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Determination of a Time-Dependent Coefficient in a Wave Equation with Unusual Boundary Condition

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Abstract. In this paper, an initial boundary value problem for a wave equation with unusual boundary condition is considered. Giving an integral over-determination condition, a time-dependent potential is determined and existence and uniqueness theorem for small times is proved. We characterize the estimates of conditional stability of the solution of the inverse problem. Also, the numerical solution of the inverse problem is studied by using finite difference method.

1. Introduction

In this paper, we consider the one dimensional wave equation

$$u_{tt} = c^2 u_{xx} + a(t)u(x,t) + f(x,t), \ \ (x,t) \in D_T, \tag{1}$$

with the initial conditions

$$u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x), \ 0 \leq x \leq 1, \tag{2}$$

Neumann boundary condition

$$u_x(1,t) = 0, \ 0 \leq t \leq T, \tag{3}$$

and unusual boundary condition

$$u_{xx}(0,t) - bu_x(0,t) = 0, \ 0 \leq t \leq T, \tag{4}$$

for $b > 0$, where $D_T = \{(x,t) : \ 0 < x < 1, \ 0 \leq t \leq T\}$ for some fixed $T > 0$ and c is a constant.

This model can be used for the motion of the longitudinal vibration of a uniform elastic bar subjected to a distributed force $f(x,t)$ per unit length, where $c^2 = \frac{E}{\rho}$, E is Young's modulus and ρ is the mass of density of elastic bar. $u = u(x,t)$ represents the displacement at the instant t of the point located at x, $a(t)$ is the time dependent potential, and the function $\varphi(x)$ specifies the initial displacement, while $\psi(x)$ specifies its initial velocity. The boundary condition (3) means that right end of uniform elastic bar is free. On the contrary of

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the common boundary conditions, the boundary condition (4) contains the term of maximal order $u_{xx}(0,t)$ which is called unusual (non-classical) boundary condition. This boundary condition arises if the left end of bar is restrained with a rotational spring and b is the rotational stiffness coefficient. In this paper, we take $c = 1$ for simplicity.

For a given function $a(t)$, $0 \le t \le T$ the problem (1) - (4) for the unknown function $u(x,t)$ is called direct (forward) problem. Direct problems for the wave equation with various boundary conditions are satisfactorily investigated in [3], [5], [14], [22] and [29]. It is important to note that the papers [2], [19], [20], [21] which investigate the solution of direct problem for the wave equation with non-local integral boundary condition. For the some numerical aspects of initial and initial-boundary value problems of the hyperbolic equations is considered for direct problem in [4].

If $a(t)$, $0 \le t \le T$ is unknown, finding the pair of solution $\{a(t), u(x,t)\}$ of the problem (1)-(4) with the additional condition

$$\int_0^1 u(x,t)dx = h(t), \; 0 \le t \le T. \tag{5}$$

is called inverse problem.

Many physical models include unknown coefficients (i.e. potential, source) in the wave equation and the solution of the inverse problems for the identification of these coefficients has become a very popular area of research in recent years. The inverse problems for the wave equation with different boundary conditions and space dependent coefficients are considered in [9], [17], [18] [23] and more recently in [10], [30] . The inverse problem for the wave equation with time dependent coefficient with integral condition is investigated in [15] and with non-classical boundary condition is studied in [1]. The time-dependent source function of a time-fractional wave equation with integral condition in a bounded domain is determined in [26].

On contrary to the inverse initial boundary value problem for the equation (1) with time-dependent potential case, the finite difference method for the inverse problem for finding space-dependent potential, or space dependent damping coefficient, or force function is well-known. The work [6] considers the inverse problem for the wave equation which consists in determining an unknown time-dependent force function by applying finite difference method. Same method is used for an unknown space-dependent force function acting on a vibrating structure in the wave equation from Cauchy boundary data in [7]. In [8], inverse problem of finding space-dependent potential or damping coefficients in the wave equation is considered.

In present paper, we consider an initial boundary value problem for a wave equation with unusual boundary condition. Giving an integral over-determination condition,we determine the time-dependent potential and prove the existence and uniqueness theorem for small T, and we characterize the estimations of conditional stability of the solution of the inverse problem. Also, we use the finite difference method to the inverse initial boundary value problem for the equation (1) with time-dependent potential.

The article is organized as following: In Section 2, we present auxiliary spectral problem of this problem and its properties. In Section 3, the series expansion method in terms of eigenfunctions converts the inverse problem to a fixed point problem in a suitable Banach space. Under some consistency, regularity conditions on initial and boundary data the existence and uniqueness of the inverse problem is shown by the way that the fixed point problem has unique solution for small T. In this section, we also give the theorem for continuous dependence upon the data in a certain class of data. In section 4, the inverse problem of finding time-dependent potential is studied by using the finite difference method.

2. Auxiliary Spectral Problem

We attempt to apply the Fourier method of eigenfunction expansion to the problem (1)-(5). Auxiliary spectral problem of the problem (1)-(4) is

$$\begin{cases} X''(x) + \lambda X(x) = 0, \ 0 \leq x \leq 1, \\ X'(1) = 0, \ bX'(0) + \lambda X(0) = 0. \end{cases} \tag{6}$$

The problem (6) is considered in [12] and has eigenfunctions

$$X_n(x) = \sqrt{2}\cos\sqrt{\lambda_n}(1-x), n = 0, 1, 2, ... \tag{7}$$

with positive eigenvalues λ_n determined from the equation

$$\tan\sqrt{\lambda} = \frac{-\sqrt{\lambda}}{b}.$$

The zero index is assigned to an arbitrary eigenfunction and all remaining eigenfunctions are numbered increasing order of eigenvalues. This characteristic equation has no roots outside the positive part of the real line on the complex plane. The asymptotic formula for the eigenvalues has the form

$$\sqrt{\lambda_n} = \mu_n + \frac{b}{\mu_n} + O(\frac{1}{n^2})$$

for sufficiently large n and $\mu_n = \frac{(2n-1)}{2}\pi$.

