Filomat 34:1 (2020), 187–196
https://doi.org/10.2298/FIL2001187E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$\gamma-$ Operation & Decomposition of Some Forms of Fuzzy Soft Mappings on Fuzzy Soft Ideal Topological Spaces

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Abstract. In this paper, we extend the notion of $\gamma-$operation by using the fuzzy soft ideal notions. Furthermore, we introduce some forms of fuzzy soft continuity in fuzzy soft topological spaces via fuzzy soft ideals. The decomposition of these forms is investigated.

1. Introduction

It is well known that the conception of fuzzy sets, firstly defined by Zadeh [39] in 1965. Chronologically in 1968 and 1976, Chang [8] and Lowen [25] redound the concept of fuzzy topological spaces to literature substantively by using this conception. In 1999, Molodtsov [29] introduced the concept of soft set theory which is a completely new approach for modeling uncertainty. In his paper, Molodtsov established the fundamental results of this new theory and successfully applied the soft set theory into several directions, such as smoothness of functions, operations research, Riemann integration, game theory, theory of probability and so on. Maji et al. [27] defined and studied several basic notions of soft set theory in 2003. Shabir and Naz [33] introduced the concept of soft topological space and studied neighborhoods and separation axioms.

Maji et al. [28] initiated the study involving both fuzzy sets and soft sets. In his paper, the notion of fuzzy soft sets was introduced as fuzzy generalizations of soft sets and some basic properties of fuzzy soft sets are discussed in detail. In 2011, Tanay et al. [34] gave the topological structure of fuzzy soft sets. Kandil et al. introduced the concept of fuzzy soft connected sets[19–21], fuzzy soft hyperconnected spaces [22].

On the other hand; some properties of the concept of ideal or topological ideal obtained by delineate of Vaidyanathaswamy [38] in 1945 and Kuratowski [24] in 1966. In 1960 Vaidyanathaswamy [37], and Jankovic´ [12] in 1990 studied intensity on this subject. In 2015, Kandil et al.[16] intiated the notion of soft ideal topological spaces for the first time and studied its properties. Recently, in 2016, Kandil et al. [17, 18] introduced the concepts of fuzzy soft ideal and fuzzy soft local function. These concepts are discussed with a view to find new fuzzy soft topologies from the original one, called fuzzy soft topological spaces via fuzzy soft ideal $(X, \tau, E, \widetilde{I})$.

2010 *Mathematics Subject Classification*. 03E99, 54C05, 54C08, 54C50
Keywords. Fuzzy soft set, Fuzzy soft topological space, Fuzzy soft ideal, Fuzzy soft ideal topological space, $\gamma-$ operation, fuzzy soft γ-open, fuzzy soft γ-closed sets, fussy soft γ-continuous function
Received: 14 Febraury 2019; Revised: 15 July 2019; Accepted: 17 September 2019
Communicated by Biljana Popović
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Our aim of this paper is to introduce and study some types of subsets of fuzzy soft topological spaces due to [10] to fuzzy soft topological spaces via fuzzy soft ideals. Also, we extend the idea of decomposition of continuity due to [9, 11, 23, 31, 40] to fuzzy soft topological spaces via fuzzy soft ideals.

2. Preliminaries

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in X, and the set of all subsets of X will be denoted by $P(X)$. In this section, we present the basic definitions and results of fuzzy soft set theory which will be needed in the sequel.

Definition 2.1. [8] A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \to [0;1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for $x \in X$. Let I^X denotes the family of all fuzzy sets on X.

Definition 2.2. [29] Let A be a non-empty subset of E . A pair (F, A) denoted by F_A is called a soft set over X , where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X . For a particular $e \in A$, $F(e)$ may be considered the set of e -approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \phi$ i.e $F = \{F(e) : e \in A \subseteq E, F : A \to P(X)\}$.

Proposition 2.1. [7] Every fuzzy set may be considered a soft set.

Definition 2.3. [28] Let $A \subseteq E$. A pair (f, A) , denoted by f_A, is called fuzzy soft set over X , where f is a mapping given by $f : A \longrightarrow I^X$ defined by $f_A(e) = \mu^e_{f_A}$; where $\mu^e_{f_A} = \overline{0}$ if $e \notin A$, and $\mu^e_{f_A} \neq \overline{0}$ if $e \in A$, where $\overline{0}(x) = 0 \forall\, x \in X$. The family of all these fuzzy soft sets over X denoted by $FSS(X)_E$. Note that, a fuzzy soft set is a hybridization of fuzzy sets and soft sets, in which soft set is defined over fuzzy set. The family of all fuzzy soft sets over X with a fixed set of parameter E is denoted by $FSS(X)_E$.

Definition 2.4 [28] The complement of a fuzzy soft set (f, A) , denoted by $(f, A)^c$, is defined by $(f, A)^c = (f^c, A)$, $(f_A)^c : A \longrightarrow I^X$ is a mapping given by $\mu^e_{(f_A)^c} = 1 - \mu^e_{f_A}$ $\forall e \in A$, where $\overline{1}(x) = 1 \forall x \in X$. Clearly, $((f_A)^c)^c = f_A$.

Definition 2.5. [34] A fuzzy soft set f_E over X is said to be a null- fuzzy soft set, denoted by $\widetilde{0}_E$, if for all $e \in E$, $f_E(e) = \overline{0}$.

Definition 2.6. [34] A fuzzy soft set f_E over X is said to be an absolute fuzzy soft set, denoted by $\widetilde{1}_E$, if $f_E(e) = \overline{1}$ $\forall e \in E$. Clearly we have $(\widetilde{0}_E)^c = \widetilde{1}_E$ and $(\widetilde{1}_E)^c = \widetilde{0}_E$.

Definition 2.7.[34] Let $f_A, g_B \in FSS(X)_E$. Then f_A is fuzzy soft subset of g_B , denoted by $f_A \widetilde{\sqsubseteq} g_B$, if $A \subseteq B$ and $\mu^e_{f_A}(x) \leq \mu^e_{g_B}(x)$ $\forall x \in X$ and $\forall e \in E$. Also, g_B is called fuzzy soft superset of f_A denoted by $g_B \widetilde{\sqsupseteq} f_A$.

Definition.2.8. [36] Two fuzzy soft sets f_A and g_B on X are called equal if $f_A \widetilde{\sqsubseteq} g_B$ and $g_B \widetilde{\sqsubseteq} f_A$.

Definition 2.9. [32] The union of two fuzzy soft sets f_A and g_B over the common universe X , denoted by $f_A \sqcup g_B$, is also a fuzzy soft set h_C , where $C = A \cup B$ and for all , $e \in C, h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \vee \mu^e_{g_B}$ $\forall e \in C$.

