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On $I-$ Deferred Statistical Convergence of Order α

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Abstract. The idea of $I-$convergence of real sequences was introduced by Kostyrko et al. [*Kostyrko, P., Šalát, T. and Wilczyński, W. I−convergence, Real Anal. Exchange 26(2) (2000/2001), 669-686*] and also independently by Nuray and Ruckle [*Nuray, F. and Ruckle, W. H. Generalized statistical convergence and convergence free spaces. J. Math. Anal. Appl. 245(2) (2000), 513–527*]. In this paper we introduce $I-$deferred statistical convergence of order α and strong $I-$deferred Cesàro convergence of order α and investigated between their relationship.

1. Introduction, Definitions and Preliminaries

The concept of statistical convergence was introduced by Steinhaus [29] and Fast [12] and later reintroduced by Schoenberg [24] independently. Later on it was further investigated from the sequence spaces point of view and linked with summability theory by Çınar et al. ([6],[8]), Connor [5], Çolak [17], Çakallı et al. ([2],[3],[4]), Et et al. ([9],[10],[11],[28]), Fridy [13], Işık et al. ([14],[15],[16]), Küçükaslan and Yılmaztürk ([19],[30]), Savaş and Et [23], Şengül et al. ([25],[26],[27]) and many others.

The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset E of $\mathbb{N}$ is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ provided the limit exists,}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

The idea of $I-$convergence of real sequences was introduced by Kostyrko et al. [18] and also independently by Nuray and Ruckle [20] (who called it generalized statistical convergence) as a generalization of statistical convergence. Later $I-$convergence was studied by Das et al. [7], Salat et al. ([21], [22]), Şengül and Et ([25],[26]) and many others.

Let X be non-empty set. Then a family sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I additive *i.e.* $A, B \in I$ implies $A \cup B \in I$ and hereditary, *i.e.* $A \in I$, $B \subset A$ implies $B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a *filter* of X if and only if (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$ and (iii) $A \in F$, $A \subset B$ implies $B \in F$.

An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$.

A non-trivial ideal I is said to be *admissible* if $I \supset \{\{x\} : x \in X\}$.

If I is a non-trivial ideal in X, $X \neq \phi$, then the family of sets

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$F(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter of X, called the *filter associated with I*.

Throughout the paper I will stand for a non-trivial admissible ideal of $\mathbb{N}$.

The deferred Cesàro mean of sequences was introduced by Agnew [1] such as:

$$\left(D_{p,q}x\right)_n = \frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = +\infty.$$

Let K be a subset of $\mathbb{N}$ and denote the set $\{k : p(n) < k \leq q(n), k \in K\}$ by $K_{p,q}(n)$. Deferred density of K is defined by

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{(q(n) - p(n))} \left|K_{p,q}(n)\right|, \text{ provided the limit exists.} \tag{1}$$

The vertical bars in (1) indicate the cardinality of the set $K_{p,q}(n)$.

It is clear that, if $K \subseteq M$, then $\delta_{p,q}(K) \leq \delta_{p,q}(M)$ and if $q(n) = n$, $p(n) = 0$, then deferred density coincides natural density of K.

2. Main Results

In this section, we introduce the concepts of $I-$deferred statistical convergence of order α and strong $I-$deferred Cesàro convergence of order α and investigated between their relationship.

Definition 2.1. *Let $\{p(n)\}$ and $\{q(n)\}$ be two sequences as above and $\alpha \in (0,1]$ be given. The sequence $x = (x_k)$ is said to be I−deferred statistically convergent of order α (or $DS^{\alpha}_{p,q}(I)-$convergent) to L if, for every $\varepsilon, \delta > 0$*

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in I.$$

In this case we write $DS^{\alpha}_{p,q}(I) - \lim x_k = L$ or $x_k \to L\left(DS^{\alpha}_{p,q}(I)\right)$. The set of all I−deferred statistically convergent sequences of oder α will be denoted by $DS^{\alpha}_{p,q}(I)$. If $q(n) = n$, $p(n) = 0$, then I−deferred statistical convergence of order α coincides I−statistical convergence of order α and also if $q(n) = n$, $p(n) = 0$ and $\alpha = 1$, then I−deferred statistical convergence of order α coincides I−statistical convergence.

