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Embeddings in the Fell and Wijsman Topologies

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Abstract. It is shown that if a T_2 topological space X contains a closed uncountable discrete subspace, then the spaces $(\omega_1+1)^\omega$ and $(\omega_1+1)^{\omega_1}$ embed into $(CL(X), \tau_F)$, the hyperspace of nonempty closed subsets of X equipped with the Fell topology. If (X, d) is a non-separable perfect topological space, then $(\omega_1+1)^\omega$ and $(\omega_1+1)^{\omega_1}$ embed into $(CL(X), \tau_{w(d)})$, the hyperspace of nonempty closed subsets of X equipped with the Wijsman topology, giving a partial answer to the Question 3.4 in [2].

1. Introduction

Throughout this paper, let 2^X ($CL(X)$) denote the family of all (nonempty) closed subsets of a given T_2 topological space. For $M \in CL(X)$, put

$$M^- = \{A \in CL(X) : A \cap M \neq \emptyset\}, M^+ = \{A \in CL(X) : A \subseteq M\}$$

and denote $M^c = X \setminus M$. The Vietoris topology [12] τ_V on $CL(X)$ has as subbase elements of the form U^-, V^+, where U, V are open in X.

The Fell topology [5] τ_F on $CL(X)$ has as a subbase the collection

$$\{U^- : U \text{ open in } X\} \cup \{(K^c)^+ : K \text{ compact in } X\}.$$

It is known [1] that $(CL(X), \tau_F)$ is Hausdorff (regular, Tychonoff, respectively) if and only if X is locally compact. The normality problem of the Fell hyperspace was settled by Holá, Levi and Pelant in [7], where they showed that $(CL(X), \tau_F)$ is normal if and only if X is locally compact and Lindelöf.

For a metric space (X, d), let $d(x, A) = \inf\{d(x, a) : a \in A\}$ denote the distance between a point $x \in X$ and a nonempty subset A of (X, d).

A net $\{A_\alpha : \alpha \in \lambda\}$ in $CL(X)$ is said to be Wijsman convergent to some A in $CL(X)$ if $d(x, A_\alpha) \to d(x, A)$ for every $x \in X$. The Wijsman topology on $CL(X)$ induced by d, denoted by $\tau_{w(d)}$, is the weakest topology such that for every $x \in X$, the distance functional

$$d(x, .) : CL(X) \to R^+$$

is continuous. It can be seen easily that the Wijsman topology on $CL(X)$ induced by d has the family

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$$\{U^- : U \text{ open in } X\} \cup \{\{A \in CL(X) : d(x, A) > \epsilon\} : x \in X, \epsilon > 0\}$$

as a subbase [1].

The above type of convergence was introduced by Wijsman in [13] for sequences of closed convex sets in Euclidean space R^n, when he considered optimum properties of the sequential probability ratio test.

It is known [11] that if (X, d) is a metric space, then $(CL(X), \tau_{w(d)})$ is metrizable if and only if (X, d) is separable.

One of the fundamental problems in the study of the Wijsman topology is to determine when it is normal. The problem was first mentioned by Di Maio and Meccariello in [3], where it was asked whether the normality of the Wijsman topology is equivalent to its metrizability. Partial solutions to this problem were found by Holá and Novotný in [8] and by Cao and Junnila in [2]. Holá and Novotný in [8] showed that for a normed linear space (X, d), $(CL(X), \tau_{w(d)})$ is normal iff it is metrizable. Cao and Junnila in [2] proved that a Wijsman hyperspace is hereditarily normal if and only if it is metrizable.

2. Main Result

In [2] the following Question 3.4 is posed.

Question 3.4. ([2]) Let (X, d) be a non-separable metric space. Does $(CL(X), \tau_{w(d)})$ contain a copy of $(\omega_1 + 1)^{\omega}$ or $(\omega_1 + 1)^{\omega_1}$?

