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Some General Families of q-Starlike Functions Associated with the Janowski Functions

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Abstract. By making use of the concept of basic (or q-) calculus, various families of q-extensions of starlike functions, which are associated with the Janowski functions in the open unit disk $\mathbb{U}$, were introduced and studied from many different viewpoints and perspectives. In this paper, we first investigate the relationship between various known families of q-starlike functions which are associated with the Janowski functions. We then introduce and study a new subclass of q-starlike functions which involves the Janowski functions and is related with the conic domain. We also derive several properties of such families of q-starlike functions with negative coefficients including (for example) sufficient conditions, inclusion results and distortion theorems. In the last section on conclusion, we choose to point out the fact that the results for the q-analogues, which we consider in this article for $0 < q < 1$, can easily (and possibly trivially) be translated into the corresponding results for the (p, q)-analogues (with $0 < q < p \leqq 1$) by applying some obvious parametric and argument variations, the additional parameter p being redundant.

1. Introduction and Definitions

We denote by $\mathcal{H}(\mathbb{U})$ the class of functions which are analytic in the open unit disk $\mathbb{U}$ given by

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

where $\mathbb{C}$ is the set of complex numbers. Also let $\mathcal{A}$ denote a subclass of analytic functions f in $\mathcal{H}(\mathbb{U})$, satisfying the following normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

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In other words, a function $f \in \mathcal{A}$ has Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

We denote by $\mathcal{S}$ a subclass of $\mathcal{A}$, consisting of analytic functions which are also univalent in $\mathbb{U}$. Furthermore, the class of all normalized starlike functions is denoted by $\mathcal{S}^*$. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$ if it satisfies the condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

A function f is said to be subordinate to a function g, written simply as $f \prec g$, if there exists a Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$, such that

$$f(z) = g\big(w(z)\big).$$

In particular, if the function g is univalent in $\mathbb{U}$ and $f(0) = g(0)$, then $f(\mathbb{U}) \subset g(\mathbb{U})$. As a matter of fact, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1. (see [5]) A given function h with $h(0) = 1$ is said to belong to the class $\mathcal{P}[A, B]$ if and only if

$$h(z) \prec \frac{1 + Az}{1 + Bz} \qquad (-1 \leqq B < A \leqq 1).$$

The analytic function class $\mathcal{P}[A, B]$ was introduced by Janowski [5], who showed that $h(z) \in \mathcal{P}[A, B]$ if and only if there exists a function $p \in \mathcal{P}$ such that

$$h(z) = \frac{(A+1)\,p(z) - (A-1)}{(B+1)\,p(z) - (B-1)} \qquad (-1 \leqq B < A \leqq 1).$$

Remark 1. It should be noted that

$$\mathcal{P}[1, -1] =: \mathcal{P}$$

where $\mathcal{P}$ is the well-known class of Carathéodory functions.

Definition 2. A function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{S}^*[A, B]$ if and only if there exists a function $p \in \mathcal{P}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{(A+1)\,p(z) - (A-1)}{(B+1)\,p(z) - (B-1)} \qquad (-1 \leqq B < A \leqq 1). \tag{2}$$

Remark 2. It can be seen that

$$\mathcal{S}^*[1, -1] = \mathcal{S}^*,$$

where $\mathcal{S}^*$ is the well-known class of normalized starlike functions with respect to the origin.

Recently, Kanas *et al.* (see [7] and [8]; see also [6]) introduced the conic domain Ω_k $(k \geqq 0)$, which we recall here as follows:

$$\Omega_k = \left\{u + iv : u > k\ \sqrt{(u-1)^2 + v^2}\right\}. \tag{3}$$

By using the conic domain Ω_k, Kanas *et al.* (see [7] and [8]; see also [6]) also introduced and studied the class k-$\mathcal{UCV}$ of k-uniformly convex functions in $\mathbb{U}$ as well as the corresponding class k-$\mathcal{ST}$ of k-starlike functions (see, for details, [6], [7] and [8]). For fixed k, Ω_k represents the conic region bounded successively by the imaginary axis ($u = 0$). For $k = 1$, the domain Ω_k represents a parabola; for $0 < k < 1$, it represents the right branch of a hyperbola; for $k > 1$, it represents an ellipse. For these conic regions, the following functions play the rôle of extremal functions:

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z} & (k = 0) \\[2mm] 1 + \dfrac{2}{\pi^2}\left[\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2 & (k = 1) \\[2mm] 1 + \dfrac{2}{1-k^2}\,\sinh^2\left[\left(\dfrac{2}{\pi}\,\arccos k\right)\arctan\left(h\sqrt{z}\right)\right] & (0 \leqq k < 1) \\[2mm] 1 + \dfrac{1}{k^2-1}\left[1 + \sin\left(\dfrac{\pi}{2K(\kappa)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{\kappa}}} \dfrac{dt}{\sqrt{(1-t^2)(1-\kappa^2t^2)}}\right)\right] & (k > 1), \end{cases} \tag{4}$$

where

$$u(z) = \frac{z-\sqrt{\kappa}}{1-\sqrt{\kappa}z} \qquad (z \in \mathbb{U})$$

and $\kappa \in (0,1)$ is so chosen that

$$k = \cosh\left(\frac{\pi K'(\kappa)}{4K(\kappa)}\right).$$

Here $K(\kappa)$ is Legendre's complete elliptic integral of the first kind and

$$K'(\kappa) = K\left(\sqrt{1-\kappa^2}\right),$$

that is, $K'(\kappa)$ is the complementary integral of Legendre's complete elliptic integral $K(\kappa)$ of the first kind (see, for example, [20, p. 326, Eq. 9.4 (209)]).

The above-mentioned function classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ are defined as follows.

Definition 3. A function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{UCV}$ if and only if

$$\frac{\left(zf'(z)\right)'}{f'(z)} \prec p_k(z) \qquad (z \in \mathbb{U};\ k \geqq 0)$$

or, equivalently,

$$\Re\left(\frac{(zf'(z))'}{f'(z)}\right) > k\left|\frac{(zf'(z))'}{f'(z)} - 1\right| \qquad (z \in \mathbb{U};\ k \geqq 0).$$

Definition 4. A function $f \in \mathcal{A}$ is said to be in the class k-$\mathcal{ST}$ if and only if

$$\frac{zf'(z)}{f(z)} \prec p_k(z) \qquad (z \in \mathbb{U};\ k \geqq 0)$$

or, equivalently,

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \mathbb{U};\ k \geqq 0).$$

Recently, Noor *et al.* [14] combined the concept of Janowski [5] and Kanas *et al.* [6–8] and defined the following class.

Definition 5. A function $h \in \mathcal{P}$ is said to be in the class $k\text{-}\mathcal{P}[A, B]$ if and only if

$$h(z) \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1))p_k(z)-(B-1)} \qquad (-1 \leqq B < A \leqq 1;\ k \geqq 0), \tag{5}$$

where the unction $p_k(z)$ is defined by (4).

Geometrically, each function $h \in k\text{-}\mathcal{P}[A, B]$ takes on all values in the domain $\Omega_k[A, B]$ $(-1 \leqq B < A \leqq 1;\ k \geqq 0)$ which is defined as follows:

$$\Omega_k[A,B] = \left\{ w : \Re\left(\frac{(B-1)w-(A-1)}{(B+1))w-(A+1)}\right) > k\left|\frac{(B-1)w-(A-1)}{(B+1))w-(A+1)}-1\right|\right\}$$

or, equivalently, $\Omega_k[A, B]$ is a set of numbers $w = u + iv$ such that

$$\begin{aligned}&\left[\left(B^2-1\right)\left(u^2+v^2\right)-2\,(AB-1)\,u+\left(A^2-1\right)\right]^2\\ &\qquad > k\left[\left(-2(B+1))\left(u^2+v^2\right)+2\,(A+B+2)\,u-2(A+1)\right)^2+4(A-B)^2v^2\right].\end{aligned}$$

The above-defined domain $\Omega_k[A, B]$ represents a conic-type region (see, for details, [14]).

Definition 6. (see [14]) A function $f \in \mathcal{A}$ is said to be in the class $k\text{-}\mathcal{ST}[A, B]$ if and only if

$$\frac{zf'(z)}{f(z)} \in k\text{-}\mathcal{P}[A,B] \qquad (\forall\ z \in \mathbb{U};\ k \geqq 0).$$

We now recall some essential definitions and concepts of the basic or quantum (or q-) calculus, which are useful in our investigation. We suppose throughout this paper that $0 < q < 1$ and

$$\mathbb{N} := \{1,2,3,\cdots\} = \mathbb{N}_0 \setminus \{0\} \quad (\mathbb{N}_0 := \{0,1,2,\cdots\} = \mathbb{N} \cup \{0\}).$$