Definition 2.10. [34] The intersection of two fuzzy soft sets f_A and g_B over the common universe X , denoted by $f_A \sqcap g_B$, is also a fuzzy soft set h_C, where $C = A \cap B$ and for all , $e \in C, h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \wedge \mu^e_{g_B}$ $\forall e \in C$.

Definition 2.11. [35] The difference of two fuzzy soft sets f_A and g_B over the common universe X, denoted by $f_A \widetilde{-} g_B$, is the fuzzy soft set h_C, where $C = A \cap B \neq \phi$ and $\forall e \in C, x \in X$, $\mu^e_{h_C}(x) = \min\{\mu^e_{f_A}(x), 1 - \mu^e_{g_B}(x)\}$. Clearly, we have $f_A \widetilde{-} g_B = f_A \sqcap g^c_B$.

Definition 2.12. [30, 34, 36] Let τ be a collection of fuzzy soft sets over a universe X with a fixed set of parameters E , then τ is called fuzzy soft topology on X if:
(1) $\widetilde{0}_E$, $\widetilde{1}_E \in \tau$, where $\widetilde{0}_E(e) = \overline{0}$ and $\widetilde{1}_E(e) = \overline{1}$ $\forall e \in E$,
(2) the union of any members of τ belongs to τ.
(3) the intersection of any two members of τ belongs to τ.
The triplet (X, τ, E) is called fuzzy soft topological space over X. Also, each member of τ is called fuzzy soft open ste in (X, τ, E).

Definition 2.13. [34?] Let (X, τ, E) be a fuzzy soft topological space. A fuzzy soft set f_A over X is said to be fuzzy closed soft set in X, denoted by $f_A \in \tau^c$, if its relative complement $(f_A)^c$ is fuzzy open soft set.

Definition 2.14. [30, 32, 34?] Let (X, τ, E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. The fuzzy soft closure of f_A , denoted by $Fcl(f_A)$ is the intersection of all fuzzy closed soft super sets of f_A . i.e. $Fcl(f_A) = \sqcap\{h_C; h_C$ is fuzzy closed soft set and $f_A \,\widetilde{\sqsubseteq}\, h_C\}$. Clearly, $Fcl(f_A)$ is the smallest fuzzy soft closed set over X which contains f_A, and $Fcl(f_A)$ is fuzzy closed soft set.

Definition 2.15.[30, 32, 34?] The fuzzy soft interior of g_B , denoted by $Fint(g_B)$, is the fuzzy soft union of all fuzzy open soft subsets of g_B .i.e.$Fint(g_B) = \sqcup\{h_C; h_C$ is fuzzy open soft set and $h_C \,\widetilde{\sqsubseteq}\, g_B\}$. Clearly, $Fint(g_B)$ is the largest fuzzy soft open set contained in g_B and $Fint(g_B)$ is fuzzy open soft set.

Definition 2.16. [26] The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that

$$f_A(a) = \begin{cases} x_\alpha & ; a = e \\ \overline{0} & a \in E - \{a\} \end{cases} \quad ; 0 < \alpha \leqslant 1$$

This fuzzy soft point is denoted by x^e_α or f_e . The set of all fuzzy soft points in X will be denoted by $FSP(X)_E$.

Definition 2.17. [26] The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set f_A , denoted by $x^e_\alpha \,\widetilde{\in}\, f_A$, if for the element $e \in A$, $\alpha \leq \mu^e_{f_A}(x)$.

Definition 2.18. [17, 18] A non-empty collection $\widetilde{I}$ of fuzzy soft sets over a universe X with a fixed set of parameters E is said to a fuzzy soft ideal on X if it satisfies the following conditions:

(i) If $f_A \in \widetilde{I}$, $g_B \,\widetilde{\sqsubseteq}\, f_A \Longrightarrow g_B \in \widetilde{I}$, (heredity)

(ii) If $f_A, g_B \in \widetilde{I} \Longrightarrow f_A \sqcup g_B \in \widetilde{I}$. (finite additivity)

We denote $(X, \tau, E, \widetilde{I})$ as a fuzzy soft ideal topological space on X.

Definition 2.19. [17] Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space on X. Then $f_A{}^\star(\widetilde{I}, \tau)$ or $f_A{}^\star = \sqcup\{x^e_\alpha \,\widetilde{\in}\, FSS(X)_E; f_A \sqcap g_B \notin \widetilde{I}\ \forall g_B \in \tau(x^e_\alpha)\}$ is called the fuzzy soft local function of f_A with respect to $\widetilde{I}$ and τ where $\tau(x^e_\alpha)$ is the set of all fuzzy soft open sets contains x^e_α.

Definition 2.20.[17, 18] In a fuzzy soft ideal topological space $(X, \tau, E, \widetilde{I})$, the collection $\tau^\star(\widetilde{I})$ means an extension of fuzzy soft topological space finer than τ via fuzzy soft ideal $\widetilde{I}$ which is constructed by considering the class $\beta = \{f_A \,\widetilde{-}\, h_C; f_A \in \tau, h_C \in \widetilde{I}\}$ as a base. Each member of $\tau^\star(\widetilde{I})$ is called fuzzy soft $\star$-open set and its relative complement is called fuzzy soft $\star$-closed set. Clearly, $\tau \subseteq \tau^\star(\widetilde{I})$.

Theorem 2.1.[17, 18] Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space on X. Then the operator $Fcl^\star : FSS(X)_E \to FSS(X)_E$ defined by: $Fcl^\star(f_A) = f_A \sqcup f_A{}^\star$ is a fuzzy soft closure operator satisfies Kuratwski's axioms.

Definition 2.21.[17, 18] Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space on X. The fuzzy soft $\star$-closure of a fuzzy soft set f_A, denoted by $Fcl^\star(f_A)$, in a fuzzy soft ideal topological space $(X, \tau, E, \widetilde{I})$ defined as $Fcl^\star(f_A) = f_A \sqcup f_A{}^\star$. The fuzzy soft $\star$-interior of a fuzzy soft set f_A, denoted by $Fint^\star(f_A)$, in a fuzzy soft ideal topological space $(X, \tau, E, \widetilde{I})$ is the largest fuzzy soft open subset of f_A in $\tau^*(\widetilde{I})$.

