $|I^{(1)}|=1$ and $|I^{(1)}|=2$ can be considered in a similar way.

Under the condition of $|I^{(1)}|=0$, we get $\sum_{i=4}^{b}|I^{(i)}|=0$, $|I^{(3)}|\le 1$,

$$|T|\le |I_1|(b-1)+|I_2|(b-1)=(b-1)(|I_1|+|I_2|),$$

and

$$|S| \leq \frac{|T|+1}{a} \leq \frac{(b-1)(|I_1|+|I_2|)+1}{a}.$$

Since $|I_1|+|I_2| \geq 2$, we get

$$\frac{b^2-\Delta-1}{a} \leq t(G) \leq \frac{|U|}{\omega(G-U)} \leq \frac{|S|+|I_2|(b-2)+|I_1|(b-1)}{|I_1|+|I_2|}.$$

Hence,

$$(b^2-\Delta-1)(|I_1|+|I_2|) \leq (|I_1|+|I_2|)(b-1)+1+(ab-2a)(|I_1|+|I_2|)+|I_1|a.$$

This implies $(b-a)(b-2)(|I_1|+|I_2|) \leq 1-a|I_2|$, a contradiction.

Therefore, we complete the proof of the desired result. □

4. Conclusion and discussion

In our article, we determine that for a non-completed graph G if $t(G) \geq \frac{b^2-\Delta-1}{a}+n$, then G is an all fractional (g,f,n)-critical deleted graph. We consider the extreme case of $a=b=k$ (i.e., $\Delta=0$) and $n=0$, then the all fractional (g,f,n)-critical graph becomes fractional k-deleted graph, and original toughness condition in Theorem 1.1 becomes $t(G) \geq k-\frac{1}{k}$. Comparing Liu's results on fractional k-factor and Gao's results on fractional k-deleted graph, we know that the result obtained in this paper is sharp in this extreme circumstance. However, the tightness conditions for the all fractional (g,f,n,m)-critical deleted graph are completely open, even when $m=1$ the problem is still open. Intuitively, the smaller the values of Δ,n,m are, the result yielded in our paper is closer to the tight one; the larger the values of Δ,n,m, the less the essence can be reflected. From this point of view, we believe that the tight lower bound of the toughness for the all fractional (g,f,n,m)-critical deleted graph can be expressed as a linear function of Δ,n, and m. In this way, the determination of the coefficients associated with Δ,n, and m in the linear function is the key to cracking the entire open problem.

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