Definition 7. Let $q \in (0, 1)$. Then the q-number $[\lambda]_q$ is defined by

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\[2ex] \displaystyle\sum_{k=0}^{n-1} q^k = 1+q+q^2+\cdots+q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 8. Let $q \in (0, 1)$. Then the q-factorial $[n]_q!$ is defined by

$$[n]_q! = \begin{cases} 1 & (n=0) \\[2ex] \displaystyle\prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 9. (see [3] and [4]) The q-derivative (or the q-difference) of a function $f(z)$ of the form (1) is denoted by $(D_q f)(z)$ and defined in a given subset of $\mathbb{C}$ by

$$(D_q f)(z) = \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)\,z} & (z \neq 0) \\[2ex] f'(0) & (z=0). \end{cases} \tag{6}$$

In the limit case when $q \to 1-$, the q-derivative operator D_q approaches to the ordinary derivative operator. That is, we have

$$\lim_{q\to 1-} \left(D_q f\right)(z) = f'(z),$$

provided that $f'(z)$ exists.

The operator D_q provides an important tool that has been used in order to investigate the various subclasses of analytic functions of the form given in Definition 9. Whereas q-extension of the class of starlike functions was first introduced in [2] by means of the q-derivative operator D_q, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory by Srivastava (see, for details, [16]). Subsequently, a great deal of research work has been done by many researchers, which has played an important rôle in the development of Geometric Function Theory. In particular, Srivastava and Bansal [19] studied the close-to-convexity of the q-Mittag-Leffler functions. On the other hand, Srivastava *et al.* [18] investigated the Hankel determinant of a subclass of bi-univalent functions defined by using a symmetric q-derivative. Mahmood *et al.* [10] studied the class of q-starlike functions in the conic region, while Srivastava *et al.* [21] and Mahmood *et al.* [9] studied several families of q-starlike functions related with the celebrated Janowski functions (see also [11]). The upper bound of the third Hankel determinant for the class of q-starlike functions was investigated in [12]. Recently, Srivastava *et al.* [17] investigated the Hankel and Toeplitz determinants of a subclass of q-starlike functions. Wongsaijai and Sukantamala (see [25]) defined and studied certain new subfamilies of starlike functions in a systematic way. In fact, Srivastava *et al.* [22] successfully extended the work of Wongsaijai and Sukantamala [25] by introducing some general subfamilies of q-starlike functions related with the Janowski functions. Motivated by the above-mentioned developments, in this paper our aim is to present some subclasses of q-starlike functions related with a conic domain by using the idea of Srivastava *et al.* [22]. We propose here to introduce and study three (presumably new) subclasses of the class $\mathcal{S}_q^*$ of q-starlike functions associated with the Janowski functions and related with conic-like domains.

Definition 10. (see [2]) A function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{S}_q^*$ of q-starlike functions in $\mathbb{U}$ if

$$f(0) = f'(0) - 1 = 0 \tag{7}$$

and

$$\left|\frac{z}{f(z)}\left(D_q f\right)(z) - \frac{1}{1-q}\right| \leqq \frac{1}{1-q} \qquad (z \in \mathbb{U}). \tag{8}$$

We readily observe from Definition 10 that, as $q \to 1-$, the closed disk

$$\left|w - \frac{1}{1-q}\right| \leqq \frac{1}{1-q}$$

becomes the right-half complex plane and the class $\mathcal{S}_q^*$ of q-starlike functions in $\mathbb{U}$ reduces to the familiar class $\mathcal{S}^*$ of normalized starlike functions with respect to the origin ($z = 0$). Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (7) and (8) as follows (see [24]):

$$\frac{z}{f(z)}\left(D_q f\right)(z) \prec \widehat{p}(z) \qquad \left(\widehat{p}(z) := \frac{1+z}{1-qz}\right). \tag{9}$$

We now introduce three (presumably new) subclasses of the class $\mathcal{S}_q^*$ of q-starlike functions in $\mathbb{U}$, which are associated with the Janowski functions in the following way.

Definition 11. A function $f \in \mathcal{A}$ is said to belong to the class $k\text{-}\mathcal{S}^*_{(q,1)}[A,B]$ if and only if

$$\Re\left(\frac{(B-1)\dfrac{z(D_qf)(z)}{f(z)}-(A-1)}{(B+1)\dfrac{z(D_qf)(z)}{f(z)}-(A+1)}\right) \geqq k\left|\frac{(B-1)\dfrac{zD_qf(z)}{f(z)}-(A-1)}{(B+1)\dfrac{zD_qf(z)}{f(z)}-(A+1)}-1\right|.$$

We call $k\text{-}\mathcal{S}^*_{(q,1)}[A,B]$ the class of k-uniformly q-starlike functions of Type 1 associated with the Janowski functions.

