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Analytical Techniques for Solving the Equation Governing the Unsteady Flow of a Polytropic Gas With Time-Fractional Derivative

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Abstract. In this work, some analytical techniques viz. homotopy perturbation method, new iterative method and integral iterative method are used to solve nonlinear fractional differential equations such as the equation governing the unsteady flow of a polytropic gas with time-fractional derivative. Comparisons are made between the considered techniques and also between their results. The obtained results reveal that these techniques are very simple and effective and give the solution in series form which in closed form gives the exact solution also, reveal that the integral iterative technique is simpler and shorter in its computational procedures and time than the other techniques.

1. Introduction

Recently, many important phenomena occurring in various fields of applied sciences are frequently modeled through nonlinear fractional differential equations. However, it is still very difficult to obtain closed-form solutions for most models of real-life problems. A broad class of analytical and numerical methods were used to handle such problems such as variational iteration method [1–6], Adomian decomposition method [7–10], homotopy perturbation method [11–17], new iterative method [18–25] and integral iterative method [26, 27]. It is worth mentioning that the new iterative, homotopy perturbation and integral iterative methods are applied without any discretization, restrictive assumption or transformation and are free from round off errors. Also, the three methods are applied without calculating Adomian polynomials or Lagrange multiplier values which need much computational time. All these advantages simplify and reduce the computational procedures and time and make these methods more suitable and convenient for solving fractional differential equations.

The motivation of this work, is to extend the application of the new iterative method, the homotopy perturbation method and the integral iterative method to solve fractional differential equations, specially the equation governing the unsteady flow of a polytropic gas with time-fractional derivative.

2. Basic definitions of fractional calculus

In this section, we mention some basic definitions of fractional calculus which are used in this work.

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Definition 2.1. *The Riemann-Liouvill fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu$, $\mu \geq 1$ is defined as [28]:*

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \qquad t > 0, \tag{1}$$

$$I_t^0 f(t) = f(t). \tag{2}$$

For the Riemann-Liouvill fractional integral, we have:

$$I_t^\alpha t^\nu = \frac{\Gamma(\nu+1) t^{\nu+\alpha}}{\Gamma(\nu+1+\alpha)}. \tag{3}$$

Definition 2.2. *The fractional derivative of $f(t)$ in the Caputo sense is defined as [29]:*

$$D_t^\alpha f(t) = I_t^{m-\alpha} D_t^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \tag{4}$$

for $m-1 < \alpha \leq m$, $m \in N$, $t > 0$. For the Caputo fractional derivative, we have:

$$D_t^\alpha t^\nu = \frac{\Gamma(\nu+1) t^{\nu-\alpha}}{\Gamma(\nu+1-\alpha)}, \qquad \upsilon \geq \alpha. \tag{5}$$

For the Riemann-Liouville fractional integral and the Caputo fractional derivative, we have:

$$I_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+) \frac{t^k}{k!}, \qquad m-1 < \alpha \leq m,\ m \in N. \tag{6}$$

3. Analytical techniques.

In this section, we discuss the analysis and algorithms of the considered techniques.

3.1 Homotopy perturbation method (HPM).

3.1.1 Analysis of the method.

To illustrate the basic idea of this method, proposed first by He, consider the following general nonlinear differential equation [11–17]:

$$L(u) + N(u) = g(t), \qquad t \in \Omega, \tag{7a}$$

with the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0, \qquad t \in \Gamma, \tag{7b}$$

where L is a linear operator, N is a nonlinear operator, B is a boundary operator, $g(t)$ is a known analytic function and Γ is the boundary of the domain Ω.

By the homotopy perturbation technique, He construct a homotopy:
$v(t, p) : \Omega \times [0, 1] \to R$
which satisfies:

$$\mathcal{H}(v, p) = (1-p)\,[L(v) - L(u_0)] + p[L(v) + N(v) - g(t)] = 0, \tag{8}$$

or

$$\mathcal{H}(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - g(t)] = 0, \tag{9}$$

where $t \in \Omega$, $p \in [0,1]$ is an impeding parameter and u_0 is an initial approximation which satisfies the boundary conditions.
Obviously, from Eqs. (8) and (9), we have:

$$\mathcal{H}(v,0) = L(v) - L(u_0) = 0, \qquad \mathcal{H}(v,1) = L(v) + N(v) - g(t) = 0. \tag{10}$$

The changing process of p from zero to unity is just that of $v(t,p)$ from $u_0(t)$ to $u(t)$. In topology, this is called deformation, $L(v) - L(u_0)$ and $L(v) + N(v) - g(t)$ are called homotopic. The basic assumption is that the solution of Eqs. (8) and (9) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2v_2 + \dots . \tag{11}$$

The approximate solution of Eq. (7), therefore, can be readily obtained:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \dots . \tag{12}$$

The convergence of the series (12) has been proved in [16, 17].