Definition 2.22.[6, 10] Let $FSS(X)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over X and Y, respectively. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Then the map f_{pu} is called fuzzy soft mapping from $FSS(X)_E$ to $FSS(Y)_K$, denoted by $f_{pu} : FSS(X)_E \longrightarrow FSS(Y)_K$, such that:

(1) If $g_B \in FSS(X)_E$, then the image of g_B under the fuzzy soft mapping f_{pu} , denoted by $f_{pu}(g_B)$, is a fuzzy soft set over$Y$ defined by:

$$f_{pu}(g_B)(k)(y) = \begin{cases} \bigvee\limits_{u(x)=y} [\bigvee\limits_{p(e)=k} (g_B(e))](x) & ; \text{if } x \in u^{-1}(y), \\ 0 & ; \text{otherwise} \end{cases} \quad \forall k \in p(E), \forall y \in Y,$$

(2) If $h_C \in FSS(Y)_K$, then the inverse image of h_C under the fuzzy soft mapping f_{pu}, denoted by $f^{-1}_{pu}(h_C)$, is a fuzzy soft set over X defined by:

$$f_{pu}^{-1}(h_C)(e)(x) = \begin{cases} h_C(p(e))(u(x)) & ;\text{for } p(e) \in C, \\ 0 & ;\text{ otherwise} \end{cases} \forall e \in p^{-1}(K), \forall x \in X$$

Definition 2.23.[15] The fuzzy soft mapping f_{pu} is called surjective (respectively, injective, bijective) if p and u are surjective (respectively, injective, bijective), also f_{pu} is said to be constant if p and u are constant.
Definition 2.24. [30] Let (X, τ, E) and (Y, σ, K) be two fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \longrightarrow FSS(Y)_K$ be a fuzzy soft mapping. f_{pu} is called fuzzy soft continuous if $f_{pu}^{-1}(h_C) \in \tau$ for every $h_C \in \sigma$.
Definition 2.25. Let (X, τ, E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$.
(1) f_A is called fuzzy soft semi-open set if $f_A \widetilde{\subseteq} Fcl(Fint(f_A))$ [13**?** , 14].
(2) f_A is called fuzzy soft pre-open set if $f_A \widetilde{\subseteq} Fint(Fcl(f_A))$ [1, 5].
(3) f_A is called fuzzy soft β-open set if $f_A \widetilde{\subseteq} Fcl(Fint(Fcl(f_A))$ [2, 19].
(4) f_A is called fuzzy soft α-open set if $f_A \widetilde{\subseteq} Fint(Fcl(Fint(f_A))$ [3, 4].
The collection of all fuzzy soft semi-open (respectively, fuzzy soft pre-open, fuzzy soft β-open, fuzzy soft α-open) sets will be denoted by $FSSO(X)_E$ (respectively, $FSPO(X)_E$, $FS\beta O(X)_E$, $FS\alpha O(X)_E$).

3. Subsets of fuzzy soft topological space via fuzzy soft ideal

In this section, we extend some spacial subsets of a fuzzy soft topological space (X, τ, E) via fuzzy soft ideal.
Definition 3.1. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and $f_A \in FSS(X)_E$. Then, f_A is called fuzzy soft $\widetilde{I}$-open if $f_A \widetilde{\subseteq} Fint(f_A^*)$. A fuzzy soft set f_A is said to be fuzzy soft $\widetilde{I}$-closed if its complement is a fuzzy soft $\widetilde{I}$-open. We will denoted the set of all fuzzy soft $\widetilde{I}$-open sets (fuzzy soft $\widetilde{I}$-closed sets) by $FSIO(X)_E$ $(FSIC(X)_E)$.
Definition 3.2. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and $(X, \tau^*, E, \widetilde{I})$ be its fuzzy soft $*$-topological space. A mapping $\gamma : FSS(X)_E \longrightarrow FSS(X)_E$ is said to be an γ-operation on $FSO(X)_E$ if $f_A \widetilde{\subseteq} \gamma(f_A)$ for every $f_A \in FSO(X)_E$.

The collection of all fuzzy soft γ-open sets is denoted by $FSO(\gamma)_E = \{f_A;\ f_A \widetilde{\subseteq} \gamma(f_A),\ f_A \in FSS(X)_E\}$. The complement of fuzzy soft γ-open set is called fuzzy soft γ-closed. i.e., $FSC(\gamma)_E = \{(f_A)^c;\ f_A$ is fuzzy soft γ-open $f_A \in FSS(X)_E\}$ is the family of all fuzzy soft γ-closed sets.
Definition 3.3. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and $f_A \in FSS(X)_E$. Different cases of γ-operations on $FSS(X)_E$ are as follows:
(1) If $\gamma = Fint(Fcl^*)$, then γ is called fuzzy soft pre-$\widetilde{I}$-open operator. We will denoted the set of all fuzzy soft pre-$\widetilde{I}$-open sets by $FSPIO(X)_E$ and the set of all fuzzy soft pre-$\widetilde{I}$-closed sets by $FSPIC(X)_E$.
(2) If $\gamma = Fint(Fcl^*(Fint))$, then γ is called fuzzy soft α-$\widetilde{I}$-open operator. We will denoted the set of all fuzzy soft α-$\widetilde{I}$-open sets by $FS\alpha IO(X)_E$ and the set of all fuzzy soft α-$\widetilde{I}$-closed sets by $FS\alpha IC(X)_E$.
(3) If $\gamma = Fcl^*(Fint)$, then γ is called fuzzy soft semi-$\widetilde{I}$-open operator. We will denoted the set of all fuzzy soft semi-$\widetilde{I}$-open sets by $FSSIO(X)_E$ and the set of all fuzzy soft semi-$\widetilde{I}$-closed sets by $FSSIC(X)_E$.
(4) If $\gamma = Fcl(Fint(Fcl^*))$, then γ is called fuzzy soft β-$\widetilde{I}$-open operator. We will denoted the set of all fuzzy soft β-$\widetilde{I}$-open sets by $FS\beta IO(X)_E$ and the set of all fuzzy soft β-$\widetilde{I}$-closed sets by $FS\beta IC(X)_E$.
Theorem 3.1. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and $\gamma : FSS(X)_E \longrightarrow FSS(X)_E$ be an γ-operation on $FSO(X)_E$. If $\gamma = \{Fint(Fcl^*), Fint(Fcl^*(Fint)), Fcl^*(Fint), Fcl(Fint(Fcl^*))\}$, then:
(1) $\widetilde{1}_E$ and $\widetilde{0}_E$ are fuzzy soft γ-open sets.
(2) arbitrary union of fuzzy soft γ-open sets is a fuzzy soft γ-open.
(3) arbitrary intersection of of fuzzy soft γ-closed sets is a fuzzy soft γ-closed.
Proof.
(1) Immediate.

