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Stability of Fractional Optimal Control Problems With Parameters in the Objective Function

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Abstract. In this article solvability and stability sets is given for the fractional optimal control problem which contains general parameters in the cost function. Further, the Kuhn-Tucker optimality necessary conditions are established in the presence fractional calculus. Finally, an illustrative example is given to clarify the development results in the paper.

1. Introduction

Optimal control problems (OCP) arise naturally in various areas of science, medicine, engineering, economic and mathematics. These problems has been done in the area of integer optimal control problems whose dynamics are described by conventional integer differential equations. Fractional optimal control problems (FOCP) are a subclass of optimal control problems whose dynamics are described by fractional differential equations. There are various definitions of fractional derivatives; the two most common types of fractional derivative are the Riemann-Liouville and Caputo derivatives, [1, 2, 9, 10, 14, 15, 18] and [19]. The paper is organized as follows: In Section 2, we summarize briefly some basic concepts of fractional derivatives as well as FOCP formulation presented by Agrawal [3–5] and [6] . Section 3 devoted to formulate a fractional optimal control problem and the necessary optimality conditions, [5] . In section 4, solvability and stability sets are defined as well as qualitative analysis for stability set is presented, [7, 8, 11–13, 16, 17] and [20] . Section 5 devoted to present an algorithm to determine the stability set. Finally, an illustrative example to clarify the developed results is presented.

2. Fractional Derivatives

In this section we collect the well-known notions and some results of fractional derivatives in the sense of Riemann-Liouville see [18].

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Definition 2.1 (Fractional derivative in the sense of Riemann-Liouville).

Let f be an integrable continuous function in the interval $[a, b]$. For $t \in [a, b]$, the left the Riemann Liouville fractional derivative [LRLFD] ${}_aD_t^\alpha f(t)$ and the right Riemann-Liouville fractional derivative [RRLFD] ${}_tD_b^\alpha f(t)$, of order α are defined by:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds \tag{1} $$

$$ {}_tD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} f(s)\, ds \tag{2} $$

where $n \in N$, $n - 1 < \alpha < n$, N is the set of natural numbers and Γ is the Euler gamma function defined by:

$$ \Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \tag{3} $$

also, the usual definitions of derivatives are obtained at α equals an integer.

Remark: If α is an integer, these derivatives are defined in the usual sense, that is,

$$ {}_aD_t^\alpha f(t) = (\frac{d}{dt})^n \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds \tag{4} $$

The main point in FOCPs is to find the optimal control $u(t)$ which minimizes the performance index

$$ J(u) = \int_0^1 F(x, u, t)\, dt \tag{5} $$

subject to the dynamic constraints system

$$ {}_0D_t^\alpha x = G(x, u, t) \tag{6} $$

and satisfying the terminal condition $x(0) = x_0$

Here t denotes the time, $x(t)$ and $u(t)$ are $n_x \times 1$ state and $n_u \times 1$ control vectors, f and g are two arbitrary functions.
When $\alpha = 1$, the problem (5) reduced to a standard optimal control problem. Also, the boundaries of integration, for simplicity, were taken as 0 and 1 and $0 < \alpha < 1$.

Agrawal [5] had studied a solution scheme for FOCP and developed the necessary conditions for the optimality of the FOCP. His idea is to consider a modified performance index as:

$$ \bar{J}(u) = \int_0^1 \Big[F(x, u, t) + \lambda \Big(G(x, u, t) - {}_0D_t^\alpha x \Big) \Big]\, dt \tag{7} $$

where λ is the Lagrange multiplier which is known as a costate or an adjoint variable. Using the techniques of fractional principles, the necessary equations for the FOCP can be written as:

$$ {}_0D_t^\alpha x = G(x, u, t) \tag{8} $$

$$ {}_tD_1^\alpha \lambda = \frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} \tag{9} $$

$$\frac{\partial F}{\partial u} + \lambda \frac{\partial G}{\partial u} = 0 \tag{10}$$

$$x(0) = x_0, \ \lambda(1) = 0 \tag{11}$$

Equations (8) to (11) represent the Euler-Lagrange equations for the FOCP. These equations give the necessary conditions for the optimality of the FOCP considered here.