3.1.2 Reliable algorithm of HPM.

To illustrate the reliable algorithm of the HPM, we consider the following general nonlinear fractional differential equation of any order $\alpha > 0$:

$$D_t^\alpha u(t) = L(u,du) + N(u,du) + g(t), \qquad m-1 < \alpha \le m, \tag{13}$$

where L and N are linear and nonlinear operators (functions) of u and du (derivatives of $u$with respect to t) and $g$is a known analytic function, subject to the initial conditions:

$$\frac{d^k}{dt^k}u(0) = h_k, \qquad k = 0,1,2,\dots,m-1. \tag{14}$$

In view of the homotopy technique, we can construct the following homotopy:

$$D_t^\alpha u(t) - L(u,du) - g(t) = p[N(u,du)], \tag{15}$$

or

$$D_t^\alpha u(t) - g(t) = p[L(u,du) + N(u,du)], \tag{16}$$

where $p \in [0,1]$. The homotopy parameter p always changes from zero to unity. When $p = 0$, Eq. (15) becomes the linearized equation:

$$D_t^\alpha u(t) = L(u,du) + g(t), \tag{17}$$

and Eq. (16) becomes the linearized equation:

$$D_t^\alpha u(t) = g(t), \tag{18}$$

and when $p = 1$, Eq. (15) or Eq. (16) turns out to be the original Eq. (13). The basic assumption is that the solution of Eq. (15) or Eq. (16) can be written as a power series in p:

$$u = u_0 + pu_1 + p^2u_2 + \dots . \tag{19}$$

Finally, we approximate the solution $u(t)$ by:

$$u(t) = \sum_{i=0}^{\infty} u_i(t). \tag{20}$$

3.2 New iterative method (NIM).

3.2.1 Analysis of the method.

To illustrate the basic idea of this method, proposed first by Gejji and Jafari, consider the following general functional equation [18–25]:

$$u(t) = f(t) + N(u(t)), \tag{21}$$

where N is a nonlinear operator from a Banach space $B \to B$ and $f(t)$ is a known function (element) of a Banach space B. We are looking for a solution $u(t)$ of Eq. (21) having the series form:

$$u(t) = \sum_{i=0}^{\infty} u_i(t). \tag{22}$$

The nonlinear operator N can be decomposed as:

$$N\Big(\sum_{i=0}^{\infty} u_i\Big) = N(u_0) + \sum_{i=1}^{\infty}\Big\{N\Big(\sum_{j=0}^{i} u_j\Big) - N\Big(\sum_{j=0}^{i-1} u_j\Big)\Big\}. \tag{23}$$

From Eqs. (refeq22) and (23), Eq. (21) is equivalent to:

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\Big\{N\Big(\sum_{j=0}^{i} u_j\Big) - N\Big(\sum_{j=0}^{i-1} u_j\Big)\Big\}. \tag{24}$$

The required solution for Eq. (21) can be obtained recurrencely from the recurrence relation:

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0), \\ u_{r+1} = N\Big(\sum\limits_{i=0}^{r} u_i\Big) - N\Big(\sum\limits_{i=0}^{r-1} u_i\Big), & r = 1, 2, \dots . \end{cases} \tag{25}$$

Then

$$\sum_{i=1}^{r+1} u_i = N\Big(\sum_{i=0}^{r} u_i), \qquad r = 0, 1, 2, \dots, \tag{26}$$

and

$$\sum_{i=0}^{\infty} u_i = f + N\Big(\sum_{i=0}^{\infty} u_i\Big). \tag{27}$$

The r-term approximate solution of Eq. (21) is given by $u(t) = \sum\limits_{i=0}^{r-1} u_i$. If N is a contraction, i.e. $\| N(x) - N(y) \| \leq k \| x - y \|$, $0 < k < 1$, then:

$$\| u_{r+1} \| \leq k^{r+1} \| u_0 \|, \qquad r = 0, 1, 2, \dots \tag{28}$$

and the series $\sum_{i=0}^{\infty} u_i$ absolutely and uniformly converges to a solution of Eq. (21) [30] which is unique in view of the Banach fixed point theorem [eq31]. The convergence of the NIM has been proved in [18, 25].

