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A Result On A Question of Lü, Li and Yang

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Abstract. Let f be a transcendental meromorphic function of finite order with finitely many poles, $c \in \mathbb{C}\setminus\{0\}$ and $n, k \in \mathbb{N}$. Suppose $f^n(z) - Q_1(z)$ and $(f^n(z+c))^{(k)} - Q_2(z)$ share $(0,1)$ and $f(z)$, $f(z+c)$ share 0 CM. If $n \geq k+1$, then $(f^n(z+c))^{(k)} \equiv \frac{Q_2(z)}{Q_1(z)} f^n(z)$, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$. Furthermore, if $Q_1 = Q_2$, then $f(z) = c_1 e^{\frac{\lambda}{n}z}$, where c_1 and λ are non-zero constants such that $e^{\lambda c} = 1$ and $\lambda^k = 1$. Also we exhibit some examples to show that the conditions of our result are the best possible.

1. Introduction, Definitions and Results

In this paper, by a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function in the whole complex plane $\mathbb{C}$. Here we denote by $n(r, \infty; f)$ the number of poles of f lying in $|z| < r$, the poles are counted with their multiplicities. We call the quantity

$$N(r, \infty; f) = \int_0^r \frac{n(t, \infty; f) - n(0, \infty; f)}{t} dt + n(0, \infty; f) \log r,$$

as the integrated counting function or simply the counting function of poles of f and

$m(r, \infty; f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta$, as the proximity function of poles of f, where $\log^+ x = \log x$, if $x \geq 1$ and $\log^+ x = 0$, if $0 \leq x < 1$.

The sum $m(r, \infty; f) + N(r, \infty; f) = T(r, f)$ is called the Nevanlinna characteristic function of f. We adopt the standard notation $S(r, f)$ for any quantity satisfying the relation $\frac{S(r,f)}{T(r,f)} \to 0$ as $r \to \infty$ except possibly a set of finite linear measure.

For $a \in \mathbb{C}$, we write $N(r, a; f) = N\left(r, \infty; \frac{1}{f-a}\right)$ and $m(r, a; f) = m\left(r, \infty; \frac{1}{f-a}\right)$.

Again let us denote by $\overline{n}(r, a; f)$ the number of distinct a points of f lying in $|z| < r$, where $a \in \mathbb{C} \cup \{\infty\}$. The quantity

$$\overline{N}(r, a; f) = \int_0^r \frac{\overline{n}(t, a; f) - \overline{n}(0, a; f)}{t} dt + \overline{n}(0, a; f) \log r$$

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denotes the reduced counting function of a points of f (see, e.g., [7, 16]).

The order of f is defined by

$$\sigma(f) = \limsup_{r\longrightarrow\infty} \frac{\log T(r,f)}{\log r}.$$

Let k be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We use the notations $N_{k)}(r,a;f)$ and $N_{(k+1}(r,a;f)$ to denote the counting function of a-points of f with multiplicity not greater than k and the counting function of a-points of f with multiplicity greater than k respectively. Similarly $\overline{N}_{k)}(r,a;f)$ and $\overline{N}_{(k+1}(r,a;f)$ are their reduced functions respectively.

For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum

$$\overline{N}_{(1}(r,a;f) + \overline{N}_{(2}(r,a;f) + \ldots + \overline{N}_{(p}(r,a;f).$$

A meromorphic function a is said to be a small function of f if $T(r,a) = S(r,f)$, i.e., if $T(r,a) = o(T(r,f))$ as $r \to \infty$ except possibly a set of finite linear measure.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. If $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share $a(z)$ with CM (counting multiplicities) and if we do not consider the multiplicities then we say that $f(z)$ and $g(z)$ share $a(z)$ with IM (ignoring multiplicities).

The value distribution theory has an important role in Complex Analysis. In this theory we studies how an entire or a meromorphic function assumes some values and we discuss the influence of assuming certain values in some specific manner on a function. Perhaps the Fundamental Theorem of Classical Algebra is the most well-known value distribution theorem and the next one is Picard's theorem.

Now the uniqueness theory of entire and meromorphic functions has become an extensive subfield of the value distribution theory. In this theory we mainly studies the conditions for which there exists essentially only one function. Although a polynomial is determined by its zero points (the set on which the polynomial take zeros) except for a non-constant factor but this theory is not necessarily true for transcendental entire or meromorphic functions. For examples e^z and e^{-z} have the same ±1, 0 and ∞ (poles) points. So to determine uniquely an entire or meromorphic function is interesting although sophisticated. The uniqueness theory of meromorphic functions is devoted to study conditions that are satisfied by a few meromorphic functions only, or even determine a meromorphic function uniquely. Finnish mathematician Prof. Rolf Nevanlinna was the first who gave results of this type within the value distribution theory. These results are usually called Nevanllina's five-value, resp. four-value, theorem, meaning that whenever two meromorphic functions take five, resp. four, extended complex values at the same points in the complex plane, these two functions actually agree, resp. are Bilinear transformations of each other. These two theorems are the starting points of the uniqueness theory, essentially developed during the last four decades, being presently an extensive theory. A meromorphic function and it's derivative share some values or functions or set is an important subtopic in the uniqueness theory.

Rubel and Yang was the first to study the entire functions that share values with their derivatives. In 1977 they proved the following important theorem.

Theorem 1.1. *[14] Let a and b be complex numbers such that $b \neq a$ and let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share the values a and b CM, then $f \equiv f'$.*

From then on, this result has undergone various extensions and improvements (see [16]). In 1980, G. G. Gundersen improved Theorem 1.1 and obtained the following result.

Theorem 1.2. *[5] Let f be a non-constant meromorphic function, a and b be two distinct finite values. If f and f' share the values a and b CM, then $f \equiv f'$.*

Mues and Steinmetz [13] generalized Theorem 1.1 from sharing values CM to IM and obtained the following result.