(2) We give the proof for the case of fuzzy soft pre-$\widetilde{I}$-open operator, i.e. $\gamma = Fint(Fcl^*)$. Let $\{(f_A)_j;\ j \in J\} \subseteq FSPIO(X)_E$. Then, $\forall j \in J$, $(f_A)_j \mathrel{\widetilde{\subseteq}} Fint(Fcl^*(f_A)_j)$. It follows that $\bigsqcup_{j\in J}(f_A)_j \mathrel{\widetilde{\subseteq}} \bigsqcup_{j\in J} Fint(Fcl^*(f_A)_j) \mathrel{\widetilde{\subseteq}} Fint(\bigsqcup_{j\in J} Fcl^*(f_A)_j) = Fint(Fcl^*(\bigsqcup_{j\in J}(f_A)_j))$. Hence, $\bigsqcup_{j\in J}(f_A)_j \in FSPIO(X)_E$. The other cases are similar.

(3) It follows directly by (2).

Remark 3.1. A finite intersection of fuzzy soft γ-open sets where $\gamma = \{Fint(Fcl^*), Fcl^*(Fint), Fcl(Fint(Fcl^*))\}$ need not to be a fuzzy soft γ-open as shown by the following example:

Example 3.1.

(1) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = \{\widetilde{0}_E\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.6}, b_{0.7}\})\}, \{(e_1, \{a_{0.1}, b_{0.2}\})\}\}$. Then, $f_E = \{(e_1, \{a_{0.2}, b_{0.5}\})\}$ and $g_E = \{(e_1, \{a_{0.5}, b_{0.2}\})\}$ are fuzzy soft pre-$\widetilde{I}$-open sets but $f_E \sqcap g_E \notin FSPIO(X)_E$.

(2) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = \{\widetilde{0}_E\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.2}\})\}, \{(e_1, \{b_{0.2}\})\}, \{(e_1, \{a_{0.2}, b_{0.2}\})\}\}$.Then, $f_E = \{(e_1, \{a_{0.5}, b_{0.1}\})\}$ and $g_E = \{(e_1, \{a_{0.1}, b_{0.5}\})\}$ are fuzzy soft semi-$\widetilde{I}$-open sets but $f_E \sqcap g_E \notin FSSIO(X)_E$.

(3) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = FSS(X)_E$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.2}\})\}, \{(e_1, \{b_{0.2}\})\}, \{(e_1, \{a_{0.2}, b_{0.2}\})\}\}$.Then, $f_E = \{(e_1, \{a_{0.2}, b_{0.1}\})\}$ and $g_E = \{(e_1, \{b_{0.5}\})\}$ are fuzzy soft β-$\widetilde{I}$-open sets but $f_E \sqcap g_E \notin FS\beta IO(X)_E$.

Remark 3.2. Note that the family of all fuzzy soft γ-open sets on a fuzzy soft ideal topological space $(X, \tau, E, \widetilde{I})$ forms a fuzzy soft supra topology, which is a collection of fuzzy soft sets contains $\widetilde{1}_E$, $\widetilde{0}_E$ and closed under arbitrary union.

Proposition 3.1. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and $f_A \in FSS(X)_E$. Then, we have:

(1) If $\widetilde{I} = \{\widetilde{0}_E\}$, then f_A is fuzzy soft pre-$\widetilde{I}$- (respectively, semi-$\widetilde{I}$-, α-$\widetilde{I}$-, β-$\widetilde{I}$-) open $\Longleftrightarrow$ f_A is fuzzy soft pre- (respectively, semi-, α-, β-) open.

(2) If $\widetilde{I} = FSS(X)_E$, then f_A is fuzzy soft pre-$\widetilde{I}$- (respectively, semi-$\widetilde{I}$-, α-$\widetilde{I}$-, β-$\widetilde{I}$-) open $\Longleftrightarrow$ f_A is fuzzy soft τ-open.

Proof. As a sample we will prove the case of fuzzy soft α-$\widetilde{I}$-open operator. i.e., $\gamma = Fint(Fcl^*(Fint))$.

(1) Let $\widetilde{I} = \{\widetilde{0}_E\}$. Then, $f_A^* = Fcl(f_A)$ and hence $Fcl^*(f_A) = Fcl(f_A)$ for every $f_A \in FSS(X)_E$. Therefore, $Fint(Fcl^*(Fint(f_A))) = Fint(Fcl(Fint(f_A)))$ and hence $FS\alpha IO(X)_E = \tau_\alpha$.

(2) Let $\widetilde{I} = FSS(X)_E$. Then, $f_A^* = \widetilde{0}_E$ and hence $Fcl^*(f_A) = f_A$ for every $f_A \in FSS(X)_E$. Therefore, $Fint(Fcl^*(Fint(f_A))) = Fint(Fint(f_A)) = Fint(f_A)$ and hence $\tau = FS\alpha IO(X)_E$.

The other cases are similar.

Definition 3.3. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and $f_E \in FSS(X)_E$. Then:

(1) x_α^e is called $\gamma-$fuzzy soft interior point of f_E if $\exists\, g_E \in FSO(\gamma)_E$ such that $x_\alpha^e \mathrel{\widetilde{\in}} g_E \mathrel{\widetilde{\subseteq}} f_E$. The set of all $\gamma-$fuzzy soft interior points of f_E is called the $\gamma-$fuzzy soft interior of f_E and is denoted by $\gamma Fint(f_E)$. Consequently, $\gamma Fint(f_E) = \sqcup\{g_E;\ g_E \in FSO(\gamma)_E,\ g_E \mathrel{\widetilde{\subseteq}} f_E\}$.

(2) x_α^e is called $\gamma-$fuzzy soft cluster point of f_E if $f_E \sqcap g_E \neq \widetilde{0}_E$ $\forall g_E \in FSO(\gamma)_E$.The set of all $\gamma-$fuzzy soft cluster points of f_E is called the $\gamma-$fuzzy soft closure of f_E and is denoted by $\gamma FScl(f_E)$. Consequently, $\gamma FScl(f_E) = \sqcap\{g_E;\ g_E \in FSC(\gamma)_E,\ f_E \mathrel{\widetilde{\subseteq}} g_E\}$.

Theorem 3.2. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space, $\gamma : FSS(X)_E \longrightarrow FSS(X)_E$ be one of the operations of Definition 3.3 and f_E, $g_E \in FSS(X)_E$. Then, the following properties are satisfied for the $\gamma-$fuzzy soft interior operators denoted by $\gamma Fint(f_E)$:

(1) $\gamma Fint(\widetilde{0}_E) = \widetilde{0}_E$ and $\gamma Fint(\widetilde{1}_E) = \widetilde{1}_E$.