Theorem 2.1. *Let X be a T_2 topological space which contains an uncountable closed discrete set. Then $(CL(X), \tau_F)$ contains a copy of $(\omega_1 + 1)^{\omega}$ and $(\omega_1 + 1)^{\omega_1}$.*

Proof. We will prove the theorem for the case of $(\omega_1 + 1)^{\omega_1}$, the other case is similar. Let D be a closed uncountable discrete set in X such that $| D |= \aleph_1$. We express D as a pairwise disjoint union of closed discrete sets $\{D_i : i \in \omega_1\}$,

$$D = \bigcup_{i\in\omega_1} D_i,$$

such that $| D_i |= \aleph_1$ for every $i \in \omega_1$. Without loss of generality we can suppose that $X \neq D$ and let $l \in X \setminus D$. For each $x \in D$ fix an open neighbourhood $U(x)$ of x such that $U(x) \cap (D \cup \{l\}) = \{x\}$.

For every $i \in \omega_1$ enumerate $D_i = \{x^i_\alpha : 0 < \alpha \leq \omega_1\}$ and define a map

$$\varphi_i : \omega_1 + 1 \to 2^X$$

as follows: $\varphi_i(0) = \emptyset$, $\varphi_i(\alpha) = \{x^i_\eta : \alpha \leq \eta\}$ if $0 < \alpha \leq \omega_1$. Let $(\alpha_i)_{i\in\omega_1}$ be a point from $(\omega_1 + 1)^{\omega_1}$ its ith coordinate is α_i. Now we will define a map

$$\phi : (\omega_1 + 1)^{\omega_1} \to CL(X)$$

as follows: $\phi((\alpha_i)_{i\in\omega_1}) = \{l\}$ if $\alpha_i = 0$ for every $i \in \omega_1$ and

$$\phi((\alpha_i)_{i\in\omega_1}) = \{l\} \cup \bigcup_{i\in\omega_1} \varphi_i(\alpha_i)$$

if there is $i \in \omega_1$ such that $\alpha_i \neq 0$.

If we take distinct $A, B \in \phi((\omega_1 + 1)^{\omega_1})$, then there is $x \in D$ with $U(x)$ missing one of A, B and hitting the other. So $U(x)^- \cap \phi((\omega_1 + 1)^{\omega_1})$ and $(X \setminus \{x\})^+ \cap \phi((\omega_1 + 1)^{\omega_1})$ are disjoint $\phi((\omega_1 + 1)^{\omega_1})$ neighbourhoods of A, B.

Consequently, $(\phi((\omega_1 + 1)^{\omega_1}), \tau_F)$ is a Hausdorff space, so to show that $\phi : (\omega_1 + 1)^{\omega_1} \to (\phi((\omega_1 + 1)^{\omega_1}, \tau_F)$ is a homeomorphism it suffices to show that ϕ is continuous, since $(\omega_1 + 1)^{\omega_1}$ is compact and ϕ is one-to-one.

To prove the continuity of ϕ we first show that $\phi^{-1}(V^-)$ is open in $(\omega_1 + 1)^{\omega_1}$ for each open subset V of X. So let $V \subseteq X$ be open. If $l \in V$ then $\phi^{-1}(V^-) = (\omega_1 + 1)^{\omega_1}$. Suppose $l \notin V$. Let $(\alpha_i)_{i\in\omega_1} \in \phi^{-1}(V^-)$. Thus $\phi((\alpha_i)_{i\in\omega_1}) \cap V \neq \emptyset$. There is $i \in \omega_1$ such that $\alpha_i \neq 0$ and $\varphi_i(\alpha_i) \cap V \neq \emptyset$. It is easy to verify that $(0, \alpha_i]$ is an open neighbourhood of α_i such that $\varphi_i(\eta) \cap V \neq \emptyset$ for every $\eta \in (0, \alpha_i]$. Thus $\Pi_{j\in\omega_1} X_j$, where $X_i = (0, \alpha_i]$ and $X_j = \omega_1 + 1$ for $j \neq i$ is a neighbourhood of $(\alpha_i)_{i\in\omega_1}$ contained in $\phi^{-1}(V^-)$.