3.2.2 Solving Fractional differential equations by NIM.

To illustrate how we can solve any fractional differential equation of arbitrary order $\alpha > 0$ by NIM, we consider the general fractional differential equation:

3.2.3 Solving Fractional differential equations by NIM.

To illustrate how we can solve any fractional differential equation of arbitrary order $\alpha > 0$ by NIM, we consider the general fractional differential equation:

$$D_t^{\alpha}u(t) = L(u) + K(u) + g(t), \qquad m-1 < \alpha \leq m, \ m \in \mathbb{N}, \tag{29a}$$

$$\frac{d^k}{\partial t^k}u(0) = h_k, \qquad k = 0, 1, 2, ..., m-1, \tag{29b}$$

where L is a linear operator, K is a nonlinear operator, $g$is a nonhomogeneous term. In view of the fractional integral operators, the initial value problem (29) is equivalent to the integral equation:

$$u(t) = \sum_{k=0}^{m-1} h_k \cdot \frac{t^k}{k!} + I_t^{\alpha}[g(t)] + I_t^{\alpha}[L(u) + K(u)] = f + N(u), \tag{30}$$

where $f = \sum_{k=0}^{m-1} h_k \cdot \frac{t^k}{k!} + I_t^{\alpha}[g(t)], N(u) = I_t^{\alpha}[L(u) + K(u)]$. The required solution $u(t)$ for Eq. (30) and hence for Eq. (29) is obtained recurrencely from the recurrence relation (25).

3.3 Integral iterative method (IIM).

3.3.1 Analysis of the method.

The IIM is a new iterative method depends explicitly on the integral operator; the inverse of the differential operator in the problem under consideration [26, 27]. To illustrate the basic idea of this method, consider the following general fractional differential equation of arbitrary order $\alpha > 0$:

$$D_t^{\alpha}u(t) = L(u) + K(u) + g(t), \qquad m-1 < \alpha \leq m, \ m \in N, \tag{31a}$$

$withtheinitialconditions:$

$$\frac{d^k}{dt^k}u(0) = h_k, \qquad k = 0, 1, 2, ..., m-1, \tag{31b}$$

where D_t^{α} is the fractional differential operator of order α with respect to t, L, K are linear and nonlinear operators of orders less than α and $g(t)$ is a nonhomogeneous term. Applying the integral operator with respect to t,denoted by I_t, α times to both sides of Eq. (31a), taking in account the given initial conditions (31b), we can have the following integral equation:

$$u(t) = \sum_{k=0}^{m-1} h_k \cdot \frac{t^k}{k!} + I_t^{\alpha}[g(t)] + I_t^{\alpha}[L(u) + K(u)] = f + N(u), \tag{32}$$

where: $f = \sum_{k=0}^{m-1} h_k . \frac{t^k}{k!} + I_t^{\alpha}[g(t)]$, and $N(u) = I_t^{\alpha}[L(u) + K(u)]$.

The required solution $u(t)$ for Eq (eq32) which is also the solution for Eq. (31) can be obtained recurrencely from the recurrence relation:

$$\begin{cases} u_0 = f, \\ u_{r+1} = u_0 + N(u_r), \qquad r = 0, 1, 2, \ldots, \end{cases} \tag{33}$$

where $u(t) = \lim_{r\to\infty} u_r$.