Definition 12. A function $f \in \mathcal{A}$ is said to belong to the class $k\text{-}\mathcal{S}^*_{(q,2)}[A,B]$ if and only if

$$\left|\frac{(B-1)\dfrac{z(D_qf)(z)}{f(z)}-(A-1)}{(B+1)\dfrac{z(D_qf)(z)}{f(z)}-(A+1)}-\frac{1+k-kq}{(1-q)(k+1)}\right| < \frac{1}{(1-q)(k+1)}.$$

We call $k\text{-}\mathcal{S}^*_{(q,2)}[A,B]$ the class of k-uniformly q-starlike functions of Type 2 associated with the Janowski functions.

Definition 13. A function $f \in \mathcal{A}$ is said to belong to the class $k\text{-}\mathcal{S}^*_{(q,3)}[A,B]$ if and only if

$$\left|\frac{(B-1)\dfrac{z(D_qf)(z)}{f(z)}-(A-1)}{(B+1)\dfrac{z(D_qf)(z)}{f(z)}-(A+1)}-1\right| < \frac{1}{k+1}.$$

We call $k\text{-}\mathcal{S}^*_{(q,3)}[A,B]$ the class of k-uniformly q-starlike functions of Type 3 associated with the Janowski functions.

Each of the following special cases of the above-defined q-starlike function classes associated with the Janowski functions is worthy of note:

$$k\text{-}\mathcal{S}^*_{(q,1)}[A,B], \quad k\text{-}\mathcal{S}^*_{(q,2)}[A,B] \qquad \text{and} \qquad k\text{-}\mathcal{S}^*_{(q,3)}[A,B].$$

I. If we put

$$k=0, \quad A=1-2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B=-1$$

in Definition 11, we get the class $\mathcal{S}^*_{(q,1)}(\alpha)$ which was introduced and studied by Wongsaijai and Sukantamala (see [25, Definition 1]).

II. If we set

$$k=0, \quad A=1-2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B=-1$$

in Definition 12, we are led to the class $\mathcal{S}^*_{(q,2)}(\alpha)$ which was introduced and studied by Wongsaijai and Sukantamala (see [25, Definition 2]).

III. If we put

$$k = 0, \quad A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1$$

in Definition 13, we have the class $\mathcal{S}^*_{(q,3)}(\alpha)$ which was introduced and studied by Wongsaijai and Sukantamala (see [25, Definition 3]).

IV. If we set

$$k = 0, \quad A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1$$

in Definition 12, we obtain the class $\mathcal{S}^*_q(\alpha)$ which was introduced and studied by Agrawal and Sahoo [1].

V. If we put

$$k = 0, \quad A = 1 \qquad \text{and} \qquad B = -1$$

in Definition 12, we get the class $\mathcal{S}^*_q$ which was studied by Ismail *et al.* [2].

VI. In Definition 12, If we let $k = 0$, $q \to 1-$ and put $A = \lambda$ and $B = 0$, then we will arrived at the function class, studied by Ponnusamy and Singh (see [13]).

VII. One can easily see that

$$0\text{-}\mathcal{S}^*_{(q,1)}[A, B] = \mathcal{S}^*_{(q,1)}[A, B],$$

$$0\text{-}\mathcal{S}^*_{(q,2)}[A, B] = \mathcal{S}^*_{(q,2)}[A, B]$$

and

$$0\text{-}\mathcal{S}^*_{(q,3)}[A, B] = \mathcal{S}^*_{(q,3)}[A, B]$$

where $\mathcal{S}^*_{(q,1)}[A, B]$, $\mathcal{S}^*_{(q,2)}[A, B]$ and $\mathcal{S}^*_{(q,3)}[A, B]$ are the function classes which were introduced and studied by Srivastava *et al.* [22].

Geometrically, for $f \in k\text{-}\mathcal{S}^*_{(q,m)}[A, B] \quad (m = 1, 2, 3)$, the following quotient:

$$\frac{z(D_q f)(z)}{f(z)}$$

lies in the domains $\Omega_j \quad (j = 1, 2, 3)$ given by

$$\Omega_1 = \left\{ w : w \in \mathbb{C} \quad \text{and} \quad \Re(w) > \frac{A - 2k - 1}{B - 2k - 1} \right\},$$

$$\Omega_2 = \left\{ w : w \in \mathbb{C} \quad \text{and} \quad \left| w - \frac{2(k+1) + q(A - 2k - 1)}{(B - 2k - 1)q + (B + 2k + 3)} \right| < \frac{A + 1}{(B - 2k - 1)q + (B + 2k + 3)} \right\}$$

and

$$\Omega_3 = \left\{ w : w \in \mathbb{C} \quad \text{and} \quad \left| w - \frac{2(k+!)}{B + 2k + 3} \right| < \frac{A + 1}{B + 2k + 3} \right\},$$

respectively.