Theorem 1.3. *[13] Let a and b be complex numbers such that $b \neq a$ and let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share the values a and b IM, then $f \equiv f'$.*

In 1996, Brück [1] discussed the possible relation between f and f' when an entire function f and it's derivative f' share only one finite value CM. In this direction an interesting problem still open is the following conjecture proposed by Brück [1].

Conjecture 1.4. *Let f be a non-constant entire function. Suppose*

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a} = c, \text{ for some non-zero constant } c. \tag{1.1}$$

The conjecture had been proved by Brück [1] for the cases $a = 0$ and $N(r,0;f') = S(r,f)$. From the differential equations

$$\frac{f'-a}{f-a} = e^{z^n} \text{ and } \frac{f'-a}{f-a} = e^{e^z}, \tag{1.2}$$

we see that when $\rho_1(f)$ is a positive integer or infinite, the conjecture does not hold.

Gundersen and Yang [6] proved that the conjecture is true when f is of finite order. Further Chen and Shon [3] proved that the conjecture is also true when f is of infinite order with $\rho_1(f) < \frac{1}{2}$. Recently Cao [2] proved that the Brück conjecture is also true when f is of infinite order with $\rho_1(f) = \frac{1}{2}$. But the case $\rho_1(f) > \frac{1}{2}$ is still open. However, the conjecture for meromorphic functions fails in general (see [6]). For example if

$$f(z) = \frac{2e^z + z + 1}{e^z + 1},$$

then f and f' share the value 1 CM, but (1.1) does not hold.

It is now interesting to know what happens if f is replaced by f^n in the Brück conjecture. From (1.2) we see that the conjecture does not hold when $n = 1$. Thus we only need to discuss the problem when $n \geq 2$.

To the knowledge of the authors perhaps Yang and Zhang [15] were the first to consider the uniqueness of a power of an entire function $F = f^n$ and its derivative F' when they share certain value as this type of considerations gives most specific form of the function.

Yang and Zhang [15] proved that the Brück conjecture holds for the function f^n and the order restriction on f does not needed if n is relatively large. Actually they proved the following result.

Theorem 1.5. *[15] Let f be a non-constant entire function, $n(\geq 7)$ be an integer and let $F = f^n$. If F and F' share 1 CM, then $F \equiv F'$ and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a non-zero constant.

As a result during the last decade, growing interest has been devoted to this setting of entire functions. Improving all the results obtained in [15], Zhang [17] proved the following theorem.

Theorem 1.6. *[17] Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k + 4$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c and λ are non-zero constants such that $\lambda^k = 1$.

In 2009, Zhang and Yang [18] further improved the above result in the following manner.

Theorem 1.7. *[18] Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k+1$. Then conclusion of Theorem 1.6 holds.*

In 2010, Zhang and Yang [19] further improved the above result in the following manner.

Theorem 1.8. *[19] Let f be a non-constant entire function, n and k be positive integers. Suppose f^n and $(f^n)^{(k)}$ share 1 CM and $n \geq k+1$. Then conclusion of Theorem 1.6 holds.*

Using the theory of normal families, in 2011, Lü and Yi [10] proved the following theorem.

Theorem 1.9. *[10] Let f be a transcendental entire function, n, k be two positive integers with $n \geq k+1$, $F = f^n$ and $Q \not\equiv 0$ be a polynomial. If $F - Q$ and $F^{(k)} - Q$ share the value 0 CM, then $F \equiv F^{(k)}$ and $f(z) = ce^{wz/n}$, where c and w are non-zero constants such that $w^k = 1$.*

Remark 1.10. *Following example shows that the hypothesis of the transcendental of f in Theorem 1.9 is necessary.*

Example 1.11. *[10] Let $f(z) = z$ and $n = 2, k = 1$. Then*

$$\frac{\left(f^2\right)' - Q}{f^2 - Q} = 2$$

and $\left(f^2\right)' - Q$, $f^2 - Q$ share 0 CM, but $\left(f^2\right)' \not\equiv f^2$, where $Q(z) = 2z^2 - 2z$.

Remark 1.12. *It is easy to see that the condition $n \geq k+1$ in Theorem 1.9 is sharp by the following example.*

Example 1.13. *Let $f(z) = e^{e^z} \int\limits_0^z e^{-e^t}(1-e^t)t\, dt$ and $n = 1, k = 1$. Then*

$$\frac{f'(z) - z}{f(z) - z} = e^z$$

and $f'(z) - z$, $f(z) - z$ share 0 CM, but $f' \not\equiv f$.

Now observing the above theorem, Lü, Li and Yang [11] asked the following question:
Question 1. What can be said "if $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share the value 0 CM" ? where Q_1 and Q_2 are polynomials and $Q_1Q_2 \not\equiv 0$.

Lü, Li and Yang [11] solved the above question for $k = 1$ by giving the transcendental entire solutions of the equation

$$F' - Q_1 = Re^{\alpha}(F - Q_2), \tag{1.3}$$

where $F = f^n$, R is a rational function and α is an entire function and they obtained the following results.

Theorem 1.14. *[11] Let f be a transcendental entire function and let $F = f^n$ be a solution of equation (1.3), $n \geq 2$ be an integer, then $\frac{Q_1}{Q_2}$ is a polynomial and $f' \equiv \frac{Q_1}{nQ_2} f$.*

Theorem 1.15. *[11] Let f be a transcendental entire function, $n \geq 2$ be an integer. If $f^n - Q$ and $(f^n)' - Q$ share 0 CM, where $Q \not\equiv 0$ is a polynomial, then*

$$f(z) = ce^{z/n},$$

where c is a non-zero constant.

Also in the same paper, Lü, Li and Yang [11] posed the following conjecture.

Conjecture 1.16. *Let f be a transcendental entire function, n, k be two positive integers. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM, and $n \geq k+1$, then $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$. Further, if $Q_1 = Q_2$, then $f(z) = ce^{wz/n}$, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$ and c, w are non-zero constants such that $w^k = 1$.*

Again Lü, Li and Yang [11] asked the following question.