(2) $\gamma Fint(f_E) \mathrel{\widetilde{\subseteq}} f_E$.

(3) $\gamma Fint(f_E)$ is the largest fuzzy soft $\gamma-$open set contained in f_E.

(4) If $f_E \mathrel{\widetilde{\subseteq}} g_E$, then $\gamma Fint(f_E) \mathrel{\widetilde{\subseteq}} \gamma Fint(g_E)$.

(5) $\gamma Fint(f_E) \sqcup \gamma Fint(g_E) \mathrel{\widetilde{\subseteq}} \gamma Fint(f_E \sqcup g_E)$.

(6) $\gamma Fint(\gamma Fint(f_E)) = \gamma Fint(f_E)$.

(7) $\gamma Fint(f_E \sqcap g_E) \mathrel{\widetilde{\subseteq}} \gamma Fint(f_E) \sqcap \gamma Fint(g_E)$.

Proof. Immediate.

Theorem 3.3. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space, $\gamma : FSS(X)_E \longrightarrow FSS(X)_E$ be one of the operations of Definition 3.3 and f_E, $g_E \in FSS(X)_E$. Then, the following properties are satisfied for the

$\gamma-$fuzzy soft closure operators denoted by $\gamma FScl(f_E)$:
(1) $\gamma FScl(\widetilde{0}_E) = \widetilde{0}_E$ and $\gamma FScl(\widetilde{1}_E) = \widetilde{1}_E$.
(2) $f_E \widetilde{\subseteq} \gamma FScl(f_E)$.
(3) $\gamma FScl(f_E)$ is the smallest fuzzy soft $\gamma-$closed set contains f_E.
(4) If $f_E \widetilde{\subseteq} g_E$, then $\gamma FScl(f_E) \subseteq \gamma FScl(g_E)$.
(5) $\gamma FScl(f_E) \sqcup \gamma FScl(g_E) \widetilde{\subseteq} \gamma FScl(f_E \sqcup g_E)$.
(6) $\gamma FScl(\gamma FScl(f_E)) = \gamma FScl(f_E)$.
(7) $\gamma FScl(f_E \sqcap g_E) \widetilde{\subseteq} \gamma FScl(f_E) \sqcap \gamma FScl(g_E)$.
Proof. Immediate.

4. Relations between subsets of fuzzy soft topological space via fuzzy soft ideal

Theorem 4.1. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space. The following statements are hold:
(1) Every fuzzy soft open set is fuzzy soft semi-$\widetilde{I}$-open,
(2) Every fuzzy soft semi-$\widetilde{I}$-open set is fuzzy soft semi-open.
Proof.
(1) Let f_A be a fuzzy soft open set. Then, $Fint(f_A) = f_A$. Therefore, $f_A \widetilde{\subseteq} Fcl^\star(Fint(f_A)) = Fcl^\star(f_A)$. Hence f_A is a fuzzy soft semi-$\widetilde{I}$-open set.
(2) Let f_A be a fuzzy soft semi-$\widetilde{I}$-open set. Then, $f_A \widetilde{\subseteq} Fcl^\star(Fint(f_A)) = Fin(f_A) \sqcup [Fint(f_A)]^\star \widetilde{\subseteq} Fint(f_A) \sqcup Fcl(Fint(f_A)) = Fcl(Fint(f_A))$. Therefore, f_A is a fuzzy soft semi-open set.

The following example shows that the converse of Theorem 4.1 is not true in general.

Example 4.1. Let $X = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$, $f_E = \{(e_1, \{a_{0.1}, b_{0.1}\}), (e_2, \{a_{0.1}, b_{0.2}, c_{0.1}\})\}$ and $g_E = \{(e_1, \{a_{0.3}, b_{0.4}, c_{0.5}\}), (e_2, \{a_{0.4}, b_{0.2}, c_{0.1}\})\}$. We put $\tau = \{\widetilde{1}_E, \widetilde{0}_E, f_E\}$. If we take $\widetilde{I} = \{\widetilde{0}_E\}$, then g_E is fuzzy soft semi-$\widetilde{I}$-open set but g_E is fuzzy soft open, since $g_E^\star = Fcl(g_E)$. If we take $\widetilde{I} = FSS(X)_E$, then g_E is fuzzy soft semi-open set but is fuzzy soft semi-$\widetilde{I}$-open, since $g_E^\star = \widetilde{0}_E$.

Theorem 4.2. In fuzzy soft ideal topological space $(X, \tau, E, \widetilde{I})$, the following statements are hold:
(1) every fuzzy soft open set is a fuzzy soft pre-$\widetilde{I}$-open,
(2) every fuzzy soft pre-$\widetilde{I}$-open set is a fuzzy soft pre-open,
(3) every fuzzy soft $\widetilde{I}$-open set is a fuzzy soft pre-$\widetilde{I}$-open.
Proof.
(1) Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and $f_A \in \tau$. Then $Fint(f_A) = f_A$. Since $f_A \widetilde{\subseteq} Fcl^*(f_A)$, then $f_A = Fint(f_A) \widetilde{\subseteq} Fint(Fcl^*(f_A))$. Hence, f_A is a fuzzy soft pre-$\widetilde{I}$-open.
(2) Let $f_A \in FSPIO(X)_E$. Then, $f_A \widetilde{\subseteq} Fint(Fcl^*(f_A))$. Since $Fcl^*(f_A) \widetilde{\subseteq} Fcl(f_A)$, then $f_A \widetilde{\subseteq} Fint(Fcl(f_A))$ and so f_A is a fuzzy soft pre-open.
(3) Let $f_A \in FSIO(X)_E$. Then, $f_A \widetilde{\subseteq} Fint(f_A^*)$. Since $f_A^* \widetilde{\subseteq} Fcl^*(f_A)$, then $f_A \widetilde{\subseteq} Fint(Fcl^*(f_A))$ and so $f_A \in FSPIO(X)_E$.

The following example shows that the converse of Theorem 4.2 is not true in general.

Example 4.2. Let $X = \{a, b, c\}$, $E = \{e_1, e_2, ..., e_7\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, f_E = \{(e_1, \{a_{0.5}, b_{0.4}, c_{0.6}\})\}\}$ be a fuzzy soft topology defined on X.
(1) If $\widetilde{I} = \{\widetilde{0}_E\}$ and $g_E = \{(e_1, \{a_{0.5}, b_{0.5}, c_{0.6}\})\}$, then $g_E \in FSPIO(X)_E$. But $g_E \notin \tau$.
(2) If $\widetilde{I} = FSS(X)_E$, then $g_E = \{(e_1, \{a_{0.5}, b_{0.3}, c_{0.6}\})\}$ is a fuzzy soft pre-open set, but $g_E \notin FSPIO(X)_E$.
(3) If $\widetilde{I} = FSS(X)_E$, then $f_E \in FSPIO(X)_E$. But $f_E \notin FSIO(X)_E$.