The system $X_n(x)$, $n = 1, 2, ...$ is bi-orthogonal to the system

$$Y_n(x) = \frac{2}{1 + \frac{\cos^2\sqrt{\lambda_n}}{b}}\left[\cos\sqrt{\lambda_n}(1-x) - \frac{\cos\sqrt{\lambda_n}}{\cos\sqrt{\lambda_0}}\cos\sqrt{\lambda_0}(1-x)\right], \ n = 1, 2, ...$$

and the system $X_n(x)$, $n = 1, 2, ...$ forms a Riesz basis in $L_2[0, 1]$. Also, the system $Y_n(x)$, $n = 1, 2, ...$ is a Riesz basis in $L_2[0, 1]$ and is complete.

The pair $\{a(t), u(x, t)\}$ from the class $C[0, T] \times C^2(\overline{D}_T)$ for which the conditions (1)-(5) are satisfied, is called a classical solution of the inverse problem (1)-(5). Since we are seeking the classical solution of the inverse problem (1)-(5), the uniformly convergence of the Fourier series expansion in the system $X_n(x)$, $n = 1, 2, ...$ is important.

Lemma 2.1 (Corollary 1, [12]). *Let the function $g(x) \in C[0, 1]$ and*

$$g(0) + \frac{b}{\cos\sqrt{\lambda_0}}\int_0^1 g(x)\cos\sqrt{\lambda_0}(1-x)dx = 0$$

is satisfied. Then this function can be expanded in a Fourier series in the system $X_n(x)$, $n = 1, 2, ...$ and this expansion is uniformly convergent on $[0, 1]$.

Let us introduce the functional space

$$B^{3/2}_{2,T} = \left\{u(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n(x) : \ u_n(t) \in C[0,T], J_T(u) = \left[\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\max_{0\leq t\leq T}|u_n(t)|\right)^2\right]^{1/2} < +\infty\right\}$$

with the norm $\|u(x,t)\|_{B^{3/2}_{2,T}} \equiv J_T(u)$ which relates the Fourier coefficients of the function $u(x, t)$ by the eigenfunctions $X_n(x)$, $n = 1, 2, ...$ It is shown in [13] that $B^{3/2}_{2,T}$ is Banach space. Obviously $E^{3/2}_T = C[0, T] \times B^{3/2}_{2,T}$ with the norm $\|z\|_{E^{3/2}_T} = \|a(t)\|_{C[0,T]} + \|u(x, t)\|_{B^{3/2}_{2,T}}$ is also Banach space, where $z = \{a(t), u(x, t)\}$.

3. Solution of the Inverse Problem

In this section, we will examine the existence and uniqueness of the solution of the inverse problem (1)-(5) with time-dependent potential and conditional stability of the solution of this inverse problem.

Definition 3.1. *The pair* $\{a(t), u(x,t)\}$ *from the class* $C[0,T] \times C^{2,2}(\overline{D}_T)$ *for which the conditions (1)-(5) are satisfied is called the classical solution of the inverse problem (1)-(5).*

Since the function $a(t)$ is solely time dependent, seeking the solution of the problem (1)-(5) in the following form is suitable:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x) \tag{8}$$

where $u_n(t) = \int_0^1 u(x,t) Y_n(x) dx$, $n = 1, 2, ...$

From the equation (1) and initial condition (2), we obtain

$$\begin{cases} u_n''(t) + \lambda_n u_n(t) = F_n(t; a, u), \\ u_n(0) = \varphi_n, \;\; u_n'(0) = \psi_n, \end{cases} ,n = 1, 2, .. \tag{9}$$

where $F_n(t; a, u) = a(t)u_n(t) + f_n(t)$, $f_n(t) = \int_0^1 f(x,t) Y_n(x) dx$, $\varphi_n = \int_0^1 \varphi(x) Y_n(x) dx$, $\psi_n = \int_0^1 \psi(x) Y_n(x) dx$, $n = 1, 2, ...$

Solving the problem (9), we get

$$u_n(t) = \varphi_n \cos\sqrt{\lambda_n} t + \frac{1}{\sqrt{\lambda_n}} \psi_n \sin\sqrt{\lambda_n} t + \frac{1}{\sqrt{\lambda_n}} \int_0^t F_n(\tau; a, u) \sin\sqrt{\lambda_n}(t - \tau) d\tau. \tag{10}$$

Integrating the equation (1) from 0 to 1 with respect to x and using the over-determination condition (5) and the equality (10), we obtain the first component of the pair $\{a(t), u(x,t)\}$ as

$$a(t) = \frac{1}{h(t)} \Bigg[h''(t) - \int_0^1 f(x,t) dx + \sqrt{2} \sum_{n=1}^{\infty} \sqrt{\lambda_n} \Big(\varphi_n \cos\sqrt{\lambda_n} t$$
$$+ \frac{1}{\sqrt{\lambda_n}} \psi_n \sin\sqrt{\lambda_n} t + \frac{1}{\sqrt{\lambda_n}} \int_0^t F_n(\tau; a, u) \sin\sqrt{\lambda_n}(t - \tau) d\tau \Big) \sin\sqrt{\lambda_n} \Bigg]. \tag{11}$$

Substituting (10) into (8), the second component of the pair is

$$u(x,t) = \sum_{n=1}^{\infty} \left[\varphi_n \cos\sqrt{\lambda_n} t + \frac{1}{\sqrt{\lambda_n}} \psi_n \sin\sqrt{\lambda_n} t + \frac{1}{\sqrt{\lambda_n}} \int_0^t F_n(\tau; a, u) \sin\sqrt{\lambda_n}(t - \tau) d\tau \right] \sqrt{2} \cos\sqrt{\lambda_n}(1 - x) \tag{12}$$

Thus, the solution of the inverse problem (1)-(5) is reduced to the solution of system (11)-(12) with respect to the unknown functions $\{a(t), u(x,t)\}$.