Question 2. What can be said if the condition in the Conjecture 1.16 "$(f^n)^{(k)}$" be replaced by "$\{f(z+c_1)f(z+c_2)\ldots f(z+c_n)\}^{(k)}$" ? where $c_j (j = 1, 2, \ldots, n)$ are constants.

In 2016, Majumder [12] proved that the Conjecture 1.16 is true and obtained the following result.

Theorem 1.17. *[12] Let f be a transcendental entire function, n and k be two positive integers. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM, and $n \geq k+1$, then $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$. Further, if $Q_1 = Q_2$, then $f(z) = ce^{\frac{\lambda}{n}z}$, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$ and c, λ are non-zero constants such that $\lambda^k = 1$.*

The purpose of this paper is to give an affirmative answer of the above Question 2.

Before going to our main result, we now explain the notation of weighted sharing as introduced in [8].

Definition 1.18. *[8] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In this paper we have been able to solve the above Question 2 at the cost of considering the fact that $c_1 = c_2 = \ldots = c_n = c \in \mathbb{C} \setminus \{0\}$ and $f(z)$ is a transcendental meromorphic function of finite order with finitely many poles such that $f(z)$ and $f(z+c)$ share 0 CM.

The following theorem is the main result of this paper.

Theorem 1.19. *Let $f(z)$ be a transcendental meromorphic function of finite order with finitely many poles, $c \in \mathbb{C} \setminus \{0\}$ and $n, k \in \mathbb{N}$. Suppose $f^n(z) - Q_1(z)$ and $(f^n(z+c))^{(k)} - Q_2(z)$ share $(0, 1)$ and $f(z)$, $f(z+c)$ share 0 CM. If $n \geq k+1$, then $(f^n(z+c))^{(k)} \equiv \frac{Q_2(z)}{Q_1(z)} f^n(z)$, where Q_1 and Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$. Furthermore, if $Q_1 = Q_2$, then $f(z) = c_1 e^{\frac{\lambda}{n}z}$, where c_1 and λ are non-zero constants such that $e^{\lambda c} = 1$ and $\lambda^k = 1$.*

Remark 1.20. *It is easy to see that the condition $n \geq k+1$ in Theorem 1.19 is sharp by the following example.*

Example 1.21. *Let*

$$f(z) = e^{2z} + 1, \quad c = \pi \mathrm{i}.$$

Then $f(z) - Q_1$ and $f'(z+c) - Q_2$ share 0 CM, but $f'(z+c) \not\equiv \frac{Q_2(z)}{Q_1(z)} f(z)$, where $Q_1(z) = 3$ and $Q_2(z) = 4$.

Remark 1.22. *It is easy to see that the condition $f(z)$ and $f(z+c)$ share 0 CM in Theorem 1.19 is sharp by the following example.*

Example 1.23. *Let*

$$f(z) = e^z + 1 \quad \text{and} \quad e^c = \frac{1}{2}.$$

Clearly $f(z)$ and $f(z+c)$ do not share the value 0 CM. Also $f^2(z) - Q_1$ and $\left(f^2(z+c)\right)' - Q_2$ share 0 CM, but $\left(f^2(z+c)\right)' \not\equiv \frac{Q_2}{Q_1} f^2(z)$, where $Q_1(z) = 3$ and $Q_2(z) = 1$.

2. Lemmas

In this section we present the following lemmas which will be needed in the sequel.

Lemma 2.1. *[4] Let f be a meromorphic function of finite order σ, and let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \infty; \frac{f(z+c)}{f(z)}\right) + m\left(r, \infty; \frac{f(z)}{f(z+c)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 2.2. *[4] Let f be a meromorphic function of finite order σ, and let $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r).$$

Lemma 2.3. *Suppose that f is a transcendental meromorphic function of finite order and that*

$$f^n(z)P(z, f) = Q(z, f),$$

where $P(z, f)$ is a polynomial in $f(z+c)$ and its derivatives and $Q(z, f)$ is a polynomial in $f(z)$, $f(z+c)$ and their derivatives. Also $P(z, f)$ and $Q(z, f)$ are polynomials with meromorphic coefficient, say $\{ a_\lambda \mid \lambda \in I\}$ such that $m(r, \infty; a_\lambda) = S(r, f)$, $\forall\ \lambda \in I$. If the total degree of $Q(z, f)$ is at most n then

$$m(r, \infty; P(z, f)) = S(r, f).$$

Proof. Proof of lemma follows from proof of Theorem 2.3 [9]. □

Lemma 2.4. *[[7], Lemma 3.5] Suppose that F is meromorphic in a domain D and set $f = \frac{F'}{F}$. Then for $n \geq 1$*

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4}(f')^2 + P_{n-3}(f),$$

where $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

3. Proof of the theorem

Proof. Let $F_1(z) = \frac{f^n(z)}{Q_1(z)}$ and $G_1(z) = \frac{\left(f^n(z+c)\right)^{(k)}}{Q_2(z)}$. Clearly F_1 and G_1 share $(1, 1)$ except for the zeros of $Q_i(z)$, where $i = 1, 2$ and so $\overline{N}(r, 1; F_1) = \overline{N}(r, 1; G_1) + O(\log r)$. Let

$$F(z) = f^n(z) \quad \text{and} \quad G(z) = \left(f^n(z+c)\right)^{(k)}. \tag{3.1}$$

Clearly from Lemma 2.2, we have $S(r, f(z+c)) = S(r, f)$. Therefore by Lemma 2.1, we have $m\left(r, \infty; \frac{G}{F}\right) = S(r, f)$. Set

$$\Phi = \frac{F_1'(F_1 - G_1)}{F_1(F_1 - 1)}. \tag{3.2}$$

We now consider the following two cases.
Case 1. Suppose $\Phi \not\equiv 0$. Clearly $m(r, \infty; \Phi) = S(r, f)$.