Definition 2.2. *Let $\{p(n)\}$, $\{q(n)\}$ be given and $\alpha \in (0,1]$ and r be a positive real number. A sequence $x = (x_k)$ is said to be strongly I−deferred Cesàro convergent of order α (or strongly $Dw^{\alpha}_{r}[p,q](I)-$convergent) to L if*

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}} \sum_{p(n)+1}^{q(n)} |x_k - L|^r \geq \varepsilon\right\} \in I$$

and this is denoted by $Dw^{\alpha}_{r}[p,q](I) - \lim x_k = L$ or $x_k \to L(Dw^{\alpha}_{r}[p,q](I))$. The set of all strongly I−deferred Cesàro convergent sequences of oder α will be denoted by $Dw^{\alpha}_{r}[p,q](I)$.

The proof of each of the following results is straightforward, so we choose to state these results without proof.

Theorem 2.3. *Let $0 < \alpha \leq 1$ and $x = (x_k)$, $y = (y_k)$ be sequences of real numbers, then*
(i) If $DS^{\alpha}_{p,q}(I) - \lim x_k = x_0$ and $DS^{\alpha}_{p,q}(I) - \lim y_k = y_0$, then $DS^{\alpha}_{p,q}(I) - \lim(x_k + y_k) = x_0 + y_0$,
(ii)If $DS^{\alpha}_{p,q}(I) - \lim x_k = x_0$ and $c \in \mathbb{C}$, then $DS^{\alpha}_{p,q}(I) - \lim(cx_k) = cx_0$,
(iii) If $DS^{\alpha}_{p,q}(I) - \lim x_k = x_0$, $DS^{\alpha}_{p,q}(I) - \lim y_k = y_0$ and $x, y \in \ell_{\infty}$, then $DS^{\alpha}_{p,q}(I) - \lim(x_k y_k) = x_0 y_0$.

Theorem 2.4. *Let* $0 < \alpha \leq 1$, *then* $DS^{\alpha}_{p,q}(I) \cap \ell_{\infty}$ *is a closed subset of* ℓ_{∞}.

Proof. Suppose that $\left\{x^i\right\}_{i\in\mathbb{N}} \subseteq DS^{\alpha}_{p,q}(I) \cap \ell_{\infty}$ is convergent sequence and that it converges to $x \in \ell_{\infty}$. We need to prove that $x \in DS^{\alpha}_{p,q}(I) \cap \ell_{\infty}$. Assume that $x^i \to L_i\left(DS^{\alpha}_{p,q}(I)\right)$ for $\forall i \in \mathbb{N}$. Take a positive strictly decreasing sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ where $\varepsilon_i = \frac{\varepsilon}{2^i}$ for a given $\varepsilon > 0$. Clearly $\{\varepsilon_i\}_{i\in\mathbb{N}}$ converges to 0. Choose a positive integer i such that $\left\|x - x^i\right\|_{\infty} < \frac{\varepsilon}{4}$. Let $0 < \delta < 1$. Then

$$A = \left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}}\left|\left\{p(n) < k \leq q(n) : \left|x^i_k - L_i\right| \geq \frac{\varepsilon_i}{4}\right\}\right| < \frac{\delta}{3}\right\} \in F(I)$$

and

$$B = \left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}}\left|\left\{p(n) < k \leq q(n) : \left|x^{i+1}_k - L_{i+1}\right| \geq \frac{\varepsilon_{i+1}}{4}\right\}\right| < \frac{\delta}{3}\right\} \in F(I).$$

Since $A \cap B \in F(I)$ and $\phi \notin F(I)$, we can choose $n \in A \cap B$. Then

$$\frac{1}{(q(n) - p(n))^{\alpha}}\left|\left\{p(n) < k \leq q(n) : \left|x^i_k - L_i\right| \geq \frac{\varepsilon_i}{4}\right\}\right| < \frac{\delta}{3}$$

and

$$\frac{1}{(q(n) - p(n))^{\alpha}}\left|\left\{p(n) < k \leq q(n) : \left|x^{i+1}_k - L_{i+1}\right| \geq \frac{\varepsilon_{i+1}}{4}\right\}\right| < \frac{\delta}{3}$$

and so

$$\frac{1}{(q(n) - p(n))^{\alpha}}\left|\left\{p(n) < k \leq q(n) : \left|x^i_k - L_i\right| \geq \frac{\varepsilon_i}{4} \vee \left|x^{i+1}_k - L_{i+1}\right| \geq \frac{\varepsilon_{i+1}}{4}\right\}\right| < \delta < 1.$$