Theroem 4.3. In a fuzzy soft ideal topological space $(X, \tau, E, \widetilde{I})$, the following statements hold:
(1) Every fuzzy soft open set is fuzzy soft α-$\widetilde{I}$-open.
(2) Every fuzzy soft α-$\widetilde{I}$-open set is fuzzy soft α-open.
(3) Every fuzzy soft α-$\widetilde{I}$-open set is fuzzy soft semi-$\widetilde{I}$-open.
(4) Every fuzzy soft semi-$\widetilde{I}$-open set is fuzzy soft β-$\widetilde{I}$-open.

(5) Every fuzzy soft α-$\widetilde{I}$-open set is fuzzy soft pre-$\widetilde{I}$-open.

(6) Every fuzzy soft pre-$\widetilde{I}$-open set is fuzzy soft β-$\widetilde{I}$-open.

Proof.

(1) Let $f_A \in \tau$. Then $Fint(f_A) = f_A$ and so $Fcl^*(Fint(f_A)) = Fcl^*(f_A) \widetilde{\supseteq} f_A$. Therefore, $f_A = Fint(f_A) \widetilde{\subseteq} Fint(Fcl^*(Fint(f_A)))$. Hence, $f_A \in FS\alpha IO(X)_E$.

(2) Let $f_A \in FS\alpha IO(X)_E$. Then, we have $f_A \widetilde{\subseteq} Fint(Fcl^*(Fint(f_A)))$. Since $\tau \subseteq \tau^*$, then $Fcl^*(Fint(f_A)) \widetilde{\subseteq} Fcl(Fint(f_A))$. Therefore, $f_A \widetilde{\subseteq} Fint(Fcl^*(Fint(f_A))) \widetilde{\subseteq} Fint(Fcl(Fint(f_A)))$. This shows that f_A is a fuzzy soft α-open.

(3) Since $f_A \in FS\alpha IO(X)_E$, then $f_A \widetilde{\subseteq} Fint(Fcl^*(Fint(f_A))) \widetilde{\subseteq} Fcl^*(Fint(f_A))$. Hence, $f_A \in FSSIO(X)_E$.

(4) Let $f_A \in FSSIO(X)_E$. Then, $f_A \widetilde{\subseteq} Fcl^*(Fint(f_A))$. Since $\tau \subseteq \tau^*$, then $Fcl^*(Fint(f_A)) \widetilde{\subseteq} Fcl(Fint(f_A))$. Therefore, $f_A \widetilde{\subseteq} Fcl(Fint(f_A)) \widetilde{\subseteq} Fcl(Fint(Fcl^*(f_A))$. Hence, $f_A \in FS\beta IO(X)_E$.

(5) Let $f_A \in FS\alpha IO(X)_E$. Then, $f_A \widetilde{\subseteq} Fint(Fcl^*(Fint(f_A))) \widetilde{\subseteq} Fint(Fcl^*(f_A))$. Hence, $f_A \in FSPIO(X)_E$.

(6) Let $f_A \in FSPIO(X)_E$. Then $f_A \widetilde{\subseteq} Fint(Fcl^*(f_A)) \widetilde{\subseteq} Fcl(Fint(Fcl^*(f_A))$. Hence, $f_A \in FS\beta IO(X)_E$.

The following example shows that the converse of Theorem 4.3 is not true in general.

Example 4.3.

(1) Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = \{\widetilde{0}_E\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.1}, b_{0.1}, c_{0.1}\})\}, \{(e_1, \{a_{0.2}, b_{0.4}, c_{0.3}\})\}\}$. The fuzzy soft set $f_E = \{(e_1, \{a_{0.1}, b_{0.1}, c_{0.2}\})\}$ is a fuzzy soft α-$\widetilde{I}$-open set but f_E is not fuzzy soft open.

(2) The fuzzy soft semi-$\widetilde{I}$-open set may not be fuzzy soft α-$\widetilde{I}$-open. Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = \{\widetilde{0}_E\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.1}, b_{0.3}, c_{0.4}\})\}\}$. The fuzzy soft set $f_E = \{(e_1, \{a_{0.3}, b_{0.6}, c_{0.5}\})\}$ is fuzzy soft semi-$\widetilde{I}$-open set but it is not fuzzy soft α-$\widetilde{I}$-open.

(3) The fuzzy soft pre-$\widetilde{I}$-open set may not be fuzzy soft α-$\widetilde{I}$-open. Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = \{\widetilde{0}_E\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.1}, b_{0.2}, c_{0.2}\})\}\}$. The fuzzy soft set $f_E = \{(e_1, \{a_{0.1}, b_{0.1}, c_{0.1}\})\}$ is fuzzy soft pre-$\widetilde{I}$-open set but it is not fuzzy soft α-$\widetilde{I}$-open.

(4) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = \{\widetilde{0}_E\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.5}, b_{0.4}\})\}\}$. The fuzzy soft set $f_E = \{(e_1, \{a_{0.3}, b_{0.5}\})\}$ is a fuzzy soft β-$\widetilde{I}$-open set but f_E is not fuzzy soft semi-$\widetilde{I}$-open.

(5) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = \{\widetilde{0}_E\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.1}, b_{0.1}\})\}\}$. The fuzzy soft set $f_E = \{(e_1, \{a_{0.3}, b_{0.2}\})\}$ is a fuzzy soft β-$\widetilde{I}$-open set but f_E is not fuzzy soft pre-$\widetilde{I}$-open.

(6) Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = FSS(X)_E$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, f_E = \{(e_1, \{a_{0.5}, b_{0.4}, c_{0.6}\})\}\}$. The fuzzy soft set $g_E = \{(e_1, \{a_{0.5}, b_{0.5}, c_{0.6}\})\}$ is a fuzzy soft α-open set but g_E is not fuzzy soft α-$\widetilde{I}$-open.

(7) Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = FSS(X)_E$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, f_E = \{(e_1, \{a_{0.5}, b_{0.4}, c_{0.6}\})\}\}$. The fuzzy soft set f_E is fuzzy soft α-$\widetilde{I}$-open set but it is not fuzzy soft $\widetilde{I}$-open.