Let z_0 be a zero of f of multiplicity p such that $Q_i(z_0) \neq 0$, where $i = 1, 2$. Since $f(z)$ and $f(z+c)$ share 0 CM, it follows that z_0 must be a zero of $f(z+c)$ of multiplicity p. Consequently z_0 will a zero of F_1 and G_1 of multiplicities np and $np - k$ respectively and so from (3.2), we get

$$\Phi(z) = O\left((z - z_0)^{np-k-1}\right). \tag{3.3}$$

Since $n \geq k+1$, it follows that $\Phi(z)$ is holomorphic at z_0. Let z_1 be a zero of $F_1 - 1$ of multiplicity $q(\geq 2)$ such that $Q_i(z_1) \neq 0$, where $i = 1, 2$. Since F_1 and G_1 share $(1, 1)$, it follows that z_1 must be a zero of $G_1 - 1$ of multiplicity $r(\geq 2)$. Then in some neighbourhood of z_1, we get by Taylor's expansion

$$F_1(z) - 1 = a_q(z - z_1)^q + a_{q+1}(z - z_1)^{q+1} + \ldots, a_q \neq 0$$

$$\text{and} \quad G_1(z) - 1 = b_r(z - z_1)^r + b_{r+1}(z - z_1)^{r+1} + \ldots, b_r \neq 0.$$

Clearly

$$F_1'(z) = qa_q(z - z_1)^{q-1} + (q + 1)a_{q+1}(z - z_1)^q + \ldots.$$

Note that

$$F_1(z) - G_1(z) = \begin{cases} a_q(z - z_1)^q + \ldots, & \text{if } q < r \\ -b_r(z - z_1)^r - \ldots, & \text{if } q > r \\ (a_q - b_q)(z - z_1)^q + \ldots, & \text{if } q = r. \end{cases}$$

Clearly from (3.2), we get

$$\Phi(z) = O\left((z - z_1)^{t-1}\right), \tag{3.4}$$

where $t \geq \min\{q, r\} \geq 2$. Therefore $\Phi(z)$ is holomorphic at z_1. Thus the poles of Φ come from the zeros of Q_i, $i = 1, 2$ and the poles of f, which implies that Φ just has finitely many poles and so $N(r, \infty; \Phi) = O(\log r)$. Consequently $T(r, \Phi) = S(r, f)$. On the other hand from (3.4), we see that

$$\overline{N}_{(2}(r, 1; F_1) \leq N(r, 0; \Phi) \leq T(r, \Phi) + O(1) = S(r, f),$$

i.e., $\overline{N}_{(2}(r, 1; F_1) = S(r, f)$. Since F_1 and G_1 share $(1, 1)$ except for the zeros of $Q_i(z)$, where $i = 1, 2$, it follows that $\overline{N}_{(2}(r, 1; G_1) = S(r, f)$. Also from (3.2), we get

$$\frac{1}{F_1} = \frac{1}{\Phi} \frac{F_1'}{F_1(F_1 - 1)} \left[1 - \frac{Q_1}{Q_2} \frac{G}{F}\right]$$

and so

$$m\left(r, \infty; \frac{1}{F_1}\right) = S(r, f).$$

Hence

$$m\left(r, \infty; \frac{1}{f}\right) = S(r, f). \tag{3.5}$$

We consider the following two sub-cases.
Sub-case 1.1. Let $n > k + 1$. From (3.3), we see that

$$N(r, 0; f) \leq N(r, 0; \Phi) \leq T\left(r, \frac{1}{\Phi}\right) \leq T(r, \Phi) + O(1) = S(r, f). \tag{3.6}$$

Now from (3.5) and (3.6), we get $T(r, f) = S(r, f)$, which is a contradiction.
Sub-case 1.2. Let $n = k + 1$. From (3.3), we see that

$$N_{(2}(r, 0; f) \leq N(r, 0; \Phi) \leq T(r, \Phi) + O(1) = S(r, f).$$

Then (3.5) gives

$$T(r, f) = N_{1)}(r, 0; f) + S(r, f). \tag{3.7}$$

Note that $\overline{N}_{(2}(r, Q_1; F) = S(r, f)$ and $\overline{N}_{(2}(r, Q_2; G) = S(r, f)$.
Since $F - Q_1$ and $G - Q_2$ share $(0, 1)$, then there exists a meromorphic function β of finite order, such that

$$\frac{G - Q_2}{F - Q_1} = \beta,$$

$$\text{i.e.,} \qquad G - Q_2 = \beta(F - Q_1). \tag{3.8}$$

First we suppose β is non-constant. Since $F - Q_1$ and $G - Q_2$ share $(0, 1)$, it follows that β has a zero at z_2 if z_2 is a zero of $F - Q_1$ and $G - Q_2$ with multiplicities p and q respectively such that $p < q$ and β has a pole at z_2 if $q < p$. Since F and G have finitely many poles, it follows that $N(r, \infty; F) = O(\log r)$ and $N(r, \infty; G) = O(\log r)$. Therefore

$$\overline{N}(r, 0; \beta) \leq \overline{N}_{(2}(r, Q_2; G) + O(\log r) = S(r, f)$$

$$\text{and} \qquad \overline{N}(r, \infty; \beta) \leq \overline{N}_{(2}(r, Q_1; F) + O(\log r) = S(r, f).$$

By differentiation from (3.8), we get

$$G' - Q_2' = \beta'(F - Q_1) + \beta(F' - Q_1'). \tag{3.9}$$

Now combining (3.8) and (3.9), we get

$$\begin{aligned} G'F - \frac{\beta'}{\beta}GF - GF' &= Q_1G' - \left(\frac{\beta'}{\beta}Q_1 + Q_1'\right)G - Q_2F' \\ &\quad + \left(Q_2' - \frac{\beta'}{\beta}Q_2\right)F + \frac{\beta'}{\beta}Q_1Q_2 + Q_2Q_1' - Q_1Q_2'. \end{aligned} \tag{3.10}$$