In this paper, many properties and characteristics, such as (for example) sufficient conditions, inclusion results and distortion theorems, are discussed systematically. We also indicate relevant connections of our results with those presented in a number of other related earlier works on this subject.

2. A Set of Main Results and Their Demonstration

We first derive the inclusion results for the generalized k-uniformly q-starlike functions

$$k\text{-}\mathcal{S}^*_{(q,1)}[A,B], \quad k\text{-}\mathcal{S}^*_{(q,2)}[A,B] \qquad \text{and} \qquad k\text{-}\mathcal{S}^*_{(q,3)}[A,B],$$

which are associated with the Janowski functions.

Theorem 1. *Suppose that $k \geqq 0$. If $-1 \leqq B < A \leqq 1$, then*

$$k\text{-}\mathcal{S}^*_{(q,3)}[A,B] \subset k\text{-}\mathcal{S}^*_{(q,2)}[A,B] \subset k\text{-}\mathcal{S}^*_{(q,1)}[A,B].$$

Proof. First of all, we suppose that $f \in k\text{-}\mathcal{S}^*_{(q,3)}[A,B]$. Then, by Definition 13, we have

$$\left| \frac{(B-1)\frac{z(D_q f)(z)}{f(z)} - (A-1)}{(B+1)\frac{z(D_q f)(z)}{f(z)} - (A+1)} - 1 \right| < \frac{1}{k+1}$$

or

$$\left| \frac{(B-1)\frac{z(D_q f)(z)}{f(z)} - (A-1)}{(B+1)\frac{z(D_q f)(z)}{f(z)} - (A+1)} - 1 \right| + \frac{q}{1-q} < \frac{1}{k+1} + \frac{q}{1-q}. \tag{10}$$

Thus, by using the triangle inequality and (10), we have

$$\left| \frac{(B-1)\frac{z(D_q f)(z)}{f(z)} - (A-1)}{(B+1)\frac{z(D_q f)(z)}{f(z)} - (A+1)} - \frac{1+k(1-q)}{1-q} \right| < \frac{1}{(1-q)(1+k)}. \tag{11}$$

The last expression in (11) now implies that $f \in k\text{-}\mathcal{S}^*_{(q,2)}[A,B]$, that is, that

$$k\text{-}\mathcal{S}^*_{(q,3)}[A,B] \subset k\text{-}\mathcal{S}^*_{(q,2)}[A,B].$$

We next let $f \in k\text{-}\mathcal{S}^*_{(q,2)}[A,B]$, so that

$$\left| \frac{(B-1)\frac{z(D_q f)(z)}{f(z)} - (A-1)}{(B+1)\frac{z(D_q f)(z)}{f(z)} - (A+1)} - \frac{1+k(1-q)}{1-q} \right| < \frac{1}{(1-q)(1+k)},$$

Now, by using some elementary concepts followed by some straightforward simplifications, we have

$$\Re\left(\frac{(B-1)\frac{z(D_q f)(z)}{f(z)} - (A-1)}{(B+1)\frac{z(D_q f)(z)}{f(z)} - (A+1)} \right) > \frac{k}{k+1}.$$

This last inequality shows that $f \in k\text{-}\mathcal{S}^*_{(q,1)}[A,B]$, that is, that

$$k\text{-}\mathcal{S}^*_{(q,2)}[A,B] \subset k\text{-}\mathcal{S}^*_{(q,1)}[A,B].$$

This completes the proof Theorem 1 □

Remark 3. If we put $k = 0$ in Theorem 1, we deduce the result which was already proved by Srivastava *et al.* [22].

As a special case of Theorem 1, if we put

$$k = 0, \qquad A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1,$$

we get the following known result given by Wongsaijai and Sukantamala (see [25]).

Corollary 1. (see [25]) *For* $0 \leqq \alpha < 1$, *it is asserted that*

$$\mathcal{S}^*_{q,3}(\alpha) \subset \mathcal{S}^*_{q,2}(\alpha) \subset \mathcal{S}^*_{q,1}(\alpha).$$

We next present remarkably simple characterizations of functions in the classes $k\text{-}\mathcal{S}^*_{(q,j)}[A,B] \quad (j = 1,2,3)$ of q-starlike functions associated with the Janowski functions.