3. Problem Formulation

Let us consider the fractional optimal control problem with general parameter ' ω ' in the objective function as follows:
$P(\omega)$:

$$min \quad J(u(t), \omega) = \int_0^1 F(x, u, \omega, t)\, dt \tag{12}$$

subject to

$$M = \left\{ u \in R^n : \ {}_0D_t^{\alpha} x = G(x, u, t) \ \right\}, \quad x(0) = x_0 \tag{13}$$

where $x(t) \in R^m$, $u(t) \in R^n$ are the state and control vectors respectively, t represents the time where $t \in [0, 1]$, ω is a vector parameter in R^k , F and G are arbitrary functions where

$$F : \ [0, 1] \times R^m \times R^n \times R^k \longrightarrow R \ \text{ is } \ C^1 \quad , \quad G : \ [0, 1] \times R^m \times R^n \longrightarrow R^p \ \text{ is } \ C^1 \ .$$

The necessary conditions for the optimality of the FOCP (12) can be obtained as follow:

Define a modified cost function

$$\bar{J}(u, \omega) = \int_0^1 \left[\ F(x, u, \omega, t) + \lambda \Big(G(x, u, t) - \ {}_0D_t^{\alpha} x \Big) \right] dt \tag{14}$$

The variation of $\bar{J}(u, \omega)$ will be

$$\delta\bar{J}(u, \omega) = \int_0^1 \left\{ \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial \omega}\delta\omega + \delta\lambda\Big(G(x, u, t) - \ {}_0D_t^{\alpha} x \Big) + \lambda\left(\frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial u}\delta u + \delta({}_0D_t^{\alpha} x) \right) \right\} dt \tag{15}$$

Use the formula for fractional integration by parts

$$\int_a^b \Big\{ \Big({}_aD_t^{\alpha} \ h(t) \Big) \, k(t) \Big\} dt = \int_a^b \Big\{ h(t) \ \Big({}_tD_b^{\alpha} \ k(t) \Big) \Big\} \ dt \tag{16}$$

then

$$\int_0^1 \Big\{ \lambda \ . \ \delta \ \Big({}_0D_t^{\alpha} x \Big) \Big\} dt = \int_0^1 \Big\{ \delta x \ . \ \Big({}_tD_1^{\alpha} \lambda \Big) \Big\} \ dt \tag{17}$$

so

$$\delta\bar{J}(u, \omega) = \int_0^1 \left\{ \left(\frac{\partial F}{\partial x} \ + \ \lambda\frac{\partial G}{\partial x} \ - \ {}_tD_1^{\alpha} \lambda \ \right) \delta x \ + \ \left(\frac{\partial F}{\partial u} \ + \ \lambda\frac{\partial G}{\partial u} \ \right) \delta u + \Big(G(x, u, t) \ - \ {}_0D_t^{\alpha} x \ \Big) \, \delta\lambda \ + \ \frac{\partial F}{\partial \omega}\delta\omega \right\} dt \tag{18}$$

To make a minimization for $\delta\bar{J}(u,\omega)$ (and hence a minimization for $J(u,\omega)$), the coefficients of δx , δu , $\delta\lambda$ and $\delta\omega$ must equals zero. So, we get equations (8) to (11) and

$$\frac{\partial F}{\partial \omega} = 0 \tag{19}$$

4. The solvability and stability sets

In this section, we give the definition of the solvability set and the stability set of the first kind for the problem $P(\omega)$. Qualitative and quantitative analysis of some basic notions in parametric optimization [16, 17] , such as the solvability set and the stability set of the first kind are defined and analyzed qualitatively and quantitatively for some classes of FOCP problems.

Definition 4.1 The solvability set of the problem $P(\omega)$ which is denoted by '**B**' is defined by:

$$\mathbf{B} = \left\{ \omega \in R^k \ : \ \min_{u\in M} J(u,\omega) \ \ exists \right\}$$

Definition 4.2 Suppose that $\omega^* \in \mathbf{B}$ with a corresponding optimal control u^* , then the stability set of the first kind of $P(\omega)$ is denoted by $\mathbf{S}(u^*)$, defined by:

$$\mathbf{S}(u^*) = \left\{ \omega \in \mathbf{B} \ : \ J(u^*,\omega) \ = \ \min_{u\in M} \int_0^1 F(x,u,\omega,t)\, dt \right\}$$

Theorem 4.1 If $J(u,\omega)$ is linear in ω then the set $\mathbf{S}(u^*)$ is convex.