Let $\xi = \frac{\beta'}{\beta}$. Consequently $T(r, \xi) = S(r, f)$. Since $f(z)$ and $f(z + c)$ share 0 CM and $f(z)$ has finitely many poles, it follows that

$$f(z) = f(z + c)\alpha(z)e^{\eta(z)}, \tag{3.11}$$

where $\alpha(z)$ is a rational function and $\eta(z)$ is a polynomial and so

$$\frac{f'(z)}{f(z)} = \frac{f'(z + c)}{f(z + c)} + p(z), \tag{3.12}$$

where $p(z) = \frac{\alpha'(z)}{\alpha(z)} + \eta'(z)$. Using Lemma 2.1, we get from (3.11) that $m(r, \infty; \alpha e^{\eta}) = m\left(r, \infty; \frac{f(z)}{f(z+c)}\right) = S(r, f)$. Consequently

$$T(r, \alpha e^{\eta}) = N(r, \infty; \alpha e^{\eta}) + m(r, \infty; \alpha e^{\eta}) = O(\log r) + S(r, f) = S(r, f). \tag{3.13}$$

By induction, we deduce from (3.1) that

$$(f^n(z + c))' = nf^{n-1}(z + c)f'(z + c)$$

$$(f^n(z + c))'' = n(n - 1)f^{n-2}(z + c)\left(f'(z + c)\right)^2 + nf^{n-1}(z + c)f''(z + c)$$

$$\begin{aligned} (f^n(z + c))''' &= n(n - 1)(n - 2)f^{n-3}(z + c)\left(f'(z + c)\right)^3 \\ &\quad + 3n(n - 1)f^{n-2}(z + c)f'(z + c)f''(z + c) + nf^{n-1}(z + c)f'''(z + c) \end{aligned}$$

and so on. Thus in general, we have

$$\begin{aligned} &G(z) \\ = \; &(k + 1)!f(z + c)\left(f'(z + c)\right)^k + \frac{k(k - 1)}{4}(k + 1)!f^2(z + c)\left(f'(z + c)\right)^{k-2} f''(z + c) \\ &+ \ldots + (k + 1)f^k(z + c)f^{(k)}(z + c). \end{aligned} \tag{3.14}$$

Therefore from (3.12) and (3.14), we have

$$\begin{aligned} & \frac{f'(z)}{f(z)}G(z) \\ = & \left\{\frac{f'(z+c)}{f(z+c)}+p(z)\right\}\Big\{(k+1)!f(z+c)\left(f'(z+c)\right)^k+ \\ & +\frac{k(k-1)}{4}(k+1)!f^2(z+c)\left(f'(z+c)\right)^{k-2}f''(z+c)+\ldots+(k+1)f^k(z+c)f^{(k)}(z+c)\Big\} \\ = & (k+1)!\left(f'(z+c)\right)^{k+1}+(k+1)!p(z)f(z+c)\left(f'(z+c)\right)^k \\ & +\frac{k(k-1)}{4}(k+1)!f(z+c)\left(f'(z+c)\right)^{k-1}f'(z+c) \\ & +\frac{k(k-1)}{4}(k+1)!p(z)f^2(z+c)\left(f'(z+c)\right)^{k-2}f''(z+c)+\ldots \\ & +(k+1)f^{k-1}(z+c)f'(z+c)f^{(k)}(z+c)+(k+1)p(z)f^k(z+c)f^{(k)}(z+c). \end{aligned} \tag{3.15}$$

Again from (3.14), we have

$$\begin{aligned} & G'(z) \\ = & (k+1)!\left(f'(z+c)\right)^{k+1}+\frac{k(k+1)}{2}(k+1)!f(z+c)\left(f'(z+c)\right)^{k-1}f''(z+c) \\ & +\ldots+(k+1)f^k(z+c)f^{(k+1)}(z+c). \end{aligned} \tag{3.16}$$

Substituting (3.1), (3.14), (3.15) and (3.16) into (3.10), we have

$$f^n(z)P(z)=Q(z), \tag{3.17}$$

where $Q(z)$ is a differential polynomial in $f(z), f(z+c)$ of degree n and

$$\begin{aligned} P(z) &= G'-\xi G-n\frac{f'}{f}G \\ &= -k(k+1)!\left(f'(z+c)\right)^{k+1} \\ &\quad -(k+1)![\xi+(k+1)p(z)]f(z+c)\left(f'(z+c)\right)^k \\ &\quad +\frac{k(k+1)(3-k)(k+1)!}{4}f(z+c)\left(f'(z+c)\right)^{k-1}f''(z+c)+\ldots \\ &\quad +(k+1)f^k(z+c)f^{(k+1)}(z+c) \end{aligned} \tag{3.18}$$

is a differential polynomial in $f(z+c)$ of degree $k+1$.
First we suppose $P\not\equiv 0$. Then by Lemma 2.3, we get $m(r,\infty;P)=S(r,f)$ and so

$$T(r,P)=S(r,f) \text{ and } T(r,P')=S(r,f). \tag{3.19}$$

Note that from (3.18), we get

$$P'(z)=A_1\left(f'(z+c)\right)^k f''(z+c)+B_1(z)\left(f'(z+c)\right)^{k+1}+S_1(z), \tag{3.20}$$

is a differential polynomial in $f(z+c)$, where $A_1=-\frac{1}{4}k(k+1)^2(k+1)!$, $B_1=-(k+1)![\xi+(k+1)p]$ and $S_1(z)$ is a differential polynomial in $f(z+c)$. In particular every monomial of S_1 has the form

$$S(\xi,\xi',p,p')f^{r_0^\lambda}(z+c)\left(f'(z+c)\right)^{r_1^\lambda}\ldots\left(f^{(k+1)}(z+c)\right)^{r_{k+1}^\lambda},$$

where $r_0^\lambda, \ldots, r_{k+1}^\lambda$ are non-negative integers satisfying $\sum_{j=0}^{k+1} r_j^\lambda = n$ and $1 \leq r_0^\lambda \leq n-1$, $S(\xi, \xi', p, p')$ is a polynomial in ξ, ξ', p and p' with constant coefficients.
Let z_3 be a simple zero of $f(z+c)$ such that $\beta(z_3), \beta'(z_3) \neq 0$. Then from (3.18) and (3.20), we have

$$P(z_3) = -k(k+1)!\left(f'(z_3+c)\right)^{k+1}$$

and $$P'(z_3) = A_1\left(f'(z_3+c)\right)^k f''(z_3+c) + B_1(z_3)\left(f'(z_3+c)\right)^{k+1}.$$