(8) Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = \{\widetilde{0}_E\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.1}, b_{0.2}\})\}\}$. The fuzzy soft set $f_E = \{(e_1, \{a_{0.1}, b_{0.1}\})\}$ is a fuzzy soft $\widetilde{I}$-open set but f_E is not fuzzy soft α-$\widetilde{I}$-open.

Lemma 4.1. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and $f_A \in FSS(X)_E$. Then, f_A is a fuzzy soft α-$\widetilde{I}$-open set if and only if f_A is both fuzzy soft semi-$\widetilde{I}$-open and fuzzy soft pre-$\widetilde{I}$-open.

Proof. Necessity. This is obvious by Theroem 4.3 (3, 5).

Sufficiency. Let f_A be fuzzy soft semi-$\widetilde{I}$-open and fuzzy soft pre-$\widetilde{I}$-open set. Then, $f_A \widetilde{\subseteq} Fint(Fcl^*(f_A)) \widetilde{\subseteq} Fint(Fcl^*(Fcl^*(Fint(f_A)))) \widetilde{\subseteq} Fint(Fcl^*(Fint(f_A)))$. This shows that f_A is a fuzzy soft α-$\widetilde{I}$-open set.

Theorem 4.4. In any fuzzy soft ideal topological space $(X, \tau, E, \widetilde{I})$, the following properties are hold:

(1) Every fuzzy soft open set is a fuzzy soft β-$\widetilde{I}$-open.

(2) Every fuzzy soft β-$\widetilde{I}$-open set is a fuzzy soft β-open.

(3) Every fuzzy soft pre-$\widetilde{I}$-open set is a fuzzy soft β-$\widetilde{I}$-open.

(4) Every fuzzy soft semi-open set is a fuzzy soft β-$\widetilde{I}$-open.

(5) Every fuzzy soft $\widetilde{I}$-open set is a fuzzy soft β-$\widetilde{I}$-open.

Proof.

(1) Let $f_A \in \tau$. Then, we have $f_A = Fint(f_A) \widetilde{\subseteq} Fcl(Fint(f_A)) \widetilde{\subseteq} Fcl(Fint(f_A \sqcup f_A^*)) = Fcl(Fint(Fcl^*(f_A)))$. This

shows that $f_A \in FS\beta IO(X)_E$.
(2) Let $f_A \in FS\beta IO(X)_E$. Then, $f_A \widetilde{\sqsubseteq} Fcl(Fint(Fcl^*(f_A))) = Fcl(Fint\ (f_A \sqcup f_A^*)) \widetilde{\sqsubseteq} Fcl(Fint[f_A \sqcup Fcl(f_A)]) = Fcl(Fint(Fcl(f_A)))$. This shows that f_A is a fuzzy soft β-open.
(3) Let $f_A \in FSPIO(X)_E$. Then, $f_A \widetilde{\sqsubseteq} Fint(Fcl^*(f_A)) \widetilde{\sqsubseteq} Fcl(Fint(Fcl^*\ (f_A)))$.This shows that $f_A \in FS\beta IO(X)_E$.
(4) Let $f_A \in FSSIO(X)_E$. Then, $f_A \widetilde{\sqsubseteq} Fcl(Fint(f_A)) \widetilde{\sqsubseteq} Fcl(Fint(f_A \sqcup f_A^*)) = Fcl(Fint(Fcl^*(f_A)))$. This shows that $f_A \in FS\beta IO(X)_E$.
(5) Let $f_A \in FSIO(X)_E$. Then, $f_A \widetilde{\sqsubseteq} Fint(f_A^*) \widetilde{\sqsubseteq} Fcl(Fint(f_A^*)) \widetilde{\sqsubseteq} Fcl(Fint(f_A \sqcup f_A^*)) = Fcl(Fint(Fcl^*(f_A)))$. This shows that $f_A \in FS\beta IO(X)_E$.

The following example show that the converses of Theorem 4.4 is not true in general.

Example 4.4.
(1) Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $\widetilde{I} = \{\widetilde{0}_E\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.5}, b_{0.4}, c_{0.6}\})\}\}$. The fuzzy soft set $f_E = \{(e_1, \{a_{0.5}, b_{0.5}, c_{0.6}\})\}$ is a fuzzy soft β-$\widetilde{I}$-open set in $(X, \tau, \widetilde{I}, E)$ but $f_E \notin \tau$.
(2) Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $s_E = \{(e_1, \{a_{0.2}, b_{0.2}, c_{0.5}\})\}$, $u_E = \{(e_1, \{a_{0.8}, b_{0.7}, c_{0.4}\})\}$ are fuzzy soft sets in X. We put $\tau = \{\widetilde{1}_E, \widetilde{0}_E, s_E, u_E, s_E \sqcap u_E, s_E \sqcup u_E\}$ and we find that:

(i) If we take $\widetilde{I} = FSS(X)_E$, then $f_E = \{(e_2, \{a_{0.2}, b_{0.3}, c_{0.4}\})\}$ is a fuzzy soft β-open but $f_E \notin FS\beta IO(X)_E$.

(ii) If we take $\widetilde{I} = \{\widetilde{0}_E\}$, then $g_E = \{(e_1, \{a_{0.8}, b_{0.8}, c_{0.6}\})\}$ is a fuzzy soft β-$\widetilde{I}$-open set in X but $g_E \notin FSIO(X)_E$ and $g_E \notin FSPIO(X)_E$.

(iii) If we take $\widetilde{I} = \{\widetilde{0}_E\}$, then $h_E = \{(e_1, \{a_{0.1}, b_{0.9}, c_{0.8}\})\}$ is a fuzzy soft β-$\widetilde{I}$-open set in X but $h_E \notin FSSO(X)_E$ and so $h_E \notin FSSIO(X)_E$.

Proposition 4.1. Figure 1 shows the relation between different types of fuzzy soft open subsets of fuzzy soft topological space via fuzzy soft ideal.

$$
\begin{array}{ccccccccc}
FSO(X)_E & \rightarrow & FS\alpha IO(X)_E & \rightarrow & FSSIO(X)_E & \rightarrow & FS\beta IO(X)_E & \rightarrow & FS\beta O(X)_E \\
 & & & \searrow & & \nearrow & & \nearrow & \\
 & & FSIO(X)_E & \rightarrow & FSPIO(X)_E & \rightarrow & FSPO(X)_E & &
\end{array}
$$

5. Decompositions of some types of mappings on fuzzy soft topological spaces via fuzzy soft ideals

In this section, we introduce some types of continuity of fuzzy soft topological space with fuzzy soft ideal and study the relations between them.