The IIM may be considered as a new approach for Picard method (PM), where in PM the terms f and $N(u)$ in Eq. (32) take the forms: $f = \sum_{k=0}^{m-1} h_k \cdot \frac{t^k}{k!}$ and $N(u) = I_t^{\alpha}[g(t) + L(u) + K(u)]$. By this change, the r-order term approximate solution for Eq. (31) by IIM is the same r-term approximate solution for it by NIM but without calculating the values: $N\left(\sum_{i=0}^{r} u_i\right) - N\left(\sum_{i=0}^{r-1} u_i\right)$, $r = 1, 2, ...$ and the r-term approximate solution from the relation $u(t) = \sum_{i=0}^{r-1} u_i$, $r = 1, 2, ...$ which reduce and simplify the computational procedures and time. Also, by this change, there is no need to calculate the integral of the given function $g(t)$ for every iteration r, $r = 1, 2, ...$ as done in PM which also reduce and simplify the computational procedures and time. So, IIM is more convenient for solving fractional differential equations.

3.3.2 Existence and convergence analysis of IIM.

In this subsection, we prove the existence and uniqueness of the solution and convergence of the IIM by using the following definitions and assumptions [32].

Definition 3.1. *Let $X = C[a, b]$ be the set of all continuous functions defined on the closed interval $[a, b]$. The distance function between an arbitrary functions $u(t), v(t) \in X$ is defined in the form $D(u(t), v(t)) = \max_{a \le t \le b} \mid u(t) - v(t) \mid$.*

It is known that (X, D) is a complete metric space and the following properties are well known:

$$D(u, v) = 0, \quad \text{if and only if} \quad u = v \forall u, v \in X, \tag{34}$$

$$D(u + w, v + w) = D(u, v), \quad \forall u, v, w \in X, \tag{35}$$

$$D(u + v, w + e) \le D(u, w) + D(v, e), \quad \forall u, v, w, e \in X. \tag{36}$$

Consider $g(t)$ is bounded for all $t \in [a, b]$. Also, we suppose the linear and nonlinear operators $L(u)$ and $K(u)$ satisfy Lipschitz conditions with:

$$D(L(t, u(x)), L(t, u(y))) \le M_1 D(u(x), u(y)), \quad M_1 > 0, \tag{37}$$

$$D(K(t, u(x)), K(t, u(y))) \le M_2 D(u(x), u(y)), \quad M_2 > 0. \tag{38}$$

Let $m = M_1 + M_2$.

Theorem 3.2. *Let $0 < m < 1$, then Eq. (31) have a unique solution when $u(t)$ is differentiable of order $\alpha > 0$ with respect to t.*

Proof. Let $u(t)$ and $u^*(t)$be two different solutions for Eq. (31), then:

$$\begin{aligned}
D(u(t), u^*(t)\Big) &= D\Big(\sum_{k=0}^{m-1} h_k \cdot \frac{t^k}{k!} + I_t^\alpha[g(t)] + I_t^\alpha[L(t, u(t)) + K(t, u(t))], \\
&\qquad \sum_{k=0}^{m-1} h_k \cdot \frac{t^k}{k!} + I_t^\alpha[g(t)] + I_t^\alpha[L(t, u^*(t)) + K(t, u^*(t))]\Big) \\
&= D(I_t^\alpha[L(t, u(t)) + K(t, u(t))], I_t^\alpha[L(t, u^*(t)) + K(t, u^*(t))]) \\
&\le D(I_t^\alpha[L(t, u(t))], I_t^\alpha[L(t, u^*(t))]) + D(I_t^\alpha[K(t, u(t))], I_t^\alpha[K(t, u^*(t))]) \\
&\le (M_1 + M_2) D(u(t), u^*(t)) \\
&= m D(u(t), u^*(t)).
\end{aligned}$$

From which we get $(1 - m)D(u(t), u^*(t)) \le 0$. Since, $0 < m < 1$, then $D(u(t), u^*(t)) = 0$. Implies $u(t) = u^*(t)$ and completes the proof. □

Theorem 3.3. *The solution $u_r(t)$ obtained from (32) using IIM (33) converges to the exact solution $u(t)$ of the problem (31) when $0 < m < 1$.*

Proof.