Now let K be a compact set in X. We show that $\phi^{-1}((K^c)^+)$ is open in $(\omega_1 + 1)^{\omega_1}$. Let $(\alpha_i)_{i\in\omega_1} \in \phi^{-1}((K^c)^+)$. There is a finite set $J \subset \omega_1$ such that $K \cap D_j \neq \emptyset$ for every $j \in J$. For every $j \in J$, $\varphi_j(\alpha_j) \cap K = \emptyset$, thus there is an open neighbourhood $O(\alpha_j)$ of α_j such that $\varphi_j(\eta) \cap K = \emptyset$ for every $\eta \in O(\alpha_j)$. For every $i \in \omega_1$ put $X_i = O(\alpha_i)$ if $i \in J$ and $X_i = \omega_1 + 1$ otherwise. Then $\Pi_{i\in\omega_1} X_i$ is a neighbourhood of $(\alpha_i)_{i\in\omega_1}$ contained in $\phi^{-1}((K^c)^+)$. □

Notice that in the paper [6] we proved that if X is a T_2 topological space which contains an uncountable closed discrete set, then $\omega_1 \times (\omega_1 + 1)$ embeds into $(CL(X), \tau_F)$ as a closed set. It was shown in [9] that if X is a T_2 topological space which contains an uncountable closed discrete set, then also the Tychonoff plank embeds into $(CL(X), \tau_F)$ as a closed subspace.

3. Concerning Question 3.4 in [2]

In this part we will give a partial answer to Question 3.4 in [2].

It is known [1] that if a metric space (X, d) has nice closed balls, then the Fell topology τ_F and the Wijsman topology $\tau_{w(d)}$ on $CL(X)$ coincide.

A metric space (X, d) is said to have nice closed balls [1] provided whenever B is a closed ball in X that is a proper subset of X, then B is compact. The class of metric spaces that have nice closed balls includes those metric spaces in which closed and bounded sets are compact, as well as all $0-1$ metric spaces [1].

Theorem 3.1. *Let (X, d) be a non-separable metric space with nice closed balls. Then $(CL(X), \tau_{w(d)})$ contains a copy of $(\omega_1 + 1)^\omega$ and $(\omega_1 + 1)^{\omega_1}$.*

The following theorem gives a better partial answer to the Question 3.4 in [2] than Theorem 3.1.

Theorem 3.2. *Let (X, d) be a metric space such that every closed proper ball in X is totally bounded. If X is non-separable, then $(CL(X), \tau_{w(d)})$ contains a copy of $(\omega_1 + 1)^\omega$ and $(\omega_1 + 1)^{\omega_1}$.*

Proof. We will prove the theorem for the case of $(\omega_1 + 1)^{\omega_1}$, the other case is similar. Since (X, d) is non-separable there exist $\epsilon > 0$ and a set $D \subset X$ with $|D| = \aleph_1$ which is ϵ-discrete, that is, $d(x, y) \geq \epsilon$ for all distinct $x, y \in D$. We express D as a pairwise disjoint union of closed discrete sets $\{D_i : i \in \omega_1\}$,

$$D = \bigcup_{i\in\omega_1} D_i,$$

such that $|D_i| = \aleph_1$ for every $i \in \omega_1$. Without loss of generality we can suppose that $X \neq D$ and let $l \in X \setminus D$.

For every $i \in \omega_1$ enumerate $D_i = \{x^i_\alpha : 0 < \alpha \leq \omega_1\}$ as in the proof of Theorem 2.1 and we will proceed as in the proof of Theorem 2.1. We claim that $\phi : (\omega_1 + 1)^{\omega_1} \to (CL(X), \tau_{w(d)})$ is an embedding. Of course, it is sufficient to prove that ϕ is continuous.