This shows that z_3 is a zero of $P(z)f''(z+c) - [K_1(z)P'(z) - K_2(z)P(z)]f'(z+c)$, where $K_1 = \frac{-k(k+1)!}{A_1}$ and $K_2 = \frac{B_1}{A_1}$. Also $T(r, K_1) = S(r, f)$ and $T(r, K_2) = S(r, f)$. Let

$$\Phi_1(z) = \frac{P(z)f''(z+c) - [K_1(z)P'(z) - K_2(z)P(z)]f'(z+c)}{f(z+c)}. \tag{3.21}$$

Suppose $\Phi_1(z) \not\equiv 0$. Clearly $T(r, \Phi_1) = S(r, f)$. From (3.21), we obtain

$$f''(z+c) = \alpha_1(z)f(z+c) + \beta_1(z)f'(z+c), \tag{3.22}$$

where

$$\alpha_1 = \frac{\Phi_1}{P} \text{ and } \beta_1 = K_1\frac{P'}{P} - K_2. \tag{3.23}$$

Note that (3.22) is also true even when $\Phi_1(z) \equiv 0$. Actually in that case $\alpha_1(z) \equiv 0$. From (3.22), we have

$$f^{(i)}(z+c) = \alpha_{i-1}(z)f(z+c) + \beta_{i-1}(z)f'(z+c), \tag{3.24}$$

where $i \geq 2$ and $T(r, \alpha_{i-1}) = S(r, f)$, $T(r, \beta_{i-1}) = S(r, f)$. Now from (3.18), (3.20) and (3.24), we have respectively

$$P(z) = -k(k+1)!\left(f'(z+c)\right)^{k+1} + \sum_{j=1}^{k+1} t_j(z)f^j(z+c)\left(f'(z+c)\right)^{k+1-j} \tag{3.25}$$

and $$P'(z) = (A_1(z)\beta_1(z) + B_1(z))\left(f'(z+c)\right)^{k+1} + \sum_{j=1}^{k+1} s_j(z)f^j(z+c)\left(f'(z+c)\right)^{k+1-j}, \tag{3.26}$$

where $T(r, t_j) = S(r, f)$ and $T(r, s_j) = S(r, f)$. Also (3.23) yields

$$P' = \left(\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\right)P \tag{3.27}$$

and so

$$\beta_1 = K_1\frac{P'}{P} - K_2 = \frac{-k(k+1)!}{A_1}\frac{P'}{P} - \frac{B_1}{A_1},$$

$$\text{i.e.,}\quad A_1\beta_1 + B_1 + k(k+1)!\frac{P'}{P} = 0. \tag{3.28}$$

Now we consider following two sub-cases.
Sub-case 1.2.1. Let $k = 1$. Now from (3.18) and (3.22), we have

$$\begin{aligned} P(z) &= -2\left(f'(z+c)\right)^2 - 2(\xi + 2p)f(z+c)f'(z+c) + 2f(z+c)f''(z+c) \\ &= -2\left(f'(z+c)\right)^2 + (2\beta_1 - 2\xi - 4p)f(z+c)f'(z+c) + 2\alpha_1(f(z+c))^2 \end{aligned}$$

and so

$$\begin{aligned} P'(z) = {} & (-2\beta_1 - 2\xi - 4p)\left(f'(z+c)\right)^2 \\ & + (2\beta_1' - 2\xi' + 2\beta_1^2 - 2\beta_1\xi - 4p' - 4\beta_1 p)f(z+c)f'(z+c) \\ & + (2\alpha_1\beta_1 - 2\alpha_1\xi - 4\alpha_1 p + 2\alpha_1')(f(z+c))^2. \end{aligned}$$

Note that $K_1 = 1$ and $K_2 = \xi + 2p$ and so from (3.27), we have

$$\begin{aligned}&\left(\beta_1' - \xi' - 2p' - \beta_1\xi + \xi^2 + 4\xi p - 2\beta_1 p + 4p^2\right) f'(z+c)\\ &\qquad + \left(-2\alpha_1\xi - 4\alpha_1 p + \alpha_1'\right) f(z+c) \equiv 0.\end{aligned} \tag{3.29}$$

Suppose first that $\beta_1' - \xi' - 2p' - \beta_1\xi + \xi^2 + 4\xi p - 2\beta_1 p + 4p^2 \equiv 0$. Then by integration, we have $\beta = d_0 \frac{\beta_1 - \xi - 2p}{(\alpha e^{\eta})^2}$, provided $\beta_1 - \xi - 2p \not\equiv 0$, where $d_0 \in \mathbb{C} \setminus \{0\}$. So using (3.13), we have $T(r,\beta) = S(r,f)$. Now from (3.8), we have

$$\left(f^2(z+c)\right)' - \beta(z) f^2(z) = Q_2(z) - Q_1(z)\beta(z). \tag{3.30}$$

Since $f(z)$ and $f(z+c)$ share 0 CM, from (3.30), we have $N_{1)}(r,0;f) = S(r,f)$ and so from (3.7) we arrive at a contradiction. If $\beta_1 - \xi - 2p \equiv 0$, then by integration, we have $\beta^2 = \hat{d}_0 \frac{p}{(\alpha e^{\eta})^4}$, where $\hat{d}_0 \in \mathbb{C} \setminus \{0\}$. Then proceeding as above, from (3.7), we arrive at a contradiction.