Let us denote $z = [a(t), u(x,t)]^T$ and consider the operator equation

$$z = \Phi(z). \tag{13}$$

The operator Φ is determined in the set of functions z and has the form $[\phi_1, \phi_2]^T$, where

$$\phi_1(z) = \frac{1}{h(t)}\left[h''(t) - \int_0^1 f(x,t)dx + \sqrt{2}\sum_{n=1}^{\infty}\sqrt{\lambda_n}\Big(\varphi_n \cos\sqrt{\lambda_n}t + \frac{1}{\sqrt{\lambda_n}}\psi_n \sin\sqrt{\lambda_n}t + \frac{1}{\sqrt{\lambda_n}}\int_0^t F_n(\tau;a,u)\sin\sqrt{\lambda_n}(t-\tau)d\tau\Big)\sin\sqrt{\lambda_n}\right]. \quad (14)$$

$$\phi_2(z) = \sum_{n=1}^{\infty}\left[\varphi_n \cos\sqrt{\lambda_n}t + \frac{1}{\sqrt{\lambda_n}}\psi_n \sin\sqrt{\lambda_n}t + \frac{1}{\sqrt{\lambda_n}}\int_0^t F_n(\tau;a,u)\sin\sqrt{\lambda_n}(t-\tau)d\tau\right]\sqrt{2}\cos\sqrt{\lambda_n}(1-x) \quad (15)$$

Let us show that Φ maps $E_T^{3/2}$ onto itself continuously. In other words, we need to show $\phi_1(z) \in C[0,T]$ and $\phi_2(z) \in B_{2,T}^{3/2}$ for arbitrary $z = [a(t), u(x,t)]^T$ with $a(t) \in C[0,T]$, $u(x,t) \in B_{2,T}^{3/2}$.

We will use the following assumptions on the data of problem (1)-(5):

(A_1) $\varphi(x) \in C^3[0,1]$, $\varphi(0) + \frac{b}{\cos\sqrt{\lambda_0}}\int_0^1 \varphi(x)\cos\sqrt{\lambda_0}(1-x)dx = 0$, $\varphi'(0) = \varphi''(0) = 0$, $\varphi'(1) = 0$,

(A_2) $\psi(x) \in C^2[0,1]$, $\psi(0) + \frac{b}{\cos\sqrt{\lambda_0}}\int_0^1 \psi(x)\cos\sqrt{\lambda_0}(1-x)dx = 0$, $\psi'(0) = \psi'(1) = 0$,

(A_3) $h(t) \in C^2[0,T]$, $h(t) \neq 0, \forall t \in [0,T]$, $h(0) = \int_0^1 \varphi(x)dx$, $h'(0) = \int_0^1 \psi(x)dx$,

(A_4) $f(x,t) \in C(\overline{D}_T)$, $f_x, f_{xx} \in C[0,1], \forall t \in [0,T]$, $f_x(0,t) = f_x(1,t)$, $f(0,t) + \frac{b}{\cos\sqrt{\lambda_0}}\int_0^1 f(x,t)\cos\sqrt{\lambda_0}(1-x)dx = 0$.

By using integration by parts under the assumptions (A_1)-(A_4), it easy to see that

$$\begin{aligned}
\varphi_n &= \frac{1}{\lambda_n^{3/2}}\frac{-\sqrt{2}}{1+\frac{\cos^2\sqrt{\lambda_n}}{b}}\int_0^1 \varphi'''(x)\sin\sqrt{\lambda_n}(1-x)dx,\\
\psi_n &= \frac{1}{\lambda_n}\frac{-\sqrt{2}}{1+\frac{\cos^2\sqrt{\lambda_n}}{b}}\int_0^1 \varphi''(x)\cos\sqrt{\lambda_n}(1-x)dx,\\
f_n(t) &= \frac{1}{\lambda_n}\frac{-\sqrt{2}}{1+\frac{\cos^2\sqrt{\lambda_n}}{b}}\int_0^1 f_{xx}(x,t)\cos\sqrt{\lambda_n}(1-x)dx.
\end{aligned}$$

From these equalities, we have

$$\begin{aligned}
&\textstyle\sum_{n=1}^{\infty}\sqrt{\lambda_n}\left|\varphi_n\right| \le C_1\left\|\varphi'''(x)\right\|_{L_2[0,1]},\\
&\textstyle\sum_{n=1}^{\infty}\left|\psi_n\right| \le C_1\left\|\psi''(x)\right\|_{L_2[0,1]},\\
&\textstyle\sum_{n=1}^{\infty}\left|f_n(t)\right| \le C_1\left\|f_{xx}(x,t)\right\|_{L_2(D_T)},
\end{aligned} \quad (16)$$

by using Cauchy-Schwartz inequality and Bessel inequality where $C_1 = \left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\right)^{1/2}$.

First, let us show that $\phi_1(z) \in C[0,T]$. Under the assumptions (A$_1$)-(A$_4$) and considering the estimates (16), we obtain from (14)

$$\max_{0\le t\le T}\left|\phi_1(z)\right| \le R_1(T) + R_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^{3/2}_{2,T}} \tag{17}$$

where $R_1(T) = \frac{1}{\|h(t)\|_{C[0,T]}}(\|h''(t)\|_{C[0,T]} + \left\|f_{int}(t)\right\|_{C[0,T]} + 2\sqrt{2}C_1(\left\|\varphi'''(x)\right\|_{L_2[0,1]} + \left\|\psi''(x)\right\|_{L_2[0,1]} + T\left\|f_{xx}(x,t)\right\|_{L_2(D_T)}))$, $R_2(T) = \frac{2\sqrt{2}C_1C_2T}{\|h(t)\|_{C[0,T]}}$, $f_{int}(t) = \int_0^1 f(x,t)dx$, and $C_2 = \left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}\right)^{1/2}$. Since the right hand side is bounded, $\phi_1(z) \in C[0,T]$.