It is sufficient to verify that if $d(x, \phi((\alpha^*_i)_{i\in\omega_1})) > r$ for some $x \in X$ and $r > 0$, then $d(x, \phi((\alpha_i)_{i\in\omega_1})) > r$ for all $(\alpha_i)_{i\in\omega_1}$ from a neighbourhood of $(\alpha^*_i)_{i\in\omega_1}$. However, it is clear, since the closed proper ball with center x and the radius r can contain only finitely many points of the set D. □

Theorem 3.3. *Let (X, d) be a metric space. If the set X', the derived set of X, is non-separable, then $(\omega_1 + 1)^\omega$ and $(\omega_1 + 1)^{\omega_1}$ embed into $(CL(X), \tau_{w(d)})$.*

Proof. We will prove the theorem for the case of $(\omega_1 + 1)^{\omega_1}$, the other case is similar. Since (X', d) is non-separable, there exist $\epsilon > 0$ and a set $D \subset X'$ with $|D| = \aleph_1$ which is ϵ-discrete, that is, $d(x, y) \geq \epsilon$ for all distinct $x, y \in D$. For every $x \in D$ there is a point $t_x \in X$ such that $d(x, t_x) < \epsilon/10$. For every $x \in D$ put $\eta(x) = d(x, t_x)$. Set

$$H = X \setminus \bigcup_{x\in D} S(x, \eta(x)),$$

where $S(x, \eta(x)) = \{s \in X : d(x, s) < \eta(x)\}$. We express D as a pairwise disjoint union of closed discrete sets $\{D_i : i \in \omega_1\}$,

$$D = \bigcup_{i\in\omega_1} D_i,$$

such that $|D_i| = \aleph_1$ for every $i \in \omega_1$. For every $i \in \omega_1$ enumerate $D_i = \{x^i_\alpha : 0 < \alpha \leq \omega_1\}$ and define a map

$$\varphi_i : \omega_1 + 1 \to 2^X$$

as follows: $\varphi_i(0) = \emptyset$, $\varphi_i(\alpha) = \{x^i_\eta : \alpha \leq \eta\}$ if $0 < \alpha \leq \omega_1$. Let $(\alpha_i)_{i\in\omega_1}$ be a point from $(\omega_1+1)^{\omega_1}$ its ith coordinate is α_i. Now we will define a map

$$\phi : (\omega_1 + 1)^{\omega_1} \to CL(X)$$

as follows: $\phi((\alpha_i)_{i\in\omega_1}) = H$ if $\alpha_i = 0$ for every $i \in \omega_1$ and

$$\phi((\alpha_i)_{i\in\omega_1}) = H \cup \bigcup_{i\in\omega_1} \varphi_i(\alpha_i)$$

if there is $i \in \omega_1$ such that $\alpha_i \neq 0$.

To show that $\phi : (\omega_1 + 1)^{\omega_1} \to (\phi((\omega_1 + 1)^{\omega_1}, \tau_{w(d)})$ is a homeomorphism it suffices to show that ϕ is continuous, since $(\omega_1 + 1)^{\omega_1}$ is compact, $(\phi((\omega_1 + 1)^{\omega_1}, \tau_{w(d)})$ is a Hausdorff space and ϕ is one-to-one.

To prove the continuity of ϕ we first show that $\phi^{-1}(V^-)$ is open in $(\omega_1 + 1)^{\omega_1}$ for each open subset V of X. So let $V \subseteq X$ be open. If $H \cap V \neq \emptyset$ then $\phi^{-1}(V^-) = (\omega_1 + 1)^{\omega_1}$. Suppose $H \cap V = \emptyset$.