Hence, there exists a $k \in (p(n), q(n)]$ for which $\left|x^i_k - L_i\right| \geq \frac{\varepsilon_i}{4}$ and $\left|x^{i+1}_k - L_{i+1}\right| \geq \frac{\varepsilon_{i+1}}{4}$. Then, we can write

$$\begin{aligned} |L_i - L_{i+1}| &\leq \left|L_i - x^i_k\right| + \left|x^i_k - x^{i+1}_k\right| + \left|x^{i+1}_k - L_{i+1}\right| \\ &\leq \left|x^i_k - L_i\right| + \left|x^{i+1}_k - L_{i+1}\right| + \left\|x - x^i\right\|_{\infty} + \left\|x - x^{i+1}\right\|_{\infty} \\ &\leq \frac{\varepsilon_i}{4} + \frac{\varepsilon_{i+1}}{4} + \frac{\varepsilon_i}{4} + \frac{\varepsilon_{i+1}}{4} \leq \varepsilon_i. \end{aligned}$$

This implies that $\{L_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and so there is a real number L such that $L_i \to L$, as $i \to \infty$. We need to prove that $x \to L\left(DS^{\alpha}_{p,q}(I)\right)$. For any $\varepsilon > 0$, choose $i \in \mathbb{N}$ such that $\varepsilon_i < \frac{\varepsilon}{4}$, $\left\|x - x^i\right\|_{\infty} < \frac{\varepsilon}{4}$, $|L_i - L| < \frac{\varepsilon}{4}$. Then

$$\begin{aligned} &\frac{1}{(q(n) - p(n))^{\alpha}}\left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right| \\ \leq\ &\frac{1}{(q(n) - p(n))^{\alpha}}\left|\left\{p(n) < k \leq q(n) : \left|x^i_k - L_i\right| + \left\|x_k - x^i_k\right\|_{\infty} + |L_i - L| \geq \varepsilon\right\}\right| \\ \leq\ &\frac{1}{(q(n) - p(n))^{\alpha}}\left|\left\{p(n) < k \leq q(n) : \left|x^i_k - L_i\right| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq \varepsilon\right\}\right| \\ \leq\ &\frac{1}{(q(n) - p(n))^{\alpha}}\left|\left\{p(n) < k \leq q(n) : \left|x^i_k - L_i\right| \geq \frac{\varepsilon}{2}\right\}\right|. \end{aligned}$$

This implies that

$$\begin{aligned} &\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}}\left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right| < \delta\right\} \\ \supseteq\ &\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}}\left|\left\{p(n) < k \leq q(n) : \left|x^i_k - L_i\right| \geq \frac{\varepsilon}{2}\right\}\right| < \delta\right\} \in F(I). \end{aligned}$$

So

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right| < \delta\right\} \in F(I),$$

and so

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in I.$$

This gives that $x \to L\left(DS^{\alpha}_{p,q}(I)\right)$, and this completes the proof of the theorem. □

Theorem 2.5. *Let $\alpha \in (0,1]$. Then $Dw^{\alpha}_{r}[p,q](I) \subseteq DS^{\alpha}_{p,q}(I)$ and the inclusion is strict.*

Proof. First part of proof is easy, so omitted. To show the strictness of the inclusion, choose $q(n) = n$ and $p(n) = 0$ and define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} \sqrt{n}, & k = n^2 \\ 0, & k \neq n^2 \end{cases}.$$

Then for every $\varepsilon > 0$ and $\frac{1}{2} < \alpha \leq 1$, we have

$$\frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - 0| \geq \varepsilon\}\right| \leq \frac{\left[\sqrt{n}\right]}{n^{\alpha}},$$

and for any $\delta > 0$ we get

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - 0| \geq \varepsilon\}\right| \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \frac{\left[\sqrt{n}\right]}{n^{\alpha}} \geq \delta\right\}.$$

Since the set on the right-hand side is a finite set and so belongs to I, it follows that for $\frac{1}{2} < \alpha \leq 1$, $x_k \to 0\left(DS^{\alpha}_{p,q}(I)\right)$.