Theorem 2. *Let* $f \in \mathcal{A}$. *Then*

1. $f \in k\text{-}\mathcal{S}^*_{(q,1)}[A,B]$ *if and only if*

$$\Re\left(\frac{f(qz)}{f(z)}\right) \leqq \frac{(B-A)+q(A-2k-1)}{B-2k-1};$$

2. $f \in k\text{-}\mathcal{S}^*_{(q,2)}[A,B]$ *if and only if*

$$\left|\frac{f(qz)}{f(z)} - \frac{\sigma}{(B-2k-1)q+(B+2k+3)}\right| \leqq \frac{(A+1)(1+q)}{(B-2k-1)q+(B+2k+3)}$$

$$\left(\sigma := (A-2k-1)q^2+(B-A+2k+2)q+B+1\right).$$

3. $f \in k\text{-}\mathcal{S}^*_{(q,3)}[A,B]$ *if and only if*

$$\left|\frac{f(qz)}{f(z)} - \frac{2(k+1)q+b+1}{B+2k+3}\right| \leqq \frac{(A+1)(1-q)}{A+2k+3}.$$

Proof. The proof of Theorem 2 can easily accomplished by using the fact that

$$\frac{z(D_qf)(z)}{f(z)} = \left(\frac{1}{1-q}\right)\left(1 - \frac{f(qz)}{f(z)}\right)$$

and by the definitions of the classes $k\text{-}\mathcal{S}^*_{(q,j)}[A,B] \quad (j = 1,2,3)$ of q-starlike functions associated with the Janowski functions. □

Finally, by means of a coefficient inequality, we give a sufficient condition for the class $k\text{-}\mathcal{S}^*_{(q,3)}[A,B]$ of generalized q-starlike functions of Type 3, which also provides a corresponding sufficient condition for the classes $k\text{-}\mathcal{S}^*_{(q,1)}[A,B]$ and $k\text{-}\mathcal{S}^*_{(q,2)}[A,B]$ of Type 1 and Type 2, respectively.

Theorem 3. *A function* $f \in \mathcal{A}$ *of the form* (1) *is in the unction class* $\mathcal{S}^*_{(q,3)}[A,B]$ *if it satisfies the following coefficient inequality*:

$$\sum_{n=2}^{\infty}\left(2q(k+1)[n-1]_q + [n]_q(B+1)+(A+1)\right)|a_n| < |B-A|. \tag{12}$$

3. Analytic Functions with Negative Coefficients

In this section, we introduce new subclasses of q-starlike functions associated with the Janowski functions, which involve negative coefficients.

Let $\mathcal{T}$ be a subset of $\mathcal{A}$ consisting of functions with negative coefficients, that is,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{13}$$

We also let

$$k\text{-}\mathcal{TS}^{*}_{(q,j)}[A,B] := k\text{-}\mathcal{S}^{*}_{(q,j)}[A,B] \cap \mathcal{T} \qquad (j = 1,2,3). \tag{14}$$

Theorem 4. *Let* $-1 \leqq B < A \leqq 1$ *and* $k \geqq 0$. *Then*

$$k\text{-}\mathcal{TS}^{*}_{(q,1)}[A,B] \equiv k\text{-}\mathcal{TS}^{*}_{(q,2)}[A,B] \equiv k\text{-}\mathcal{TS}^{*}_{(q,3)}[A,B].$$

Proof. In view of Theorem 1, it is sufficient here to show that

$$k\text{-}\mathcal{TS}^{*}_{(q,1)}[A,B] \subset k\text{-}\mathcal{TS}^{*}_{(q,3)}[A,B].$$

Indeed, if we assume that $f \in k\text{-}\mathcal{TS}^{*}_{(q,1)}[A,B]$, then we have

$$\Re\left(\frac{(B-1)\frac{z(D_q f)(z)}{f(z)} - (A-1)}{(B+1)\frac{z(D_q f)(z)}{f(z)} - (A+1)}\right) \geqq \frac{k}{k+1}.$$

or

$$\Re\left(\frac{(B-1)\frac{z(D_q f)(z)}{f(z)} - (A-1)}{(B+1)\frac{z(D_q f)(z)}{f(z)} - (A+1)} - 1\right) \geqq -\frac{1}{k+1}.$$