Now, let us show that $\phi_2(z) \in B^{3/2}_{2,T}$, i.e. we need to verify that

$$J_T(\phi_2) = \left[\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\max_{0\le t\le T}\left|\phi_{2n}(t)\right|\right)^2\right]^{1/2} < +\infty,$$

where

$$\phi_{2n}(t) = \varphi_n\cos\sqrt{\lambda_n}t + \frac{1}{\sqrt{\lambda_n}}\psi_n\sin\sqrt{\lambda_n}t + \frac{1}{\sqrt{\lambda_n}}\int_0^t F_n(\tau;a,u)\sin\sqrt{\lambda_n}(t-\tau)d\tau.$$

After some manipulations on the last equality under the assumptions (A$_1$)-(A$_4$), we get

$$\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\max_{0\le t\le T}\left|\phi_{2n}(t)\right|\right)^2 \le \widetilde{R}_1(T) + \widetilde{R}_2(T)\left(\max_{0\le t\le T}|a(t)|\right)^2\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\max_{0\le t\le T}|u_n(t)|\right)^2 \tag{18}$$

where $\widetilde{R}_2(T) = 4T^2$,and $\widetilde{R}_1(T) = 4\sum_{n=1}^{\infty}|\alpha_n|^2 + 4\sum_{n=1}^{\infty}\left|\beta_n\right|^2 + 4T^2\sum_{n=1}^{\infty}\left(\max_{0\le t\le T}\left|\eta_n(t)\right|\right)^2$ with

$$\begin{aligned}
\alpha_n &= \frac{-\sqrt{2}}{1+\frac{\cos^2\sqrt{\lambda_n}}{b}}\int_0^1\varphi'''(x)\sin\sqrt{\lambda_n}(1-x)dx,\\
\beta_n &= \frac{-\sqrt{2}}{1+\frac{\cos^2\sqrt{\lambda_n}}{b}}\int_0^1\varphi''(x)\cos\sqrt{\lambda_n}(1-x)dx,\\
\eta_n(t) &= \frac{-\sqrt{2}}{1+\frac{\cos^2\sqrt{\lambda_n}}{b}}\int_0^1 f_{xx}(x,t)\cos\sqrt{\lambda_n}(1-x)dx.
\end{aligned}$$

From the Bessel inequality and $\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\max_{0\le t\le T}|u_n(t)|\right)^2 < +\infty$, series on the right side of (18) are convergent. Thus $J_T(\phi_2) < +\infty$ and ϕ_2 is belongs to the space $B^{3/2}_{2,T}$.

Now, let z_1 and z_2 be any two elements of $E_T^{3/2}$. We know that $\|\Phi(z_1)-\Phi(z_2)\|_{E_T^{3/2}} = \left\|\phi_1(z_1)-\phi_1(z_2)\right\|_{C[0,T]} + \left\|\phi_2(z_1)-\phi_2(z_2)\right\|_{B^{3/2}_{2,T}}$. Here $z_i = [a^i(t),u^i(x,t)]^T$, $i=1,2$.

Under the assumptions (A$_1$)-(A$_4$) and considering (17)-(18), we obtain

$$\|\Phi(z_1)-\Phi(z_2)\|_{E_T^{3/2}} \le A(T)C(a^1,u^2)\,\|z_1-z_2\|_{E_T^{3/2}}$$

where $A(T) = 2T\left(1+\frac{\sqrt{2}C_1C_2}{\|h(t)\|_{C[0,T]}}\right)$ and $C(a^1,u^2)$ is the constant includes the norms of $\left\|a^1(t)\right\|_{C[0,T]}$ and $\left\|u^2(x,t)\right\|_{B^{3/2}_{2,T}}$.

For sufficiently small T, $0 < A(T) < 1$. This implies that the operator Φ is contraction mapping which maps $E_T^{3/2}$ onto itself continuously. Then according to Banach fixed point theorem there exists unique solution of (13).

Thus, we proved the following theorem:

Theorem 3.2 (Existence and uniqueness). *Let the assumptions (A_1)-(A_4) be satisfied. Then, the inverse problem (1)-(5) has a unique solution for small T.*

Now, let us investigate the stability of the solution of the inverse problem. Because of the presence of the term $a(t)u(x,t)$ in the equation (1), finding the pair of solution $\{a(t), u(x,t)\}$ of the inverse problem (1)-(5) is non-linear. Therefore we can not apply the standard stability criteria but we can characterize the estimation of conditional stability. Thus we can obtain a stability estimate under a priori assumption on the smallness of $a(t)$. This type of stability results are studied by V.G. Romanov in [23] and more recently in [11], [24], [25], [27], [28].

Such an estimate can be obtained by setting a certain class of data $\Im(\alpha, N_0, N_1, N_2, N_3)$ for the functions $\varphi(x), \psi(x), h(t), f(x,t)$ and a class $\aleph(M_0)$ for the function $a(t)$ if they satisfy

$$\begin{aligned} \left\|f\right\|_{C(\overline{D}_T)} &\leq N_0,\ \left\|\varphi\right\|_{C^3[0,1]} \leq N_1,\ \left\|\psi\right\|_{C^2[0,1]} \leq N_2, \\ \|h\|_{C^2[0,T]} &\leq N_3,\ 0 < \alpha \leq |h(t)|, \end{aligned}$$

and

$$\|a(t)\|_{C[0,T]} \leq M_0,$$

respectively.