Proof

Suppose that $\omega_1, \omega_2 \in \mathbf{S}(u^*)$, then

$$J(u^*,\omega_1) \ \le \ J(u,\omega_1) \tag{20}$$

$$J(u^*,\omega_2) \ \le \ J(u,\omega_2) \tag{21}$$

with

$$\left.\frac{\partial F}{\partial \omega}\right|_{\omega=\omega_1} = 0 \ , \ \left.\frac{\partial F}{\partial \omega}\right|_{\omega=\omega_2} = 0 \ , \ {}_0D_t^\alpha x = G(x,u^*,t) \ , \ x(0) = x_0 \tag{22}$$

Now, multiply both sides of (20) by σ where $0 \le \sigma \le 1$ and (21) by $1-\sigma$, Then from assumption that J is linear in ω we have

$$J\left(u^*,\sigma\omega_1\right) \ \le \ J\left(u,\sigma\omega_1\right) \tag{23}$$

$$J\left(u^*,(1-\sigma)\omega_2\right) \ \le \ J\left(u,(1-\sigma)\omega_2\right) \tag{24}$$

adding (23) and (24)

$$J\left(u^*,\sigma\omega_1 \ + \ (1-\sigma)\omega_2\right) \ \le \ J\left(u,\sigma\omega_1 \ + \ (1-\sigma)\omega_2\right) \tag{25}$$

with

$$\left.\frac{\partial F}{\partial \omega}\right|_{\omega=\sigma\omega_1 \ + \ (1-\sigma)\omega_2} = 0 \ , \ {}_0D_t^\alpha x = G(x,u^*,t) \ , \ x(0) = x_0 \tag{26}$$

Hence $\sigma\omega_1 \ + \ (1-\sigma)\omega_2 \in \mathbf{S}(u^*)$ and $\mathbf{S}(u^*)$ is convex.

Theorem 4.2 If $J(u,\omega)$ is continuous on the parametric space R^k and $\frac{\partial F}{\partial \omega}\Big|_{\omega=\omega_n}$ is continuous for any $n \in N$ then $\mathbf{S}(u^*)$ is closed.
Proof
Let $\{\omega_n\}$ be a sequence in $\mathbf{S}(u^*)$ such that $\{\omega_n\}$ converges to ω_0 as n tends to ∞ , then

$$J(u^*,\omega_n) \;\leq\; J(u,\omega_n) \tag{27}$$

with

$$_0D_t^{\alpha}x = G(x,u^*,t) \;\;,\;\; x(0) = x_0 \;\; and \quad \frac{\partial F}{\partial \omega}\bigg|_{\omega=\omega_n} = 0 \tag{28}$$

thus

$$\lim_{n\longrightarrow\infty} J(u^*,\omega_n) \;\leq\; \lim_{n\longrightarrow\infty} J(u,\omega_n) \tag{29}$$

Since $J(u,\omega_n)$ is continuous on R^k and $\frac{\partial F}{\partial \omega}\Big|_{\omega=\omega_n}$ is continuous for any $n \in N,$ then

$J(u^*, \lim\limits_{n\longrightarrow\infty}\omega_n) \;\leq\; J(u, \lim\limits_{n\longrightarrow\infty}\omega_n) \;\longrightarrow\; J(u^*,\omega_0) \;\leq\; J(u,\omega_0)$, therefore

$\omega_0 \in \mathbf{S}(u^*)$. This is the required result.

Theorem 4.3 If $J(u,\omega)$ is strictly convex with respect to u for each $\omega \in R^k$ then $\mathbf{S}(u_1^*) \cap \mathbf{S}(u_2^*) = \phi$, where u_1^* , u_2^* are different solutions corresponding to ω_1 , $\omega_2 \in \mathbf{B}$ respectively.
Proof
Let $\mathbf{S}(u_1^*) \cap \mathbf{S}(u_2^*) \neq \phi$, then

$\exists\;\bar{\omega} \in \mathbf{S}(u_1^*),\; \bar{\omega} \in \mathbf{S}(u_2^*)$ i.e. $\bar{\omega}$ corresponds to both solutions u_1^* , u_2^* ,

$\bar{\omega} \in \mathbf{S}(u_1^*)$ then $J(u_1^*,\bar{\omega}) = \min\limits_{u\in M} \int_0^1 F(x,u,\bar{\omega},t)\,dt,$ also

$\bar{\omega} \in \mathbf{S}(u_2^*)$ then $J(u_2^*,\bar{\omega}) = \min\limits_{u\in M} \int_0^1 F(x,u,\bar{\omega},t)\,dt$

i.e. $J(u_1^*,\bar{\omega}) = J(u_2^*,\bar{\omega})$.

