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On Operators with Complex Gaussian Kernels over L^p Spaces

B. J. González[a], E. R. Negrín[a]

[a]*Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de La Laguna (ULL). Campus de Anchieta. ES-38271 La Laguna (Tenerife), España*

Abstract. In this paper we study new L^p-boundedness properties and Parseval-type relations concerning the operators with complex Gaussian kernels over the spaces $L^p(\mathbb{R}, w(x)dx)$, $1 \leq p \leq \infty$, where w represents any function greater than or equal to one almost everywhere on $\mathbb{R}$. Here, the Gauss-Weierstrass semigroup is considered as a particular case of this analysis.

1. Introduction

In this paper we consider the integral operator with complex Gaussian kernel of a suitable complex-valued function f defined on $\mathbb{R}$ by

$$(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y) = \int_{-\infty}^{+\infty} \exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x\right] f(x)dx, \tag{1}$$

where $y \in \mathbb{R}$ and $\beta, \varepsilon, \delta, \xi, \gamma \in \mathbb{C}$.

This type of integral operators are present in analysis, probability theory, and mathematical physics in numerous contexts. There are several types of examples as the Fourier transform, the Poisson formula for a solution of the heat equation, and the Mehler formula for the time evolution of a harmonic oscillator (cf. [2], [3], [8], [11], [12] and [13], amongst others).

Our main goal in this paper is to establish new L^p-boundedness properties and Parseval-type relations for the operators given by (1) over the spaces $L^p(\mathbb{R}, w(x)dx)$, $1 \leq p \leq \infty$, where w is any function greater than or equal to one a.e. on $\mathbb{R}$. For this purpose we make use of previous results obtained in [4].

Moreover, under suitable conditions, for $f, g \in L^1(\mathbb{R}, w(x)dx)$, $w \geq 1$ a.e. on $\mathbb{R}$, then one has the following Parseval-type relation

$$\int_{-\infty}^{+\infty} \left(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\right)(x)\, g(x)\, dx = \int_{-\infty}^{+\infty} f(x) \left(\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi} g\right)(x)\, dx. \tag{2}$$

Let $\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi}$ be the adjoint of the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}$, i.e.,

$$\left\langle \mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi} f, g \right\rangle = \left\langle f, \mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi} g \right\rangle.$$

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Email addresses: bjglez@ull.es (B. J. González), enegrin@ull.es (E. R. Negrín)

According to the results of [6], the previously mentioned Parseval-type relation (2) allows us to obtain an interesting connection between the operator $\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi}$ and the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$. Indeed, we conclude that the operator $\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi}$ is the natural extension of the integral operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$, i.e.,

$$\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi} T_f = T_{\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f}$$

where T_f is given by

$$< T_f, g >= \int_{-\infty}^{+\infty} f(x)g(x)dx.$$

In section 3 we consider the Gauss-Weierstrass semigroup as a particular case of this analysis.

We also point out relevant connections of our work with various earlier related results (see [5], [7], [9], [10] and [14]).

2. The operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ over the spaces $L^p(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$, $1 \le p \le \infty$

In this section we study the behaviour of the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ on the spaces $L^p(\mathbb{R}, w(x)dx)$, $1 \le p \le \infty$, $w \ge 1$ a.e. on $\mathbb{R}$.

Indeed, by following [4, Proposition 2.1, Proposition 3.1 and Proposition 4.1], we draw Theorem 2.1 below.

Theorem 2.1. *Assume that w is a function greater than or equal to one almost everywhere on $\mathbb{R}$. We get*

(i) For $\Re\varepsilon > 0$, $(\Re\delta)^2 \le \Re\varepsilon\Re\beta$ and $\Re\varepsilon\Re\xi + \Re\delta\Re\gamma = 0$, then the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^p(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$, $1 < p < \infty$.

(ii) For $\Re\beta \ge |\Re\delta|$, $\Re\varepsilon \ge |\Re\delta|$ and $\Re\xi = \Re\gamma = 0$, then the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^1(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$.