It is easy to seen that, since $\varphi(x), \psi(x), h(t), f(x,t) \in \Im(\alpha, N_0, N_1, N_2, N_3)$ and $a(t) \in \aleph(M_0)$,

$$\|u(x,t)\|_{B^{3/2}_{2,T}} \leq M_1$$

where $M_1 = \frac{4}{1-4T^2C_2M_0}(T^2N_0 + N_1 + N_2)$.

Let $\{a(t), u(x,t)\}$ and $\{\overline{a}(t), \overline{u}(x,t)\}$ be the solutions of (1)-(5) corresponding to data $\varphi(x), \psi(x), h(t), f(x,t)$ and $\overline{\varphi}(x), \overline{\psi}(x), \overline{h}(t), \overline{f}(x,t)$, respectively. Then, we obtain from (11) and (12)

$$\begin{aligned} a(t) - \overline{a}(t) = \tfrac{1}{h(t)\overline{h}(t)} \Big\{ \overline{h}(t) \Big[h''(t) - \textstyle\int_0^1 f(x,t)dx + \sqrt{2}\sum_{k=1}^{\infty} \sqrt{\lambda_n} \Big(\varphi_n \cos\sqrt{\lambda_n}t \\ + \tfrac{1}{\sqrt{\lambda_n}}\psi_n \sin\sqrt{\lambda_n}t + \tfrac{1}{\sqrt{\lambda_n}} \textstyle\int_0^t F_n(\tau; a, u) \sin\sqrt{\lambda_n}(t-\tau)d\tau \Big) \sin\sqrt{\lambda_n} \Big] \\ -h(t) \Big[\overline{h}''(t) - \textstyle\int_0^1 \overline{f}(x,t)dx + \sqrt{2}\sum_{k=1}^{\infty} \sqrt{\lambda_n} \Big(\overline{\varphi}_n \cos\sqrt{\lambda_n}t \\ + \tfrac{1}{\sqrt{\lambda_n}}\overline{\psi}_n \sin\sqrt{\lambda_n}t + \tfrac{1}{\sqrt{\lambda_n}} \textstyle\int_0^t F_n(\tau; \overline{a}, \overline{u}) \sin\sqrt{\lambda_n}(t-\tau)d\tau \Big) \sin\sqrt{\lambda_n} \Big] \Big\} \end{aligned} \tag{19}$$

and

$$\begin{aligned} u(x,t) - \overline{u}(x,t) = \textstyle\sum_{n=1}^{\infty} \Big[\varphi_n \cos\sqrt{\lambda_n}t + \tfrac{1}{\sqrt{\lambda_n}}\psi_n \sin\sqrt{\lambda_n}t \\ + \tfrac{1}{\sqrt{\lambda_n}} \textstyle\int_0^t F_n(\tau; a, u) \sin\sqrt{\lambda_n}(t-\tau)d\tau \Big] \sqrt{2}\cos\sqrt{\lambda_n}(1-x) \\ - \textstyle\sum_{n=1}^{\infty} \Big[\overline{\varphi}_n \cos\sqrt{\lambda_n}t + \tfrac{1}{\sqrt{\lambda_n}}\overline{\psi}_n \sin\sqrt{\lambda_n}t \\ + \tfrac{1}{\sqrt{\lambda_n}} \textstyle\int_0^t F_n(\tau; \overline{a}, \overline{u}) \sin\sqrt{\lambda_n}(t-\tau)d\tau \Big] \sqrt{2}\cos\sqrt{\lambda_n}(1-x) \end{aligned} \tag{20}$$

where $F_n(t;\overline{a},\overline{u}) = \overline{a}(t)\overline{u}_n(t) + \overline{f}_n(t)$, $\overline{f}_n(t) = \int_0^1 \overline{f}(x,t)Y_n(x)dx$, $\overline{\varphi}_n = \int_0^1 \overline{\varphi}(x)Y_n(x)dx$, $\overline{\psi}_n = \int_0^1 \overline{\psi}(x)Y_n(x)dx$, $n = 1,2,....$

Denote the difference between two functions with the tilde ($\sim$), i.e. $\widetilde{a} = a - \overline{a}$, $\widetilde{u} = u - \overline{u}$, etc. Then,under the conditions(A_1)-(A_4) by using the estimates given above we obtain from (19) and (20)

$$\left\|\widetilde{a}(t)\right\|_{C[0,T]} \leq \frac{D_1}{\Delta(T)}\left\{\left\|\widetilde{h}\right\|_{C^2[0,T]} + \left\|\widetilde{\varphi}\right\|_{C^3[0,1]} + \left\|\widetilde{\psi}\right\|_{C^2[0,1]} + \left\|\widetilde{f}\right\|_{C(\overline{D}_T)}\right\} \tag{21}$$

where $D_1 = \max\left\{\frac{d_4}{\alpha^2}(2N_3 + N_0 + 2\sqrt{2}C_1(N_0 + N_1 + N_2) + 2TM_0M_1),\right.$
$\left.\frac{d_4}{\alpha^2}(1 + 2\sqrt{2}C_1N_3) - 4T^2d_2, \frac{d_4}{\alpha^2}2\sqrt{2}C_1N_3 - 4d_2\right\}$, $\Delta(T) = d_1d_4 - d_2d_3 \neq 0$, $d_1 = 1 - \frac{2\sqrt{2}}{\alpha^2}TC_1N_3M_0$, $d_2 = \frac{2\sqrt{2}}{\alpha^2}TC_1C_2N_3M_1$, $d_3 = 4T^2C_2M_0$, $d_4 = 1 - 4T^2C_2M_1$.