$$
\begin{aligned}
D(u_{r+1}(t),u(t)) &= D\Big(\sum_{k=0}^{m-1} h_k\cdot\frac{t^k}{k!} + I_t^{\alpha}[g(t)] + I_t^{\alpha}[L(t,u_r(t)) + K(t,u_r(t))],\\
&\quad \sum_{k=0}^{m-1} h_k\cdot\frac{t^k}{k!} + I_t^{\alpha}[g(t)] + I_t^{\alpha}[L(t,u(t)) + K(t,u(t))]\Big)\\
&= D(I_t^{\alpha}[L(t,u_r(t)) + K(t,u_r(t))], I_t^{\alpha}[L(t,u(t)) + K(t,u(t))])\\
&\le D(I_t^{\alpha}[L(t,u_r(t))], I_t^{\alpha}[L(t,u(t))]) + D(I_t^{\alpha}[K(t,u_r(t))], I_t^{\alpha}[K(t,u(t))])\\
&\le (M_1+M_2)D(u_r(t),u(t)) = mD(u_r(t),u(t)) .
\end{aligned}
$$

Since, $0<m<1$, then $D(u_r(t),u(t))\to 0$ as $r\to\infty$. Therefore, $u_r(t)\to u(t)$ and completes the proof. □

3.4 The advantages of IIM.

The advantages of the IIM over both the NIM and HPM are that in the IIM there is no need to:

(1) calculate the r-term approximate solution from the relation $u(t) = \sum_{i=0}^{r-1} u_i$ as done in both NIM and HPM,
(2) equate the terms of equal powers of the embedding parameter p as done in HPM,
(3) calculate the values $N\Big(\sum_{i=0}^{r} u_i\Big) - N\Big(\sum_{i=0}^{r-1} u_i\Big)$, $r = 1,2,...,$ as done in NIM.

These advantages make the IIM simpler in its computational procedures and shorter in its computational time than NIM and HPM. Also, the three methods are applied without calculating: Adomian polynomials as done in the Adomian decomposition method or Lagrange multiplier value as don in the variational iteration method. Moreover, these methods can be used without linearization or small perturbation as done in the perturbation methods, so these methods are more convenient and effective for solving fractional differential equations as shown in the following section

4. Applications

To illustrate the effectiveness of the mentioned methods, two test problems are carried out in this section.

Problem 4.1. *Consider the nonlinear fractional differential equation:*

$$D_t^{\alpha}u(x) - u^2(x) + 1 = 0, \qquad u(0) = 0, \qquad 0<\alpha\le 1. \tag{39}$$

In view of the HPM, the homotopy for Eq. (39), according to Eq. (16), takes the form:

$$D_t^{\alpha}u(x) + 1 = p[u^2(x)]. \tag{40}$$

Substituting (19) and the initial value $u(0) = 0$ into (40) and equating the terms of equal powers of p, we obtain the following set of fractional differential equations:

$$
\begin{aligned}
&p^0 : D_t^{\alpha}u_0 = -1, && u_0(0) = 0,\\
&p^1 : D_t^{\alpha}u_1 = u_0^2, && u_1(0) = 0,\\
&p^2 : D_t^{\alpha}u_2 = 2u_0u_1, && u_2(0) = 0,\\
&p^3 : D_t^{\alpha}u_3 = 2u_0u_2 + u_1^2, && u_3(0) = 0,\\
&\vdots
\end{aligned}
$$

The solution of the above set of equations gives the following first few components of the homotopy perturbation solution for Eq. (39):

$$
\begin{aligned}
u_0(x) &= -\frac{x^{\alpha}}{\Gamma(1+\alpha)},\\
u_1(x) &= \frac{\Gamma(1+2\alpha)x^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)},\\
u_2(x) &= -\frac{2\Gamma(1+2\alpha)\Gamma(1+4\alpha)x^{5\alpha}}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)},\\
u_3(x) &= \frac{2\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)x^{7\alpha}}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)} + \frac{\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)x^{7\alpha}}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)},\\
&\vdots
\end{aligned}
$$

and so on. In the same manner the rest of components can be obtained. The 4-term approximate solution for Eq. (39) by HPM is given by:

$$
\begin{aligned}
u(x) = \sum_{i=0}^{3} u_i(x) &= -\frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\Gamma(1+2\alpha)x^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} \frac{2\Gamma(1+2\alpha)\Gamma(1+4\alpha)x^{5\alpha}}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)},\\
&+ \left(\frac{2\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)} + \frac{\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)}\right)x^{7\alpha}.
\end{aligned} \tag{41}
$$

In view of the NIM, according to Eq. (30), the initial value problem (39) is equivalent to the fractional integral equation:

$$
u(x) = -\frac{x^{\alpha}}{\Gamma(1+\alpha)} + I_t^{\alpha}[u^2(x)]. \tag{42}
$$