(iii) For $\Re\varepsilon > 0$, $(\Re\delta)^2 \le \Re\varepsilon\Re\beta$ and $\Re\varepsilon\Re\xi + \Re\delta\Re\gamma = 0$ the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^\infty(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$.

Proof. (i) From Hölder's inequality it follows that

$$\begin{aligned}
&\left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\right| \\
&\le \int_{-\infty}^{+\infty} |f(x)| \left|\exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x\right]\right| dx \\
&= \int_{-\infty}^{+\infty} |f(x)| \left|\exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x\right]\right| w(x)^{1/p} w(x)^{-1/p} dx \\
&\le \left(\int_{-\infty}^{+\infty} |f(x)|^p w(x)dx\right)^{1/p} \\
&\times \left(\int_{-\infty}^{+\infty} \left|\exp\left[(-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x)\right]\right|^{p'} w(x)^{-p'/p} dx\right)^{1/p'} \\
&= \left\|f\right\|_p \cdot \left(\int_{-\infty}^{+\infty} \left|\exp\left[(-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x)\right]\right|^{p'} w(x)^{-p'/p} dx\right)^{1/p'}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sup_{y\in\mathbb{R}} \left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)\right| \\
&\le \left\|f\right\|_p \sup_{y\in\mathbb{R}} \left\{\left(\int_{-\infty}^{+\infty} \left|\exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta xy + \xi y + \gamma x\right]\right|^{p'} w(x)^{-p'/p} dx\right)^{1/p'}\right\}.
\end{aligned}$$

Now, since $w \geq 1$ a.e. on $\mathbb{R}$, one has

$$\sup_{y\in\mathbb{R}}\left\{\left(\int_{-\infty}^{+\infty}\left|\exp\left[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x\right]\right|^{p'} w(x)^{-p'/p}dx\right)^{1/p'}\right\}$$
$$=\sup_{y\in\mathbb{R}}\left\{\left(\int_{-\infty}^{+\infty}\exp\left[(-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x)p'\right]w(x)^{-p'/p}dx\right)^{1/p'}\right\}$$
$$\leq\sup_{y\in\mathbb{R}}\left\{\left(\int_{-\infty}^{+\infty}\exp\left[(-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x)p'\right]dx\right)^{1/p'}\right\}.$$

Now, by making use of the well-known fact that

$$(2\pi c)^{(-1/2)}\cdot\int_{-\infty}^{+\infty}\exp\left[\nu x-(x^2/2c)\right]dx=\exp(c\nu^2/2),\quad \nu\in\mathbb{C},\quad c>0, \tag{3}$$

and since $\Re\varepsilon > 0$, one has

$$\sup_{y\in\mathbb{R}}\left\{\left(\int_{-\infty}^{+\infty}\exp\left[(-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x)p'\right]dx\right)^{1/p'}\right\}$$
$$=\sup_{y\in\mathbb{R}}\left\{\left(\frac{\pi}{p'\Re\varepsilon}\right)^{1/2p'}\exp\left[-\Re\beta y^2+\frac{(\Re\delta)^2}{\Re\varepsilon}y^2+\Re\xi y+\frac{\Re\delta\Re\gamma}{\Re\varepsilon}y+\frac{(\Re\gamma)^2}{4\Re\epsilon}\right]\right\},$$

which is bounded under the hypothesis considered.

Hence, we have

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_\infty\leq C\left\|f\right\|_\infty$$

for a certain real constant C.

Therefore, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^p(\mathbb{R},w(x)dx)$ into $L^\infty(\mathbb{R},w(x)dx)$, $1<p<\infty$, $w\geq 1$ a.e. on $\mathbb{R}$.