We now assume that $\beta_1' - \xi' - 2p' - \beta_1\xi + \xi^2 + 4\xi p - 2\beta_1 p + 4p^2 \not\equiv 0$. Then from (3.29), we get $N_{1)}(r,0;f) = S(r,f)$ and so we arrive at a contradiction from (3.7).

Sub-case 1.2.2. Let $k \geq 2$. Clearly from (3.25), (3.26) and (3.27), we get

$$h_1\left(f'(z+c)\right)^k + h_2 f(z+c)\left(f'(z+c)\right)^{k-1} + \ldots + h_{k+1} f^k(z+c) \equiv 0, \tag{3.31}$$

where $h_j = s_j - \left(\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\right) t_j$ and $T(r,h_j) = S(r,f)$. If at least one of h_j' s is not identically zero then from (3.31) we get $N_{1)}(r,0;f) = S(r,f)$ and so we arrive at a contradiction from (3.7). Next we suppose $h_j(z) \equiv 0$ for $j = 1,2,\ldots,k+1$.

Note that from (3.10) and (3.17), we have

$$Q(z) = Q_1 G' - \left(\xi Q_1 + Q_1'\right) G - Q_2 F' + \left(Q_2' - \xi Q_2\right) F + \gamma, \tag{3.32}$$

where $\gamma = \xi Q_1 Q_2 + Q_2 Q_1' - Q_1 Q_2'$. Suppose $\gamma(z) \equiv 0$. Then by integration, we obtain $\beta = d_1 \frac{Q_2}{Q_1}$, where $d_1 \in \mathbb{C} \setminus \{0\}$ and so $T(r,\beta) = S(r,f)$. Therefore we arrive at a contradiction from (3.8). Consequently $\gamma(z) \not\equiv 0$. Similarly we have $\xi Q_1 + Q_1' \not\equiv 0$. On the other hand $T(r,\gamma) = S(r,f)$.

Differentiating (3.32), we have

$$\begin{aligned}Q'(z) &= Q_1' G' + Q_1 G'' - \left(\xi Q_1 + Q_1'\right) G' - \left(\xi Q_1 + Q_1'\right)' G - Q_2' F' - Q_2 F''\\ &\quad + \left(Q_2' - \xi Q_2\right)' F + \left(Q_2' - \xi\right) F' + \gamma'.\end{aligned} \tag{3.33}$$

Let z_4 be a simple zero of $f(z)$ such that $\beta(z_4), \beta'(z_4), P(z_4), P'(z_4) \neq 0$. Then from (3.17), (3.32) and (3.33), we have

$$\begin{aligned}&\gamma(z_4) = A(z_4)\left(f'(z_4+c)\right)^{k+1}\\ \text{and}\quad &\gamma'(z_4) = A_2(z_4)\left(f'(z_4+c)\right)^k f''(z_4+c) + B_2(z_4)\left(f'(z_4+c)\right)^{k+1},\end{aligned}$$

where $A(z) = -(k+1)!Q_1(z)$, $A_2(z) = -\frac{(k+1)(k+2)}{2}(k+1)!Q_1(z)$ and $B_2(z) = (k+1)!\xi(z)Q_1(z)$. This shows that z_4 is a zero of $\gamma(z)f''(z+c) - [K_3(z)\gamma'(z) - K_4(z)\gamma(z)]f'(z+c)$, where $K_3 = \frac{A}{A_2}$ and $K_2 = \frac{B_2}{A_2}$. Also $T(r,K_3) = S(r,f)$ and $T(r,K_4) = S(r,f)$. Let

$$\Phi_2(z) = \frac{\gamma(z)f''(z+c) - [K_3(z)\gamma'(z) - K_4(z)\gamma(z)]f'(z+c)}{f(z+c)}. \tag{3.34}$$

Suppose $\Phi_2(z) \not\equiv 0$. Clearly $T(r,\Phi_2) = S(r,f)$. From (3.34), we obtain

$$f''(z+c) = \phi_1(z) f(z+c) + \psi_1(z) f'(z+c), \tag{3.35}$$

where

$$\phi_1 = \frac{\Phi_2}{\gamma} \quad \text{and} \quad \psi_1 = K_3 \frac{\gamma'}{\gamma} - K_4. \tag{3.36}$$

Note that (3.35) is also true even when $\Phi_2(z) \equiv 0$. Actually in that case $\phi_1(z) \equiv 0$. Now we claim that $\psi_1(z) \not\equiv \beta_1(z)$. If $\psi_1(z) \equiv \beta_1(z)$ then from (3.23) and (3.36), we have

$$\frac{2}{(k+1)(k+2)}\frac{\gamma'}{\gamma} + \frac{2}{(k+1)(k+2)}\xi \equiv \frac{4}{(k+1)^2}\frac{P'}{P} - \frac{4}{k(k+1)^2}[\xi + (k+1)p],$$

i.e.,
$$2k(k+2)\frac{P'}{P} - k(k+1)\frac{\gamma'}{\gamma} - (k^2+3k+4)\frac{\beta'}{\beta} - 2(k+1)(k+2)\frac{\alpha'}{\alpha} \equiv 2(k+1)(k+2)\eta'.$$