On the other hand, for $0 < \alpha < 1$ and $r = 1$,

$$\frac{1}{(q(n)-p(n))^{\alpha}} \sum_{p(n)+1}^{q(n)} |x_k - 0|^r = \frac{\left[\sqrt{n}\right]\left[\sqrt{n}\right]}{n^{\alpha}} \to \infty$$

and for $\alpha = 1$,

$$\frac{\left[\sqrt{n}\right]\left[\sqrt{n}\right]}{n^{\alpha}} \to 1.$$

Then

$$\begin{aligned} \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))^{\alpha}} \sum_{p(n)+1}^{q(n)} |x_k - 0|^r \geq 1\right\} &= \left\{n \in \mathbb{N} : \frac{\left[\sqrt{n}\right]\left[\sqrt{n}\right]}{n^{\alpha}} \geq 1\right\} \\ &= \{m, m+1, m+2, ...\} \end{aligned}$$

for some $m \in \mathbb{N}$ which belongs to $F(I)$, since I is admissible. So $x_k \nrightarrow 0\left(Dw^{\alpha}_{r}[p,q](I)\right)$. □

Theorem 2.6. *Let $0 < \alpha \leq 1$, $\liminf_n \frac{q(n)}{p(n)} > 1$ and $q(n) - p(n) < p(n)$, then $S^{\alpha}(I) \subset DS^{\alpha}_{p,q}(I)$.*

Proof. Suppose that $\liminf_n \frac{q(n)}{p(n)} > 1$; then there exists a $a > 0$ such that $\frac{q(n)}{p(n)} \geq 1 + a$ for sufficiently large n, which implies that

$$\frac{q(n) - p(n)}{p(n)} \geq \frac{a}{1+a} \Longrightarrow \left(\frac{q(n) - p(n)}{p(n)}\right)^{\alpha} \geq \left(\frac{a}{1+a}\right)^{\alpha} \Longrightarrow \frac{1}{p(n)^{\alpha}} \geq \frac{a^{\alpha}}{(1+a)^{\alpha}} \frac{1}{(q(n) - p(n))^{\alpha}}.$$

If $x_k \to L(S^{\alpha}(I))$, then for every $\varepsilon > 0$ and for sufficiently large n, we have

$$\begin{aligned} \frac{1}{p(n)^{\alpha}} \left|\{k \leq p(n) : |x_k - L| \geq \varepsilon\}\right| &\geq \frac{1}{p(n)^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right| \\ &\geq \frac{a^{\alpha}}{(1+a)^{\alpha}} \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right|. \end{aligned}$$

For $\delta > 0$, we have

$$\begin{aligned} &\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \\ \subseteq &\left\{n \in \mathbb{N} : \frac{1}{p(n)^{\alpha}} \left|\{k \leq p(n) : |x_k - L| \geq \varepsilon\}\right| \geq \frac{\delta a^{\alpha}}{(1+a)^{\alpha}}\right\} \in I \end{aligned}$$

this proves the proof. □

Theorem 2.7. *If* $\lim_{n\to\infty} \inf \frac{(q(n)-p(n))^{\alpha}}{n} > 0$ *and* $q(n) < n$, *then* $S(I) \subseteq DS^{\alpha}_{p,q}(I)$.

Proof. Let $\lim_{n\to\infty} \inf \frac{(q(n)-p(n))^{\alpha}}{n} > 0$, then for each $\varepsilon > 0$ the inclusion

$$\{k \leq n : |x_k - L| \geq \varepsilon\} \supset \{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}$$

is satisfied and so we have the following inequality

$$\begin{aligned} \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| &\geq \frac{1}{n} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right| \\ &= \frac{(q(n) - p(n))^{\alpha}}{n} \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right|. \end{aligned}$$

Hence we can write

$$\begin{aligned} &\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \\ \subseteq &\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| \geq \delta \frac{(q(n) - p(n))^{\alpha}}{n}\right\} \in I. \end{aligned}$$

Therefore $S(I) \subseteq DS^{\alpha}_{p,q}(I)$. □

Theorem 2.8. *Let* α *and* β *be two real numbers such that* $0 < \alpha \leq \beta \leq 1$, *then* $Dw^{\alpha}_r[p,q](I) \subseteq Dw^{\beta}_r[p,q](I)$ *and the inclusion is strict.*