Let $N(u) = I_t^{\alpha}[u^2(x)]$. According to (25), we have the following first few components of the new iterative solution for Eq. (39):

$$
\begin{aligned}
u_0(x) &= -\frac{x^{\alpha}}{\Gamma(1+\alpha)},\\
u_1(x) &= N(u_0) = \frac{\Gamma(1+2\alpha)x^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)},\\
u_2(x) &= N(u_0+u_1) - N(u_0) = -\frac{2\Gamma(1+2\alpha)\Gamma(1+4\alpha)x^{5\alpha}}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} + \frac{\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)x^{7\alpha}}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)},\\
u_3(x) &= N\Big(\sum_{i=0}^{2} u_i\Big) - N\Big(\sum_{i=0}^{1} u_i\Big)\\
&= \frac{4\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)x^{7\alpha}}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)} - \frac{4\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)\Gamma(1+8\alpha)x^{9\alpha}}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)\Gamma(1+9\alpha)} + \dots\\
&\vdots
\end{aligned}
$$

and so on. In the same manner the rest of components can be obtained. The 4-term approximate solution for Eq. (39) by NIM is given by:

$$\begin{aligned}
u(x) = \sum_{i=0}^{3} u_i(x) &= -\frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\Gamma(1+2\alpha)x^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} - \frac{2\Gamma(1+2\alpha)\Gamma(1+4\alpha)x^{5\alpha}}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\
&+ \left(\frac{\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)} + \frac{4\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)}\right)x^{7\alpha} \\
&- \left(\frac{4\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)\Gamma(1+8\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)\Gamma(1+9\alpha)} + \frac{2\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)\Gamma(1+8\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)\Gamma(1+9\alpha)}\right)x^{9\alpha} \\
&+ \left(\frac{4\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)^2\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)^2\Gamma(1+11\alpha)} + \frac{2\Gamma(1+2\alpha)^3\Gamma(1+6\alpha)\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+3\alpha)^3\Gamma(1+7\alpha)\Gamma(1+11\alpha)}\right)x^{11\alpha}, \\
&- \frac{4\Gamma(1+2\alpha)^3\Gamma(1+4\alpha)\Gamma(1+6\alpha)\Gamma(1+12\alpha)x^{13\alpha}}{\Gamma(1+\alpha)^7\Gamma(1+3\alpha)^3\Gamma(1+5\alpha)\Gamma(1+7\alpha)\Gamma(1+13\alpha)} + \frac{\Gamma(1+2\alpha)^4\Gamma(1+6\alpha)^2\Gamma(1+14\alpha)x^{15\alpha}}{\Gamma(1+\alpha)^8\Gamma(1+3\alpha)^4\Gamma(1+7\alpha)^2\Gamma(1+15\alpha)}.
\end{aligned} \tag{43}$$

In view of the IIM, according to Eq. (32), the initial value problem (39) is equivalent to the fractional integral equation:

$$u_{r+1}(x) = u_0 + I_t^{\alpha}[u_r^2(x)], \qquad u_0(x) = -\frac{x^{\alpha}}{\Gamma(1+\alpha)}, \qquad r = 0, 1, 2, \ldots. \tag{44}$$

Let $N(u_r) = I_t^{\alpha}[u_r^2(x)]$. Therefore, from Eq. (33), we can obtain the following first few components of the integral iterative solution for Eq. (39):