Since $J(u,\omega)$ is strictly convex with respect to u for each $\omega \in \mathbf{B}$, then

$$J\left[\sigma u_1^* + (1-\sigma)u_2^*, \bar{\omega}\right] < \sigma J(u_1^*,\bar{\omega}) + (1-\sigma)J(u_2^*,\bar{\omega}) \;\;,\;\; 0 < \sigma < 1,$$

hence

$J\left[\sigma u_1^* + (1-\sigma)u_2^*, \bar{\omega}\right] < J(u_1^*,\bar{\omega})$. This is a contradiction.

Thus $\quad \mathbf{S}(u_1^*) \cap \mathbf{S}(u_2^*) = \phi.$

5. Determination the stability set of the first kind

To determine the stability set of the first kind will be follow the following steps:

STEP 1: Choose any $\omega^* \in \mathbf{B}$ in the problem (12).
STEP 2: For any α solve the system (8)-(11) to obtain the corresponding optimal control u^* , state x^* and λ^* .
STEP 3: Consider any ω in the problem (12) and construct the system (8)-(11), (19).
STEP 4: Solve the system (8)-(11), (19) for (u^*, x^*, λ^*) to obtain ω for any α .

Illustrative Example

min

$$J(u,\omega) = \int_0^1 \left[3(x+\omega)^2 + 2(u-\omega)^2\right] dt \tag{30}$$

subject to

$$_0D_t^{\alpha} x = G(x,u,t) = t + x + u \ , \quad x(0) = x_0 \tag{31}$$

solving the problem (30) subject to (31) for certain $\omega = \omega^*$ i.e.

min

$$J(u,\omega^*) = \int_0^1 \left[3(x+\omega^*)^2 + 2(u-\omega^*)^2\right] dt \tag{32}$$

The optimality conditions for (32) are obtained from (8) to (10) to be

$$_0D_t^{\alpha} x(t) = t + x + u \tag{33}$$

$$_tD_1^{\alpha} \lambda(t) = 6x + 6\omega^* + \lambda \tag{34}$$

$$4u - 4\omega^* + \lambda = 0 \tag{35}$$

from (35)

$$u = \omega^* - \frac{\lambda}{4} \tag{36}$$

substitute with (36) in (33), hence

$$_0D_t^{\alpha} x(t) = t + x + \omega^* - \frac{\lambda}{4} \tag{37}$$

Now, apply a fractional integration of order α on (34) and (37), see [10], we have the following system

$$x(t) = x_0 + {}_0I_t^{\alpha}\left[t + x + \omega^* - \frac{\lambda}{4}\right] \tag{38}$$

$$\lambda(t) = -\ {}_1I_t^{\alpha}\left[6x + \lambda + 6\omega^*\right] \tag{39}$$

we can find approximated solution by applying Picard method on (38) and (39) to be

$$x_{n+1}(t) = x_0 + {}_0I_t^{\alpha}\left[t + x_n + \omega^* - \frac{\lambda_n}{4}\right], \quad \lambda_0 = 0 \tag{40}$$

$$\lambda_{n+1}(t) = -\ {}_1I_t^{\alpha}\left[6x_n + \lambda_n + 6\omega^*\right] \tag{41}$$

For $n = 0$, we have:

$$x_1 = x_0 + {}_0I_t^{\alpha}\left[t + x_0 + \omega^* - \frac{\lambda_0}{4}\right], \quad \lambda_0 = 0 \tag{42}$$

$$x_1 = x_1^* = x_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}\,(\omega^* + x_0)}{\Gamma(\alpha+1)} \tag{43}$$

$$\lambda_1 = -\ {}_1I_t^{\alpha}\left[6x_0 + \lambda_0 + 6\omega^*\right] \tag{44}$$

$$\lambda_1 = \lambda_1^* = \frac{-6\,(t-1)^{\alpha}\,(\omega^* + x_0)}{\Gamma(\alpha+1)} \tag{45}$$