Thus, after a simple calculation, we find that

$$\Re\left(\frac{2\big(f(z) - z(D_q f)(z)\big)}{(B+1)\,z(D_q f)(z) - (A+1)\,f(z)}\right) \geqq -\frac{1}{k+1}$$

that is, that

$$\Re\left(\frac{2\sum\limits_{n=2}^{\infty}\big([n]_q - 1\big)|a_n| z^n}{-\left(-(B-A) - \sum\limits_{n=2}^{\infty}\big\{[n]_q(B+1) + (A+1)\big\}|a_n| z^n\right)}\right) \geqq -\frac{1}{k+1},$$

which can be rewritten as follows:

$$\Re\left(\frac{2\sum\limits_{n=2}^{\infty}\big([n]_q - 1\big)|a_n| z^n}{-(B-A) - \sum\limits_{n=2}^{\infty}\big\{[n]_q(B+1) + (A+1)\big\}|a_n| z^n}\right) \leqq \frac{1}{k+1}. \tag{15}$$

The inequality in (15) implies that

$$\frac{2\sum_{n=2}^{\infty}\left([n]_q-1\right)|a_n|\cdot|z|^n}{|B-A|-\sum_{n=2}^{\infty}[n]_q(B+1)+(A+1)|a_n|\cdot|z|^n}\leqq\frac{1}{k+1}\qquad(k\geqq 0),$$

which satisfies the condition (12).

Finally, by applying Theorem 3, the proof of Theorem 4 is completed. □

Remark 4. If we put $k=0$, in Theorem 4, we get the result which was already proved by Srivastava *et al.* [22].

In its special case when

$$k=0,\quad A=1-2\alpha\quad(0\leqq\alpha<1)\qquad\text{and}\qquad B=-1.$$

Theorem 4 reduces to the following known result.

Corollary 2. (see [25, Theorem 8]) *If* $0\leqq\alpha<1$, *then*

$$\mathcal{TS}^*_{(q,1)}(\alpha)\equiv\mathcal{TS}^*_{(q,2)}(\alpha)\equiv\mathcal{TS}^*_{(q,3)}(\alpha).$$

The assertions of Theorem 4 imply that Type 1, Type 2 and Type 3 of the general q-starlike functions associated with the Janowski functions are exactly the same. For convenience, therefore, we state the following distortion theorem by using the notation $k\text{-}\mathcal{TS}^*_{(q,j)}[A,B]$ in which it is tacitly assumed that $j=1,2,3$.

Theorem 5. *If* $f\in k\text{-}\mathcal{TS}^*_{(q,j)}[A,B]\quad(j=1,2,3)$, *then*

$$r-\left(\frac{|B-A|}{\Lambda(2,A,B,k,q)}\right)r^2\leqq\left|f(z)\right|\leqq r+\left(\frac{|B-A|}{\Lambda(2,A,B,k,q)}\right)r^2\qquad(k\geqq 0)$$

$$\Big(|z|=r\quad(0<r<1)\Big),$$

where

$$\Lambda(n,A,B,k,q)=2(k+1)\left([n]_q-1\right)+[n]_q(B+1)+(A+1)\qquad(n\in\mathbb{N}\setminus\{1\}).\tag{16}$$

Proof. We note that the following inequality follows from Theorem 3:

$$\Lambda(2,A,B,k,q)\sum_{n=2}^{\infty}|a_n|\leqq\sum_{n=2}^{\infty}\Lambda(n,A,B,k,q)|a_n|<|B-A|,$$

which yields

$$\left|f(z)\right|\leqq r+\sum_{n=2}^{\infty}|a_n|r^n\leqq r+r^2\sum_{n=2}^{\infty}|a_n|\leqq r+\frac{|B-A|}{\Lambda(2,A,B,k,q)}r^2.$$

Similarly, we have

$$\left|f(z)\right|\geqq r-\sum_{n=2}^{\infty}|a_n|r^n\geqq r-r^2\sum_{n=2}^{\infty}|a_n|\geqq r-\left(\frac{|B-A|}{\Lambda(2,A,B,k,q)}\right)r^2.$$

We have thus completed the proof of Theorem 5. □

In its special case when

$$k = 0, \quad A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1,$$

if we let $q \to 1-$, Theorem 5 reduces to the following known result.

Corollary 3. (see [15]) *If $f \in \mathcal{TS}^{*}(\alpha)$, then*

$$r - \left(\frac{1-\alpha}{2-\alpha}\right) r^2 \leqq \left|f(z)\right| \leqq r + \left(\frac{1-\alpha}{2-\alpha}\right) r^2$$

$$\Big(|z| = r \quad (0 < r < 1)\Big).$$

The following result (Theorem 6) can be proved by using arguments similar to those that are already presented in the proof of Theorem 5, so we choose to omit the details of our proof of Theorem 6.