(ii) Observe that

$$\left|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(y)\right|$$
$$\leq\int_{-\infty}^{+\infty}|f(x)||\exp\left[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x\right]|w(x)w(x)^{-1}dx$$
$$\leq\operatorname*{ess\,sup}_{x\in\mathbb{R}}\left\{\frac{|\exp\left[-\beta y^2-\varepsilon x^2+2\delta xy+\xi y+\gamma x\right]|}{w(x)}\right\}\int_{-\infty}^{+\infty}|f(x)|w(x)dx,$$

and so

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_\infty$$
$$\leq\left\|f\right\|_1\sup_{y\in\mathbb{R}}\operatorname*{ess\,sup}_{x\in\mathbb{R}}\left\{\frac{\exp\left[-\Re\beta y^2-\Re\varepsilon x^2+2\Re\delta xy+\Re\xi y+\Re\gamma x\right]}{w(x)}\right\}.$$

Since $w\geq 1$ a.e. on $\mathbb{R}$ and $2\Re\delta xy\leq|\Re\delta|x^2+|\Re\delta|y^2$, one has

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_\infty$$
$$\leq\left\|f\right\|_1$$
$$\times\sup_{y\in\mathbb{R}}\left\{\exp\left[-(\Re\beta-|\Re\delta|)y^2+\Re\xi y\right]\right\}\sup_{x\in\mathbb{R}}\left\{\exp\left[-(\Re\varepsilon-|\Re\delta|)x^2+\Re\gamma x\right]\right\},$$

which, from the hypothesis considered, yields to

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty \le C \left\|f\right\|_1$$

for a certain real constant C.

Therefore, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from the spaces $L^1(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$, $w \ge 1$ a.e. on $\mathbb{R}$.

(iii) Observe that

$$|(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(y)| \le \operatorname*{ess\,sup}_{x\in\mathbb{R}}\{|f(x)|\} \cdot \int_{-\infty}^{+\infty} \left|\exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta x y + \xi y + \gamma x\right]\right| dx,$$

and so

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty \le \left\|f\right\|_\infty \cdot \sup_{y\in\mathbb{R}} \left\{ \int_{-\infty}^{+\infty} \left|\exp\left[-\beta y^2 - \varepsilon x^2 + 2\delta x y + \xi y + \gamma x\right]\right| dx \right\}$$

$$= \left\|f\right\|_\infty \cdot \sup_{y\in\mathbb{R}} \left\{ \int_{-\infty}^{+\infty} \exp\left[-\Re\beta y^2 - \Re\varepsilon x^2 + 2\Re\delta x y + \Re\xi y + \Re\gamma x\right] dx \right\}.$$

By virtue of (3) and since $\Re\varepsilon > 0$, this expression is equal to

$$\left\|f\right\|_\infty \cdot \sup_{y\in\mathbb{R}} \left\{ \left(\frac{\pi}{\Re\varepsilon}\right)^{1/2} \exp\left[-\Re\beta y^2 + \frac{(\Re\delta)^2}{\Re\varepsilon} y^2 + \Re\xi y + \frac{\Re\delta\Re\gamma}{\Re\varepsilon} y + \frac{(\Re\gamma)^2}{4\Re\epsilon}\right] \right\}$$

which is bounded under the hypothesis considered.

Hence, we have

$$\|\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_\infty \le C \left\|f\right\|_\infty$$

for a certain real constant C.

Consequently, the operator $\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from $L^\infty(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$, $w \ge 1$ a.e. on $\mathbb{R}$. □

As a consequence of (ii) in Theorem 2.1, it follows that for $\Re\varepsilon \ge |\Re\delta|$, $\Re\beta \ge |\Re\delta|$ and $\Re\gamma = \Re\xi = 0$, then the operator $\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi}$ is bounded from the spaces $L^1(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$, $w \ge 1$ a.e. on $\mathbb{R}$.