Similarly, we get the estimate

$$\left\|\widetilde{u}(x,t)\right\|_{B^{3/2}_{2,T}} \leq \frac{D_2}{\Delta(T)}\left\{\left\|\widetilde{h}\right\|_{C^2[0,T]} + \left\|\widetilde{\varphi}\right\|_{C^3[0,1]} + \left\|\widetilde{\psi}\right\|_{C^2[0,1]} + \left\|\widetilde{f}\right\|_{C(\overline{D}_T)}\right\} \tag{22}$$

where D_2 is dependent only the parameters $\alpha, N_0, N_1, N_2, N_3, M_0$ and M_1.

Theorem 3.3 (continuous dependence upon the data). *Let $\{a(t), u(x,t)\}$ and $\{\overline{a}(t), \overline{u}(x,t)\}$ be two solutions of the inverse problem (1)-(5) with the data $\varphi(x), \psi(x), h(t), f(x,t)$ and $\overline{\varphi}(x), \overline{\psi}(x), \overline{h}(t), \overline{f}(x,t)$, respectively, which are satisfied the conditions of the Theorem 3.2. Then the estimates (21)-(22)are true for small T. The constants D_1 and D_2 depend only on the choice of the classes $\mathfrak{I}(\alpha, N_0, N_1, N_2, N_3)$ and $\aleph(M_0)$.*

4. Numerical Method and Examples

In this section,we describe the numerical method applied to the inverse initial boundary value problem (1)-(5).

The discrete form of our problem is as follows: We divide the domain $(0,1) \times (0,T)$ into nx and nt subintervals of equal length hx and ht, where $hx = 1/nx$ and $ht = T/nt$, respectively. We denote by $U_j^n := U(x_j, t_n)$, $a^n := a(t_n)$ and $f_j^n := f(x_j, t_n)$, where $x_j = jhx$, $t_n = nht$ for $j = 0,...,nx$, $n = 0,...,nt$. Then, a central difference approximation to the equations (1)-(4) at the mesh points (x_j, t_n) is

$$U_j^{n+1} = r^2U_{j+1}^n + 2(1-r^2)U_j^n + r^2U_{j-1}^n - U_j^{n-1} + (ht)^2(a^nU_j^n + f_j^n), \tag{23}$$

$$j = 1,...,nx-1,\ n = 1,...,nt-1,$$

$$U_j^0 = \varphi_j,\ \ j = 0,...,nx,\ \frac{U_j^1 - U_j^{-1}}{2ht} = \psi_j,\ j = 1,...,nx-1, \tag{24}$$

$$U_2^n - (2 + bhx)U_1^n + (1 + bhx)U_0^n = 0,\ \frac{U_{nx}^n - U_{nx-1}^n}{hx} = 0,\ n = 0,...,nt, \tag{25}$$

where $r = \frac{ht^2}{hx^2}$. Equation (23) represents an explicit finite difference method which is stable for $r \leq 1$. Putting $n = 0$ in the equation (23) and using (24), we obtain

$$U_j^1 = \frac{1}{2}(r^2\varphi_{j+1} + 2(1-r^2)\varphi_j + r^2\varphi_{j-1} + 2ht\psi_j + (ht)^2(a^0\varphi_j + f_j^0)), \tag{26}$$

$$j = 1,...,nx-1.$$

Consider (5) in the equation (1), we obtain

$$a(t) = \frac{h''(t) + u_x(0,t) - f_{int}(t)}{h(t)}$$

where $f_{int}(t) = \int_0^1 f(x,t)dx$ and for the over-determination condition (5) we use trapezoidal rule approximation.

After discretizing last equation, we have

$$a^n = \frac{(h^{n+1} - 2h^n + h^{n-1})/(ht)^2 + (U_1^n - U_0^n)/hx - f_{int}^n}{h^n}, \tag{27}$$

$n = 1, ..., nt - 1$

$$a^{nt} = \frac{(h^{nt} - 2h^{nt-1} + h^{nt-2})/(ht)^2 + (U_1^{nt} - U_0^{nt})/hx - f_{int}^{nt}}{h^{nt}}, \tag{28}$$

$$a^0 = \frac{(h^2 - 2h^1 + h^0)/(ht)^2 - (U_1^0 - U_0^0)/(hx)^2 - f_{int}}{h^0}. \tag{29}$$

Now let us consider (27)-(29) in (23), we obtain the system with respect to U_j^n , $j = 0, ..., nx$, $n = 0, ..., nt$ which can be solved explicitly. Then using the calculated values of U_j^n in (27)-(29), we obtain the values of a^n , $n = 0, ..., nt$.

Example 4.1. *Consider the inverse IBVP (1)-(5) with the input data*

$$\begin{aligned} f(x,t) &= (1 + 2\pi x - \sin(2\pi x) - 8\pi^2 \sin(2\pi x))\exp(t) - (1 + 2\pi x - \sin(2\pi x)), \\ \varphi(x) &= (1 + 2\pi x - \sin(2\pi x)),\ \psi(x) = (1 + 2\pi x - \sin(2\pi x)),\ h(t) = (1+\pi)exp(t), \\ b &= 1,\ x \in [0,1],\ t \in [0,1]. \end{aligned}$$

According to the Theorem 3.2, the solution of the inverse problem exist and is unique. In fact, it can easily be checked by direct substitution that the analytical solution is given by

$$\{a(t), u(x,t)\} = \{1/\exp(t), (1 + 2\pi x - \sin(2\pi x))\exp(t)\}.$$

The direct and inverse numerical solutions for $u(x,t)$ at the interior points are shown in Figure 1 for $nx = 200$, $nt = 200$, and also, the absolute error between them is included. One can notice that an excellent agreement is obtained. Figure 2 shows the inverse numerical solution in comparison with the exact $a(t)$.