Theorem 6. *If $f \in k\text{-}\mathcal{TS}^{*}_{(q,j)}[A,B] \quad (j = 1,2,3)$, then*

$$1 - \left(\frac{2|B-A|}{\Lambda(2,A,B,k,q)}\right) r \leqq \left|f'(z)\right| \leqq 1 + \left(\frac{2|B-A|}{\Lambda(2,A,B,k,q)}\right) r$$

$$\Big(|z| = r \quad (0 < r < 1)\Big),$$

where $\Lambda(n,A,B,k,q)$ is given by (16).

In its special case when

$$k = 0, \quad A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1,$$

if we let $q \to 1-$, Theorem 5 reduces to the following known result.

Corollary 4. (see [15]) *If $f \in \mathcal{TS}^{*}(\alpha)$, then*

$$1 - \left(\frac{2(1-\alpha)}{2-\alpha}\right) r \leqq \left|f'(z)\right| \leqq 1 + \left(\frac{2(1-\alpha)}{2-\alpha}\right) r$$

$$\Big(|z| = r \quad (0 < r < 1)\Big).$$

4. Conclusion

Here, in our present investigation, we have first defined three presumably new subclasses of q-starlike functions involving conic-like domain and associated with the celebrated Janowski function. We have then successfully investigated many properties and characteristics of each of these families of q-starlike functions, such as (for example) sufficient conditions, inclusion results and distortion theorems. For verification and validity of our results, we have also pointed out relevant connections of our results with those in several earlier related works on this subject.

We now choose to point out some obvious connection between the classical q-analysis and the so-called (p,q)-analysis. Here, in this last section on conclusion, we reiterate the fact that the results for the q-analogues, which we have considered in this article for $0 < q < 1$, can easily be translated into the corresponding results for the (p,q)-analogues (with $0 < q < p \leqq 1$) by applying some obvious parametric and argument variations, the additional parameter p being redundant. Indeed, as observed earlier by Srivastava *et al.* [23], remarkably many authors have made use of the so-called (p,q)-analysis by introducing a seemingly redundant parameter p in the classical q-analysis. The so-called (p,q)-number $[n]_{p,q}$ is given (for $0 < q < p \leqq 1$) by

$$\begin{aligned}[n]_{p,q} &:= \begin{cases} \dfrac{p^n - q^n}{p - q} & (n \in \{1, 2, 3, \cdots\}) \\ \\ 0 & (n = 0) \end{cases} \\ &=: p^{n-1}\ [n]_{\frac{q}{p}}, \end{aligned} \tag{17}$$

where, for the classical q-number $[n]_q$, we have

$$\begin{aligned}[n]_q &:= \frac{1 - q^n}{1 - q} \\ &= p^{1-n}\ \left(\frac{p^n - (pq)^n}{p - (pq)}\right) \\ &= p^{1-n}\ [n]_{p,pq}. \end{aligned} \tag{18}$$

Moreover, the so-called (p, q)-derivative or the so-called (p, q)-difference of a suitable function $f(z)$ is denoted by $\left(D_{p,q}\ f\right)(z)$ and defined, in a given subset of $\mathbb{C}$, by

$$\left(D_{p,q}\ f\right)(z) = \begin{cases} \dfrac{f\,(pz) - f\,(qz)}{(p - q)\,z} & (z \in \mathbb{C} \setminus \{0\};\ 0 < q < p \leqq 1) \\ \\ f'\,(0) & (z = 0;\ 0 < q < p \leqq 1), \end{cases} \tag{19}$$

so that, clearly, we have the following connection with the familiar q-derivative or the q-difference $\left(D_q f\right)(z)$ given by Definition 9:

$$\left(D_{p,q}\ f\right)(z) = \left(D_{\frac{q}{p}}\ f\right)(pz) \qquad \text{and} \qquad \left(D_q\ f\right)(z) = \left(D_{p,pq}\ f\right)\left(\frac{z}{p}\right) \qquad (z \in \mathbb{C};\ 0 < q < p \leqq 1). \tag{20}$$

Equations (17), (18), (19) and (20) show rather clearly that, in most cases, the q-analogues which we have considered in this article as well as in a remarkably large number of other earlier q-investigations on the subjest for $0 < q < 1$ can easily (and possibly trivially) be translated into the corresponding (p, q)-analogues (with $0 < q < p \leqq 1$) by applying some obvious parametric and argument variations of the types indicated above, the additional parameter p being redundant.

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