On integration, we have

$$\beta^{k^2+3k+4} \equiv \frac{d_2 P^{2k(k+2)}}{\gamma^{k(k+1)}(\alpha e^{\eta})^{2(k+1)(k+2)}},$$

where $d_2 \in \mathbb{C} \setminus \{0\}$ and so from (3.13) and (3.19), we have $T(r,\beta) = S(r,f)$. In this case also we arrive at a contradiction from (3.8). Now from (3.35), we have

$$f^{(i)}(z+c) = \phi_{i-1}(z)f(z+c) + \psi_{i-1}(z)f'(z+c), \tag{3.37}$$

where $i \geq 2$ and $T(r,\phi_{i-1}) = S(r,f)$, $T(r,\psi_{i-1}) = S(r,f)$. Also from (3.18), (3.20) and (3.37), we have respectively

$$P(z) \;=\; -k(k+1)!\,(f'(z+c))^{k+1} + \sum_{j=1}^{k+1} T_j(z)f^j(z+c)\,(f'(z+c))^{k+1-j} \tag{3.38}$$

and
$$P'(z) \;=\; (A_1(z)\psi_1(z) + B_1(z))\,(f'(z+c))^{k+1} + \sum_{j=1}^{k+1} S_j(z)f^j(z+c)\,(f'(z+c))^{k+1-j}, \tag{3.39}$$

where $T(r,T_j) = S(r,f)$ and $T(r,S_j) = S(r,f)$. From (3.38) and (3.39), we get

$$H_0\,(f'(z+c))^{k+1} + H_1 f(z+c)\,(f'(z+c))^k + \ldots + H_{k+1}f^{k+1}(z+c) \equiv 0, \tag{3.40}$$

where

$$H_0 = P\left[A_1\psi_1 + B_1 + k(k+1)!\frac{P'}{P}\right] \tag{3.41}$$

and $H_j = PS_j - P'T_j$ for $j = 1,2,\ldots,k+1$. Note that $T(r,H_0) = S(r,f)$ and $T(r,H_j) = S(r,f)$ for $j = 1,2,\ldots,k+1$. Since $\beta_1 \not\equiv \psi_1$ and $P \not\equiv 0$, it follows from (3.28) and (3.41) that $H_0 \not\equiv 0$. Then from (3.40), we have $N_{1)}(r,0;f) = S(r,f)$ and so we arrive at a contradiction from (3.7).
Next we suppose $P(z) \equiv 0$. Then from (3.17), we have $Q(z) \equiv 0$ and so from (3.10), we get

$$G'F - \frac{\beta'}{\beta}GF - GF' \equiv 0,$$

i.e.,
$$\frac{G'}{G} \equiv \frac{\beta'}{\beta} + \frac{F'}{F}. \tag{3.42}$$

By integration, we have $G = d_3\beta F$, where d_3 is a non-zero constant. Since $n = k+1$ and $\overline{N}(r,\infty;\beta) = S(r,f)$, it follows that $\overline{N}(r,0;f) = S(r,f)$. Then from (3.7), we have $T(r,f) = S(r,f)$, which is a contradiction.
Next we suppose that β is a non-zero constant, say D. Then from (3.8), we have

$$G - DF \equiv Q_2 - DQ_1.$$

Since $n = k+1$, it follows that $\overline{N}(r,0;f) = S(r,f)$. Therefore we arrive at a contradiction from (3.7).
Case 2. Suppose $\Phi \equiv 0$. Now from (3.2), we get $F_1 \equiv G_1$, i.e., $(f^n(z+c))^{(k)} \equiv \frac{Q_2}{Q_1}f^n(z)$.
Furthermore if $Q_1 = Q_2$, then

$$f^n(z) \equiv (f^n(z+c))^{(k)}. \tag{3.43}$$

Let z_5 be a zero of $f(z)$ of multiplicity t. Since $f(z)$ and $f(z+c)$ share 0 CM, it follows that z_5 must be a zero of $f(z+c)$ of multiplicity t. Consequently z_5 will be a zero of $f^n(z)$ and $(f^n(z+c))^{(k)}$ of multiplicities nt and $nt-k$ respectively. Therefore from (3.43), we arrive at a contradiction. As a result we have $f(z) \neq 0$, $f(z+c) \neq 0$ and $(f^n(z+c))^{(k)} \neq 0$. Let $G_2(z) = f^n(z+c)$. Then $(G_2(z))^{(k)} \neq 0$.
Since $f(z)$ is a transcendental meromorphic function with finitely many poles and $f(z) \neq 0$, $f(z)$ must take the form

$$f(z) = \frac{1}{P_1(z)} e^{P_2(z)},$$

where $P_1(z)$ is a non-zero polynomial and $P_2(z)$ is a non-constant polynomial. Therefore

$$G_2(z) = \frac{1}{P_3(z)} e^{P_4(z)},$$

where $P_3(z) = P_1^n(z+c)$ and $P_4(z) = nP_2(z+c)$. Let

$$g(z) = \frac{G_2'(z)}{G_2(z)} = P_4'(z) - \frac{P_3'(z)}{P_3(z)}.$$

Therefore by Lemma 2.4, we have

$$\frac{G_2^{(k)}}{G_2} = g^k + Q_{k-1}(g),$$

where $Q_{k-1}(g)$ is a polynomial of degree $k-1$ in g and its derivatives.

If P_4' is not a constant, we see that

$$\frac{G_2^{(k)}}{G_2} \sim g^k \sim (P_4')^k \to \infty \;\; as \;\; z \to \infty.$$

We know that every non-constant rational function assumes every value in the closed complex plane. Consequently $G_2^{(k)} = 0$ somewhere in the open complex plane. Therefore we arrive at a contradiction.

Next we suppose $P_4' = \lambda \neq 0$. If $g(z)$ is non-constant, then we see that

$$g(z) = \lambda, \;\; g' = g'' = \ldots = 0$$

at ∞. Also by Lemma 2.4, we observe that $\frac{G_2^{(k)}}{G_2} = \lambda^k$ at ∞. Again $\frac{G_2^{(k)}}{G_2}$ must have a zero in the open complex plane. Consequently $g(z)$ is constant. Therefore if $P_4' \neq 0$, we must have $P_4' = \lambda = g(z)$ and so $G_2(z) = e^{\lambda z + d}$. Finally $f(z)$ assumes the form

$$f(z) = c_1 e^{\frac{\lambda}{n} z},$$

where c_1 is a non-zero constant, $e^{\lambda c} = 1$ and $\lambda^k = 1$. □

4. AN OPEN PROBLEM

Keeping other conditions intact can the sharing condition in Theorem 1.19 be relaxed to $(0,0)$ so that the conclusion remains the same?

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