Example 4.2. *Consider the inverse IBVP (1)-(5) with the input data*

$$\begin{aligned} f(x,t) &= (x^2 - 2x)(exp(t) - 1) - 2\exp(t), \\ \varphi(x) &= (x^2 - 2x),\ \psi(x) = (x^2 - 2x),\ h(t) = -\frac{2}{3}exp(t), \\ b &= -1,\ x \in [0,1],\ t \in [0,1]. \end{aligned}$$

One can easily check that the input data does not satisfy the conditions $(A_1) - (A_4)$. As the conditions of Theorem 3.2 are not satisfied we can not conclude the unique solvability of the inverse problem. However, the solution at least exists and given by

$$\{a(t), u(x,t)\} = \{1/\exp(t), (x^2 - 2x)\exp(t)\}$$

which can easily check by direct substitution. Figure 3 shows the exact and numerical solution of $\{a(t), u(x,t)\}$ for $nt = 100$ and $nx = 100$. Next, we investigate the stability of numerical solution with respect to the noisy over-determination data (5), denoted by the function $h_\gamma(t) = h(t)(1 + \gamma\theta)$ where γ is the percentage of noise and θ are *random variables generated from a uniform distribution in the interval* $[-0.5, 0.5]$ *which are generated using rand command in MATLAB. Figs. 4, 5 show the exact and numerical solutions of* $\{a(t), u(nx/2, t)\}$ *when the input data (5) is contaminated by* $\gamma = 1\%$ *and* 5% *noise. Figs. 6, 7 show the exact and numerical solutions of* $\{a(t), u(nx/2, t)\}$ *obtained after mollification, when the input data (5) is contaminated by* $\gamma = 1\%$ *and* 5% *noise. This mollification procedure has been performed using MATLAB version of the computational program supplied by D. A. Murio in [16]. From these figures it can be seen that the application of the mollification to stabilize the noisy function* $h_\gamma(t)$*, produce stable numerical solutions for*$\{a(t), u(nx/2, t)\}$*.*

5. Conclusion

The inverse problems for linear wave equations with unusual boundary conditions connected with recovery of the coefficient are scarce. The paper considers the of inverse problem of recovering a time-dependent potential in an initial boundary value problem for a wave equation. The series expansion method in terms of eigenfunction of a Sturm-Liouville problem converts the considered inverse problem to a fixed point problem in a suitable Banach space. Under some consistency and regularity conditions on initial and boundary data, the existence and uniqueness of inverse problem is shown by using the Banach fixed point theorem and conditional stability of the solution of inverse problem is shown in a certain class of data. Numerically, the inverse problem has been discretized by using finite difference method, which has been solved using the MATLAB. Numerical results show that accurate, and stable solutions have been obtained.

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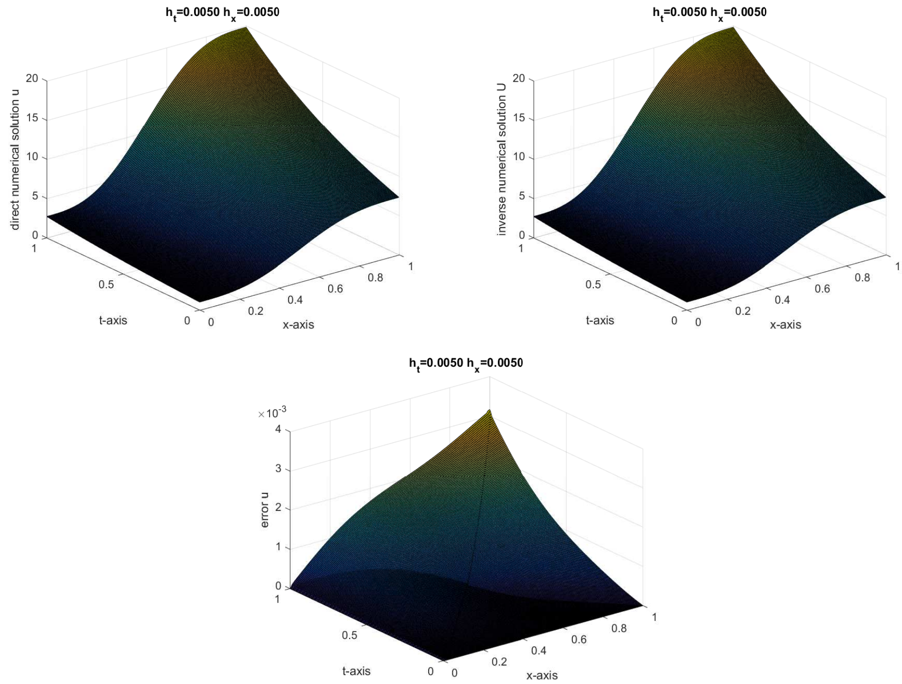

Figure 1: Direct and inverse numerical solutions for $u(x, t)$ and the absolute error for the direct and inverse numerical solutions for Example 4.1.

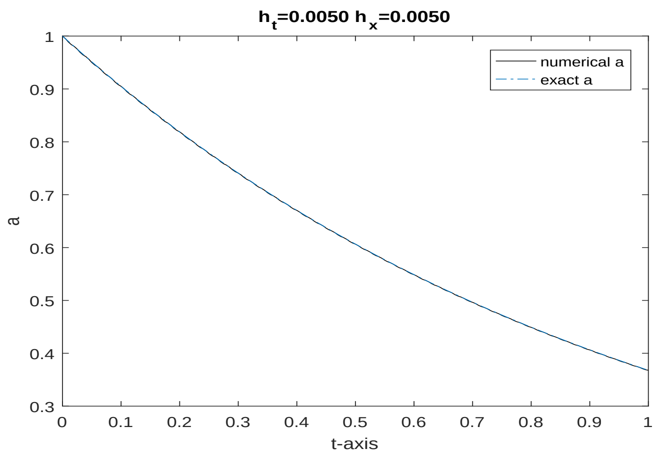

Figure 2: Exact and inverse numerical solutions for $a(t)$ for Example 4.1.