From this fact and having into account that the weight w is greater than or equal to one a.e. on $\mathbb{R}$, the Proposition 3.2 in [6] yields to

Theorem 2.2. *The following Parseval-type relation holds*

$$\int_{-\infty}^{+\infty} \left(\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\right)(x)\, g(x)\, dx = \int_{-\infty}^{+\infty} f(x) \left(\mathfrak{F}_{\varepsilon,\beta,\delta,\gamma,\xi} g\right)(x)\, dx$$

for $f, g \in L^1(\mathbb{R}, w(x)dx)$, $w \ge 1$ *a.e. on* $\mathbb{R}$, $\Re\varepsilon \ge |\Re\delta|$, $\Re\beta \ge |\Re\delta|$ *and* $\Re\gamma = \Re\xi = 0$.

Also, as a consequence of the Corollary 3.1 in [6], we get

Corollary 2.2. *Assume* $f \in L^1(\mathbb{R}, w(x)dx)$, $w \ge 1$ *a.e. on* $\mathbb{R}$. *For* $\Re\varepsilon \ge |\Re\delta|$, $\Re\beta \ge |\Re\delta|$ *and* $\Re\gamma = \Re\xi = 0$, *it follows*

$$\mathfrak{F}'_{\varepsilon,\beta,\delta,\gamma,\xi} T_f = T_{\mathfrak{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f}$$

on $\left(L^1(\mathbb{R}, w(x)dx)\right)'$.

3. The Gauss-Weierstrass semigroup

The Gauss-Weierstrass semigroup on $\mathbb{R}$ (see [1, p. 521] and [15]) is given by

$$(e^{z\Delta}f)(y) = (4\pi z)^{-1/2}\int_{-\infty}^{+\infty} \exp\left[-(y-x)^2/4z\right] f(x)dx,$$

where $\Re z \geq 0$ (and $z \neq 0$).

Except for the factor $(4\pi z)^{-1/2}$, this integral operator corresponds to the particular case when the parameters are given by

$$\beta = \varepsilon = \delta = 1/4z \text{ and } \xi = \gamma = 0.$$

Now, as a consequence of Theorem 2.1 above, it follows

Theorem 3.1. *Assume $w \geq 1$ almost everywhere on $\mathbb{R}$. For the Gauss-Weierstrass semigroup $e^{z\Delta}$, one has*

(i) For $\Re z > 0$, the operator $e^{z\Delta}$ is bounded from the spaces $L^p(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$, $1 < p < \infty$.

(ii) For $\Re z \geq 0$ (and $z \neq 0$), the operator $e^{z\Delta}$ is bounded from the spaces $L^1(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$.

(iii) For $\Re z > 0$, the operator $e^{z\Delta}$ is bounded from the spaces $L^\infty(\mathbb{R}, w(x)dx)$ into $L^\infty(\mathbb{R}, w(x)dx)$.

Also, from Theorem 2.2 above, one obtains

Theorem 3.2. *The following Parseval relation holds*

$$\int_{-\infty}^{+\infty} \left(e^{z\Delta}f\right)(x)\, g\,(x)\, dx = \int_{-\infty}^{+\infty} f\,(x) \left(e^{z\Delta}g\right)(x)\, dx \tag{4}$$

for $f, g \in L^1(\mathbb{R}, w(x)dx)$, $w \geq 1$ a.e. on $\mathbb{R}$, $\Re z \geq 0$ (and $z \neq 0$).

Moreover, from Corollary 2.2, one has

Corollary 3.1. *For $f \in L^1(\mathbb{R}, w(x)dx)$, $w \geq 1$, a.e. on $\mathbb{R}$, $\Re z \geq 0$ (and $z \neq 0$), it follows*

$$\left(e^{z\Delta}\right)' T_f = T_{e^{z\Delta}f} \tag{5}$$

on $\left(L^1(\mathbb{R}, w(x)dx)\right)'$.

4. Concluding Remarks and Observations

In our present investigation, we have systematically studied several new L^p-boundedness properties for operators with complex Gaussian kernels on the spaces $L^p(\mathbb{R}, w(x)dx)$, $1 \leq p \leq \infty$. Our main result asserted by Theorem 2.1 is believed to be new. We have also briefly considered relevant connections of our results with the Gauss-Weierstrass semigroup.

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