Proof. The inclusion part of the proof follows from the following inequality:

$$\frac{1}{(q(n) - p(n))^{\beta}} \sum_{p(n)+1}^{q(n)} |x_k - L|^r \leq \frac{1}{(q(n) - p(n))^{\alpha}} \sum_{p(n)+1}^{q(n)} |x_k - L|^r.$$

To show that the inclusion is strict define $x = (x_k)$ such that

$$x_k = \begin{cases} 1, & \text{if } k \text{ is square} \\ 0, & \text{otherwise} \end{cases}.$$

Then $x \in Dw_r^{\beta}[p,q](I)$ for $\frac{1}{2} < \beta \leq 1$ but $x \notin Dw_r^{\alpha}[p,q](I)$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 2.9. *Let $\alpha, \beta \in (0,1]$ $(0 < \alpha \leq \beta \leq 1)$, then $DS_{p,q}^{\alpha}(I) \subseteq DS_{p,q}^{\beta}(I)$ and the inclusion is strict.*

Proof. First part of proof is easy, so omitted. To show the inclusion is strict, let us define a sequence by

$$x_k = \begin{cases} 1, & k = n^2 \\ 0, & k \neq n^2 \end{cases}$$

then $x \in DS_{p,q}^{\beta}(I)$ for $\frac{1}{2} < \beta \leq 1$, but $x \notin DS_{p,q}^{\alpha}(I)$ for $0 < \alpha \leq \frac{1}{2}$, where $q(n) = 3n - 1$ and $p(n) = 2n - 1$. □

Corollary 2.10. *If a sequence is $DS_{p,q}^{\alpha}(I)$ −convergent to L, then it is $DS_{p,q}(I)$ −convergent to L.*

Theorem 2.11. *Let $0 < \alpha \leq 1$ and $0 < r < s < \infty$, then $Dw_s^{\alpha}[p,q](I) \subseteq Dw_r^{\alpha}[p,q](I)$.*

Proof. Omitted. □

Theorem 2.12. *Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be four sequences of non-negative integers such that*

$$p(n) < q(n), p'(n) < q'(n) \text{ and } q(n) - p(n) \leq q'(n) - p'(n) \text{ for all } n \in \mathbb{N} \tag{2}$$

and α, β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then
(i) If

$$\liminf_{n\to\infty} \frac{(q(n) - p(n))^{\alpha}}{(q'(n) - p'(n))^{\beta}} > 0 \tag{3}$$

then $DS_{p',q'}^{\beta}(I) \subseteq DS_{p,q}^{\alpha}(I)$,
(ii) If

$$\lim_{n\to\infty} \frac{q'(n) - p'(n)}{(q(n) - p(n))^{\beta}} = 1 \tag{4}$$

then $DS_{p,q}^{\alpha}(I) \subseteq DS_{p',q'}^{\beta}(I)$.

Proof. (i) Let (3) be satisfied. For given $\varepsilon > 0$ we have

$$\{p'(n) < k \leq q'(n) : |x_k - L| \geq \varepsilon\} \supseteq \{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\},$$

and so

$$\begin{aligned} &\frac{1}{(q'(n) - p'(n))^{\beta}} \left|\{p'(n) < k \leq q'(n) : |x_k - L| \geq \varepsilon\}\right| \\ \geq\ &\frac{(q(n) - p(n))^{\alpha}}{(q'(n) - p'(n))^{\beta}} \frac{1}{(q(n) - p(n))^{\alpha}} \left|\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}\right|. \end{aligned}$$

Hence for all $n \in \mathbb{N}$ we have

$$\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))^{\alpha}}\left|\{p(n)<k\le q(n):|x_k-L|\ge\varepsilon\}\right|\ge\delta\right\}$$
$$\subseteq\left\{n\in\mathbb{N}:\frac{1}{(q'(n)-p'(n))^{\beta}}\left|\{p'(n)<k\le q'(n):|x_k-L|\ge\varepsilon\}\right|\ge\delta\frac{(q(n)-p(n))^{\alpha}}{(q'(n)-p'(n))^{\beta}}\right\}\in I.$$

Therefore $DS^{\beta}_{p',q'}(I)\subseteq DS^{\alpha}_{p,q}(I)$.

(ii) Omitted. □

Theorem 2.13. *Let* $\{p(n)\},\{q(n)\},\{p'(n)\}$ *and* $\{q'(n)\}$*be four sequences of non-negative integers defined as in* (2) *and* α *and* β *be two real numbers such that* $0<\alpha\le\beta\le 1$.