Let $(\alpha_i)_{i\in\omega_1} \in \phi^{-1}(V^-)$. Thus $\phi((\alpha_i)_{i\in\omega_1}) \cap V \neq \emptyset$. There is $i \in \omega_1$ such that $\alpha_i \neq 0$ and $\varphi_i(\alpha_i) \cap V \neq \emptyset$. It is easy to verify that $(0, \alpha_i]$ is an open neighbourhood of α_i such that $\varphi_i(\eta) \cap V \neq \emptyset$ for every $\eta \in (0, \alpha_i]$. Thus $\Pi_{j\in\omega_1} X_j$, where $X_i = (0, \alpha_i]$ and $X_j = \omega_1 + 1$ for $j \neq i$ is a neighbourhood of $(\alpha_i)_{i\in\omega_1}$ contained in $\phi^{-1}(V^-)$.

It is sufficient to verify that if $d(x, \phi((\alpha^*_i)_{i\in\omega_1}))) > r > 0$ for some $x \in X$, then $d(x, \phi((\alpha_i)_{i\in\omega_1}))) > r$ for all $(\alpha_i)_{i\in\omega_1}$ from a neighbourhood of $(\alpha^*_i)_{i\in\omega_1}$. Suppose first that $(\alpha^*_i)_{i\in\omega_1}$ is such that $\alpha^*_i = 0$ for all $i \in \omega_1$. Thus $\phi((\alpha^*_i)_{i\in\omega_1}) = H$.

If $d(x, H) > r > 0$, then there is $z \in D$ such that $x \in S(z, \eta(z))$. There is $i \in \omega_1$ such that $z \in D_i$. Thus $\Pi_{j\in\omega_1} X_j$, where $X_i = \{0\}$ and $X_j = \omega_1 + 1$ for $j \neq i$ is a neighbourhood of $(\alpha^*_i)_{i\in\omega_1}$ such that $d(x, \phi((\alpha_i)_{i\in\omega_1}))) > r$ for all $(\alpha_i)_{i\in\omega_1}$ from the neighbourhood.

Suppose now that $d(x, \phi((\alpha^*_i)_{i\in\omega_1}))) > r > 0$ for some $x \in X$ and $(\alpha^*_i)_{i\in\omega_1}$ such that there is $j \in \omega_1$ with $\alpha^*_j \neq 0$. There is $z \in D$ with $x \in S(z, \eta(z))$. If $z \in \phi((\alpha^*_i)_{i\in\omega_1})$, then $(\omega_1 + 1)^{\omega_1}$ is a neighbourhood of $(\alpha^*_i)_{i\in\omega_1}$ such that $d(x, \phi((\alpha_i)_{i\in\omega_1}))) > r$ for every $(\alpha_i)_{i\in\omega_1}$ from the neighbourhood $(\omega_1 + 1)^{\omega_1}$.

Suppose that $z \notin \phi((\alpha^*_i)_{i\in\omega_1})$. There are $i \in \omega_1, \beta \in \omega_1$ such that $z = x^i_\beta$. If $\alpha^*_i = 0$, then $O = \Pi_{j\in\omega_1} X_j$, where $X_i = \{0\}$ and $X_j = \omega_1 + 1$ for $j \neq i$ is a neighbourhood of $(\alpha^*_i)_{i\in\omega_1}$ such that $d(x, \phi((\alpha_i)_{i\in\omega_1}))) > r$ for every $(\alpha_i)_{i\in\omega_1}$ from the neighbourhood O. If $\alpha^*_i \neq 0$, then $O = \Pi_{j\in\omega_1} X_j$, where $X_i = (\beta, \alpha^*_i]$ and $X_j = \omega_1 + 1$ for $j \neq i$ is a neighbourhood of $(\alpha^*_i)_{i\in\omega_1}$ such that $d(x, \phi((\alpha_i)_{i\in\omega_1}))) > r$ for every $(\alpha_i)_{i\in\omega_1}$ from the neighbourhood O. □

A topological space X is perfect if there are no isolated points in X. The following theorem is a corollary of Theorem 3.3.

Theorem 3.4. *Let (X, d) be a non-separable perfect metric space. Then $(\omega_1 + 1)^\omega$ and $(\omega_1 + 1)^{\omega_1}$ embed into $(CL(X), \tau_{w(d)})$.*

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