For $n = 1$, we have:

$$x_2 = x_0 + {}_0I_t^{\alpha}\left[t + x_1 + \omega^* - \frac{\lambda_1}{4}\right] \tag{46}$$

$$x_2 = x_2^* = x_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}\,(\omega^* + x_0)}{\Gamma(\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{2\alpha}\,(\omega^* + x_0)}{\Gamma(2\alpha+1)} + \frac{3\,(\omega^* + x_0)\,(t-1)^{2\alpha}}{2\Gamma(2\alpha+1)} \tag{47}$$

and

$$\lambda_2 = -\ {}_1I_t^{\alpha}\left[6x_1 + \lambda_1 + 6\omega^*\right] \tag{48}$$

$$\lambda_2 = -6\,{}_1I_t^{\alpha}\,[x_1] - {}_1I_t^{\alpha}\,[\lambda_1] - 6\,{}_1I_t^{\alpha}\,[\omega^*] \tag{49}$$

from (43), we have

$${}_1I_t^{\alpha}\,[x_1] = {}_1I_t^{\alpha}\left[x_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}\,(\omega^* + x_0)}{\Gamma(\alpha+1)}\right] \tag{50}$$

use

$$I_a\,t^c = \frac{t^{c+\alpha}}{\Gamma(\alpha)}\left(1 - \frac{a}{t}\right)^{\alpha} \sum_{n=0}^{\infty}\left[\frac{(-c)_n}{n!(\alpha+n)}\left(1 - \frac{a}{t}\right)^{n}\right] \tag{51}$$

where $0 < t < 1$, $0 \leqslant a \leqslant 1$, $(-c)_n$ = Pochhammer symbol = $\frac{\Gamma(-c+n)}{\Gamma(-c)}$
so, ${}_1I_t^{\alpha}\,[x_1]$ will be divided into

$${}_1I_t^{\alpha}\left[x_0\,(t-1)^0\right] = \frac{x_0\,\Gamma(1+0)\,(t-1)^{0+\alpha}}{\Gamma(\alpha+1)} = \frac{x_0\,(t-1)^{\alpha}}{\Gamma(\alpha+1)} \tag{52}$$

$${}_1I_t^{\alpha}\left[\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right] = \frac{t^{2\alpha+1}}{\Gamma(\alpha)\,\Gamma(\alpha+2)}\left(1 - \frac{1}{t}\right)^{\alpha} \sum_{n=0}^{\infty}\left[\frac{(-1-\alpha)_n}{n!(\alpha+n)}\left(1 - \frac{1}{t}\right)^{n}\right] \tag{53}$$

$${}_1I_t^{\alpha}\left[\frac{t^{\alpha}\,(\omega^* + x_0)}{\Gamma(\alpha+1)}\right] = \frac{(\omega^* + x_0)\,t^{2\alpha}}{\Gamma(\alpha)\,\Gamma(\alpha+1)}\left(1 - \frac{1}{t}\right)^{\alpha} \sum_{n=0}^{\infty}\left[\frac{(-\alpha)_n}{n!(\alpha+n)}\left(1 - \frac{1}{t}\right)^{n}\right] \tag{54}$$

$${}_1I_t^{\alpha}[\lambda_1] = \frac{-6\,(\omega^* + x_0)\,(t-1)^{2\alpha}}{\Gamma(2\alpha+1)} \tag{55}$$

$$-6\,{}_1I_t^{\alpha}\,[\omega^*] = \frac{-6\,\omega^*(t-1)^{\alpha}}{\Gamma(\alpha+1)} \tag{56}$$

hence,

$$\lambda_2 = \lambda_2^* = -6\left[\frac{x_0\,(t-1)^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(\alpha)\,\Gamma(\alpha+2)}\left(1-\frac{1}{t}\right)^\alpha \sum_{n=0}^{\infty}\left[\frac{(-1-\alpha)_n}{n!(\alpha+n)}\left(1-\frac{1}{t}\right)^n\right] + \frac{(\omega^*+x_0)\,t^{2\alpha}}{\Gamma(\alpha)\,\Gamma(\alpha+1)}\left(1-\frac{1}{t}\right)^\alpha \sum_{n=0}^{\infty}\left[\frac{(-\alpha)_n}{n!(\alpha+n)}\left(1-\frac{1}{t}\right)^n\right] - \frac{(\omega^*+x_0)\,(t-1)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\omega^*(t-1)^\alpha}{\Gamma(\alpha+1)}\right] \tag{57}$$