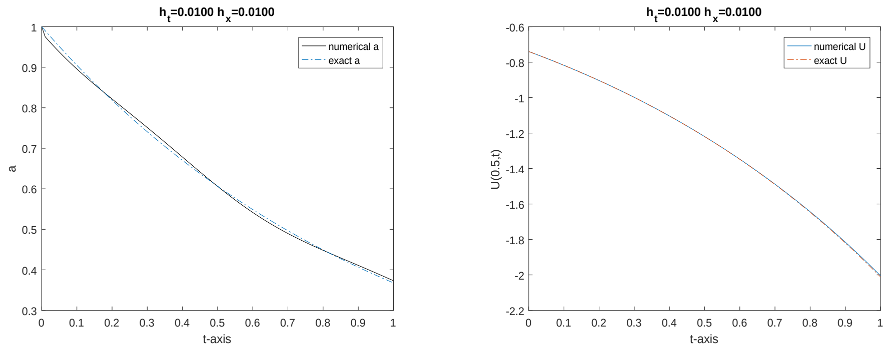

Figure 3: Exact and numerical solutions of the problem (1)-(5) for example 4.2.

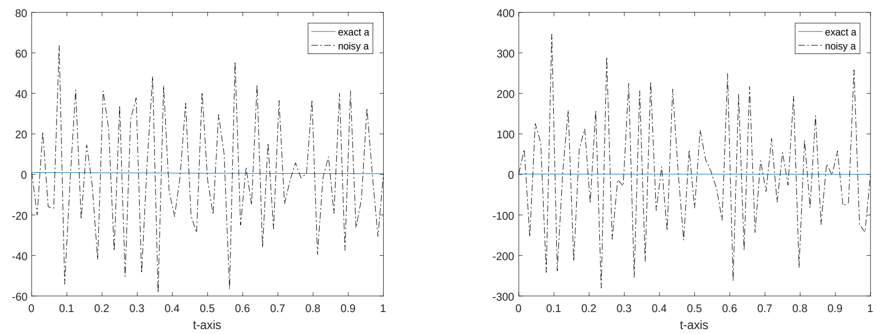

Figure 4: Exact and numerical coefficient solutions of the problem (1)-(5) for Example 4.2 with 1% and 5% noise.

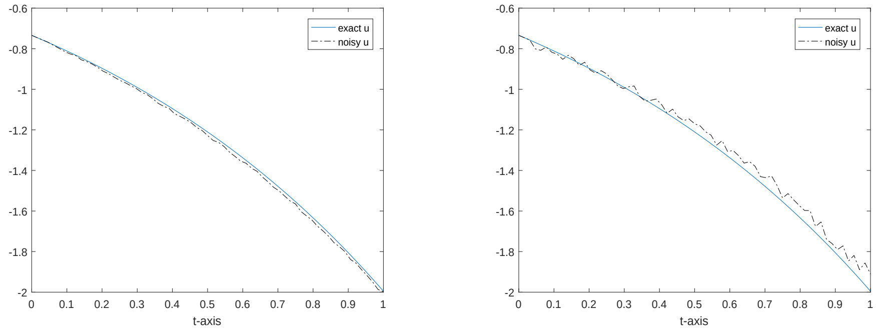

Figure 5: Exact and numerical $u(x,t)$ solutions of the problem (1)-(5) for Example 4.2 with 1% and 5% noise.

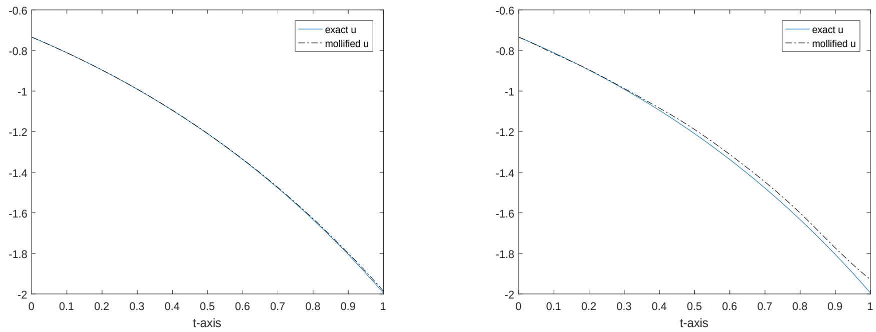

Figure 6: Exact and numerical $u(x,t)$ solutions of the problem (1)-(5) for Example 4.2 after mollification with 1% and 5% noise.

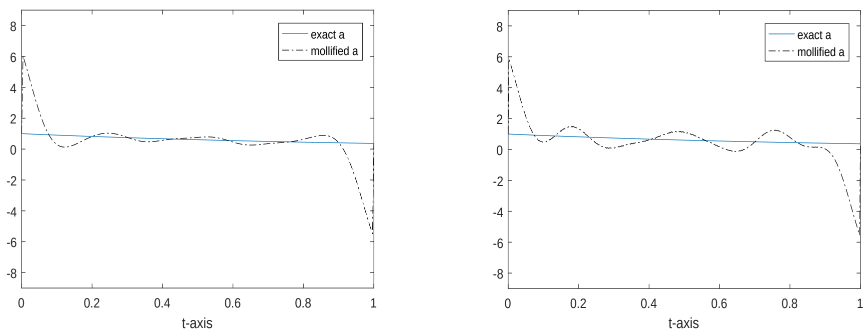

Figure 7: Exact and numerical coefficient solutions of the problem (1)-(5) for Example 4.2 after mollification with 1% and 5% noise.