(i) *If* (3) *holds then* $Dw^{\beta}_r[p',q'](I)\subset Dw^{\alpha}_r[p,q](I)$,

(ii) *If* (4) *holds and* $x=(x_k)$ *be a bounded sequence, then* $Dw^{\alpha}_r[p,q](I)\subset Dw^{\beta}_r[p',q'](I)$.

Proof. Omitted. □

Theorem 2.14. *Let* $\{p(n)\},\{q(n)\},\{p'(n)\}$ *and* $\{q'(n)\}$ *be four sequences of non-negative integers defined as in* (2) *and* α *and* β *be two real numbers such that* $0<\alpha\le\beta\le 1$. *Then*

(i) *Let* (3) *holds, if a sequence is strongly* $Dw^{\beta}_r[p',q'](I)-$*convergent to L, then it is* $DS^{\alpha}_{p,q}(I)-$*convergent to L,*

(ii) *Let* (4) *holds and* $x=(x_k)$ *be a bounded sequence, if a sequence is* $DS^{\alpha}_{p,q}(I)-$*convergent to L then it is strongly* $Dw^{\beta}_r[p',q'](I)-$*convergent to L.*

Proof. (i) Omitted.

(ii) Suppose that $DS^{\alpha}_{p,q}(I)-\lim x_k=L$ and $\{x_k\}\in\ell_{\infty}$. Then there exists some $M>0$ such that $|x_k-L|<M$ for all k, then for every $\varepsilon>0$ we may write

$$\begin{aligned}
&\frac{1}{(q'(n)-p'(n))^{\beta}}\sum_{p'(n)+1}^{q'(n)}|x_k-L|^r\\
=\;&\frac{1}{(q'(n)-p'(n))^{\beta}}\sum_{q(n)-p(n)+1}^{q'(n)-p'(n)}|x_k-L|^r+\frac{1}{(q'(n)-p'(n))^{\beta}}\sum_{p(n)+1}^{q(n)}|x_k-L|^r\\
\le\;&\frac{(q'(n)-p'(n))-(q(n)-p(n))}{(q'(n)-p'(n))^{\beta}}M^r+\frac{1}{(q'(n)-p'(n))^{\beta}}\sum_{p(n)+1}^{q(n)}|x_k-L|^r\\
\le\;&\frac{(q'(n)-p'(n))-(q(n)-p(n))^{\beta}}{(q'(n)-p'(n))^{\beta}}M^r+\frac{1}{(q'(n)-p'(n))^{\beta}}\sum_{p(n)+1}^{q(n)}|x_k-L|^r\\
\le\;&\left(\frac{q'(n)-p'(n)}{(q(n)-p(n))^{\beta}}-1\right)M^r+\frac{1}{(q'(n)-p'(n))^{\beta}}\sum_{\substack{p(n)+1\\|x_k-L|\ge\varepsilon}}^{q(n)}|x_k-L|^r\\
&+\frac{1}{(q(n)-p(n))^{\beta}}\sum_{\substack{p(n)+1\\|x_k-L|<\varepsilon}}^{q(n)}|x_k-L|^r\\
\le\;&\left(\frac{q'(n)-p'(n)}{(q(n)-p(n))^{\beta}}-1\right)M^r+\frac{M^r}{(q(n)-p(n))^{\alpha}}\left|\{p(n)<k\le q(n):|x_k-L|\ge\varepsilon\}\right|\\
&+\frac{q'(n)-p'(n)}{(q(n)-p(n))^{\beta}}\varepsilon^r
\end{aligned}$$

and so

$$
\begin{aligned}
&\left\{ n \in \mathbb{N} : \frac{1}{(q'(n) - p'(n))^{\beta}} \sum_{p'(n)+1}^{q'(n)} |x_k - L|^r \geq \delta \right\} \\
\subseteq \quad &\left\{ n \in \mathbb{N} : \frac{1}{(q(n) - p(n))^{\alpha}} \left| \{ p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon \} \right| \geq \frac{\delta}{M^r} \right\} \in I,
\end{aligned}
$$

for all $n \in \mathbb{N}$. Using (4) we obtain that $Dw_r^{\beta}[p', q'](I) - \lim x_k = L$, whenever $DS_{p,q}^{\alpha}(I) - \lim x_k = L$. □

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