now, from (36) we can get

$$u_1 = u_1^* = \omega^* - \frac{\lambda_1^*}{4} \tag{58}$$

$$u_1^* = \omega^*\left[1 + \frac{3\,(t-1)^\alpha}{2\,\Gamma(\alpha+1}\right] + \frac{3\,(t-1)^\alpha}{2\,\Gamma(\alpha+1)}\,x_0 \tag{59}$$

and

$$u_2 = u_2^* = \omega^* - \frac{\lambda_2^*}{4} \tag{60}$$

$$u_2^* = \omega^* + \frac{3}{2}\left[\frac{x_0\,(t-1)^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(\alpha)\,\Gamma(\alpha+2)}\left(1-\frac{1}{t}\right)^\alpha \sum_{n=0}^{\infty}\left[\frac{(-1-\alpha)_n}{n!(\alpha+n)}\left(1-\frac{1}{t}\right)^n\right] + \frac{(\omega^*+x_0)\,t^{2\alpha}}{\Gamma(\alpha)\,\Gamma(\alpha+1)}\left(1-\frac{1}{t}\right)^\alpha \sum_{n=0}^{\infty}\left[\frac{(-\alpha)_n}{n!(\alpha+n)}\left(1-\frac{1}{t}\right)^n\right] - \frac{(\omega^*+x_0)\,(t-1)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\omega^*(t-1)^\alpha}{\Gamma(\alpha+1)}\right] \tag{61}$$

For (30), the conditions for optimality are (8)-(11) and (19) where

$$\frac{\partial F}{\partial x} = 6x + 6\omega\ , \quad \frac{\partial F}{\partial u} = 4u - 4\omega, \quad \frac{\partial F}{\partial \omega} = 10\omega - 4u, \quad \frac{\partial G}{\partial x} = 1, \quad \frac{\partial G}{\partial u} = 1 \tag{62}$$

hence,

$${}_0D_t^\alpha\; x(t) = t + x + u \tag{63}$$

$${}_tD_1^\alpha\; \lambda(t) = 6x + 6\omega + \lambda \tag{64}$$

$$4u - 4\omega + \lambda = 0 \tag{65}$$

$$10\omega\ -\ 4u = 0 \tag{66}$$

from (66)

$$\omega = 0.4\; u \tag{67}$$

put $u = u_1^*$ from (59) in (67), then

$$\mathbf{S}_1(u) = 0.4\;\omega^*\left[1 + \frac{3\,(t-1)^\alpha}{2\,\Gamma(\alpha+1}\right] + \frac{0.6\,(t-1)^\alpha}{\Gamma(\alpha+1)}\,x_0 \tag{68}$$

also, put $u = u_2^*$ from (61) in (67), then

$$\mathbf{S}_2(u) = 0.4\;\omega^* + \frac{3}{5}\left[\frac{x_0\,(t-1)^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(\alpha)\,\Gamma(\alpha+2)}\left(1-\frac{1}{t}\right)^\alpha \sum_{n=0}^{\infty}\left[\frac{(-1-\alpha)_n}{n!(\alpha+n)}\left(1-\frac{1}{t}\right)^n\right] + \frac{(\omega^*+x_0)\,t^{2\alpha}}{\Gamma(\alpha)\,\Gamma(\alpha+1)}\left(1-\frac{1}{t}\right)^\alpha \sum_{n=0}^{\infty}\left[\frac{(-\alpha)_n}{n!(\alpha+n)}\left(1-\frac{1}{t}\right)^n\right] - \frac{(\omega^*+x_0)\,(t-1)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\omega^*(t-1)^\alpha}{\Gamma(\alpha+1)}\right] \tag{69}$$

If $x(t)$ is continuous in (t, λ) and $\lambda(t)$ is continuous in (t, x) and they have continuous first partial derivatives with respect to x, λ then the sequence x_{n+1}, λ_{n+1} yielding from Picard approximation for the system (40), (41) will lead to a convergent stability set for $0 < \alpha < 1$.

6. conclusion

In this paper, we have presented a parametric study for the fractional optimal control problem in the sense of Riemann-Liouville with general parameters in the cost function such that the optimal control for the considered problem remains steady regardless any variation in the state variable.

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