Definition 5.1. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and (Y, σ, K) be a fuzzy soft topological space. Let $u : X \rightarrow Y$ and $p : E \rightarrow K$ be mappings. Let

$$f_{pu} : (X, \tau, E, \widetilde{I}) \rightarrow (Y, \sigma, K)$$

be a function. Then,
(1) f_{pu} is called fuzzy soft $\widetilde{I}$-continuous function if $f_{pu}^{-1}(g_B) \in FSIO(X)_E$ for every $g_B \in$
sigma.
(2) f_{pu} is called fuzzy soft pre-$\widetilde{I}$-continuous function if $f_{pu}^{-1}(g_B) \in FSPIO(X)_E$ for every $g_B \in \sigma$.
(3) f_{pu} is called fuzzy soft semi-$\widetilde{I}$-continuous function if $f_{pu}^{-1}(g_B) \in FSSIO(X)_E$ for every $g_B \in \sigma$.
(4) f_{pu} is called fuzzy soft α-$\widetilde{I}$-continuous function if $f_{pu}^{-1}(g_B) \in FS\alpha IO(X)_E$ for every $g_B \in \sigma$.
(5) f_{pu} is called fuzzy soft β-$\widetilde{I}$-continuous function if $f_{pu}^{-1}(g_B) \in FS\beta IO(X)_E$ for every $g_B \in \sigma$.

Theorem 5.1. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and (Y, σ, K) be a fuzzy soft topological space. Let $u : X \rightarrow Y$ and $p : E \rightarrow K$ be mappings. Let

$$f_{pu} : (X, \tau, E, \widetilde{I}) \rightarrow (Y, \sigma, K)$$

be a function. Then every fuzzy soft $\widetilde{I}$-continuous function is a fuzzy soft pre-$\widetilde{I}$-continuous function,
Proof. This is an immediate consequence of Theorem 4.2 (3).

Theorem 5.2. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and (Y, σ, K) be a fuzzy soft topological space. Let $u : X \to Y$ and $p : E \to K$ be mappings. Let

$$f_{pu} : (X, \tau, E, \widetilde{I}) \to (Y, \sigma, K)$$

be a function. Then every fuzzy soft pre-$\widetilde{I}$-continuous function is fuzzy soft pre-continuous.
Proof. This is an immediate consequence of Theorem 4.2 (2).

Theorem 5.3. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and (Y, σ, K) be a fuzzy soft topological space. Let $u : X \to Y$ and $p : E \to K$ be mappings. A fuzzy soft function $f_{pu} : (X, \tau, E, \widetilde{I}) \longrightarrow (Y, \sigma, K)$ is a fuzzy soft α-$\widetilde{I}$-continuous function $\Longleftrightarrow$ it is fuzzy soft semi-$\widetilde{I}$-continuous and fuzzy soft pre-$\widetilde{I}$-continuous.
Proof. This is an immediate consequence of Lemma 4.1.

Theorem 5.4. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and (Y, σ, K) be a fuzzy soft topological space. Let $u : X \to Y$ and $p : E \to K$ be mappings. Let

$$f_{pu} : (X, \tau, E, \widetilde{I}) \to (Y, \sigma, K)$$

be a function. Then:
(1) every fuzzy soft α-$\widetilde{I}$-continuous function is a fuzzy soft semi-$\widetilde{I}$-continuous.
(2) every fuzzy soft α-$\widetilde{I}$-continuous function is a fuzzy soft pre-$\widetilde{I}$-continuous.
Proof. This is an immediate consequence of Theorem 4.3 (3, 5).

Theorem 5.5. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and (Y, σ, K) be a fuzzy soft topological space. Let $u : X \to Y$ and $p : E \to K$ be mappings. Let

$$f_{pu} : (X, \tau, E, \widetilde{I}) \to (Y, \sigma, K)$$

be a function. Then:
(1) every fuzzy soft pre-$\widetilde{I}$-continuous function is a fuzzy soft β-$\widetilde{I}$-continuous.
(2) every fuzzy soft semi-$\widetilde{I}$-continuous function is a fuzzy soft β-$\widetilde{I}$-continuous.
Proof. This is an immediate consequence of Theorem 4.3 (4, 6).

Proposition 5.1. Let $(X, \tau, E, \widetilde{I})$ be a fuzzy soft ideal topological space and (Y, σ, K) be a fuzzy soft topological space. Let $u : X \to Y$ and $p : E \to K$ be mappings. Let

$$f_{pu} : (X, \tau, E, \widetilde{I}) \longrightarrow (Y, \sigma, K)$$

be a fuzzy soft function. Then, the following properties hold:
(1) If $\widetilde{I} = \{\widetilde{0}_E\}$, then f_{pu} is a fuzzy soft semi-$\widetilde{I}$-continuous (respectively, fuzzy soft pre-$\widetilde{I}$-continuous, fuzzy soft β-$\widetilde{I}$-continuous, fuzzy soft α-$\widetilde{I}$-continuous) function $\Longleftrightarrow$ f_{pu} is a fuzzy soft semi-continuous (respectively, fuzzy soft pre-continuous, fuzzy soft β-continuous, fuzzy soft α–continuous).
(2) If $\widetilde{I} = FSS(X)_E$, then f_{pu} is a fuzzy soft semi–$\widetilde{I}$-continuous (respectively, fuzzy soft pre–$\widetilde{I}$-continuous, fuzzy soft $\widetilde{I}$-continuous, fuzzy soft α-$\widetilde{I}$-continuous) function $\Longleftrightarrow$ f_{pu} is a fuzzy soft continuous.
Proof. This proof is obviously by using Proposition 3.1.

Proposition 5.2. The following diagram shows the decompositions of fuzzy soft mappings on fuzzy soft topological spaces via fuzzy soft ideals.

$$\begin{array}{ccccccc}
\text{Fuzzy soft } \alpha\text{–}\widetilde{I} - \text{cts} & \to & \text{Fuzzy soft semi} - \widetilde{I} - \text{cts} & \to & \text{Fuzzy soft } \beta\text{–}\widetilde{I} - \text{cts} & \searrow & \\
 & \searrow & & \nearrow & & & \text{Fuzzy soft } \beta\text{–cts} \\
\text{Fuzzy soft } \widetilde{I} - \text{cts} & \to & \text{Fuzzy soft pre} - \widetilde{I} - \text{cts} & \to & \text{Fuzzy soft pre} - \text{cts} & \nearrow &
\end{array}$$

6. Acknowledgment

The authors would like to thank Deanship of Scientific Research at Majmaah University for supporting this work under project number R-1441/75.

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