$$\begin{aligned}
u_0(x) &= -\frac{x^{\alpha}}{\Gamma(1+\alpha)}, \\
u_1(x) &= u_0 + N(u_0) = -\frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\Gamma(1+2\alpha)x^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}, \\
u_2(x) &= u_0 + N(u_1) = -\frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\Gamma(1+2\alpha)x^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + -\frac{2\Gamma(1+2\alpha)\Gamma(1+4\alpha)x^{5\alpha}}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\
&\quad + \frac{\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)x^{7\alpha}}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)}, \\
u_3(x) &= u_0 + N(u_2) = -\frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{\Gamma(1+2\alpha)x^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} - \frac{2\Gamma(1+2\alpha)\Gamma(1+4\alpha)x^{5\alpha}}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\
&\quad + \left(\frac{\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)} + \frac{4\Gamma(1+2\alpha)\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma(1+\alpha)^4\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma(1+7\alpha)}\right)x^{7\alpha} \\
&\quad - \left(\frac{4\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)\Gamma(1+8\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)\Gamma(1+9\alpha)} + \frac{2\Gamma(1+2\alpha)^2\Gamma(1+6\alpha)\Gamma(1+8\alpha)}{\Gamma(1+\alpha)^5\Gamma(1+3\alpha)^2\Gamma(1+7\alpha)\Gamma(1+9\alpha)}\right)x^{9\alpha} \\
&\quad + \left(\frac{4\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)^2\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+3\alpha)^2\Gamma(1+5\alpha)^2\Gamma(1+11\alpha)} + \frac{2\Gamma(1+2\alpha)^3\Gamma(1+6\alpha)\Gamma(1+10\alpha)}{\Gamma(1+\alpha)^6\Gamma(1+3\alpha)^3\Gamma(1+7\alpha)\Gamma(1+11\alpha)}\right)t^{11\alpha}, \\
&\vdots \\
&\quad - \frac{4\Gamma(1+2\alpha)^3\Gamma(1+4\alpha)\Gamma(1+6\alpha)\Gamma(1+12\alpha)x^{13\alpha}}{\Gamma(1+\alpha)^7\Gamma(1+3\alpha)^3\Gamma(1+5\alpha)\Gamma(1+7\alpha)\Gamma(1+13\alpha)} + \frac{\Gamma(1+2\alpha)^4\Gamma(1+6\alpha)^2\Gamma(1+14\alpha)x^{15\alpha}}{\Gamma(1+\alpha)^8\Gamma(1+3\alpha)^4\Gamma(1+7\alpha)^2\Gamma(1+15\alpha)} \\
&\vdots
\end{aligned} \tag{45}$$

and so on. In the same manner the rest of components can be obtained. The 4order term approximate solution obtained by IIM in (45) is the same 4-term approximate solution as obtained by NIM in (43) but without calculating the values $N\left(\sum_{i=0}^{r} u_i\right) - N\left(\sum_{i=0}^{r-1} u_i\right)$ and the r-term approximate solution $u(x) = \sum_{i=0}^{r-1} u_i$. Also, the approximate solution by IIM is obtained without equating the terms of equal powers of the imbedding parameter p. Therefore, IIM is simpler and more convenient than both NIM and HPM.

The 4-term approximate solution for (39) obtained by HPM in (41) denoted by u_{HPM} and obtained by both NIM in (43) and IIM in (45) denoted by u_{IIM} are shown in Table 1 with the corresponding exact solution $u(x) = -\tanh x$ denoted by u_{Exact} for different values of α and x. It is clear that the approximate solutions converge to the exact solution as $\alpha \to 1$ and u_{IIM} is more accurate than u_{HPM}. It is evident that the efficiency of the considered methods can be increased by increasing the number of the computed terms of the approximate solution.

Table 1: Numerical values for Eq. (39) for different values of α and x

	$\alpha = 0.6$		$\alpha = 0.8$		$\alpha = 1.0$		
x	u_{HPM}	u_{IIM}	u_{HPM}	u_{IIM}	u_{HPM}	u_{IIM}	u_{Exact}
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.2	−0.38719	−0.38682	−0.28525	−0.28524	−0.19738	−0.19738	−0.19738
0.4	−0.53134	−0.52689	−0.46331	−0.46285	−0.37998	−0.37995	−0.37995
0.6	−0.61145	−0.59864	−0.59079	−0.58755	−0.53739	−053698	−0.53705
0.8	−0.62290	−0.61387	−0.68062	−0.67047	−0.66570	−0.66330	−0.66404
1.0	−0.51231	−0.56440	−0.72629	−0.71154	−0.76508	−0.75817	−0.76159

In Figs. (1-3), we have plotted the approximate solution for (39) obtained by HPM in (41), NIM in (43) and IIM in (45), for different values of α, and the corresponding exact solution $u = -\tanh x$. It is clear that as $\alpha \to 1$, the approximate solution $\to$ the exact solution. Also, it is important to note that the rate of convergence can be increased by increasing the number of iterations.

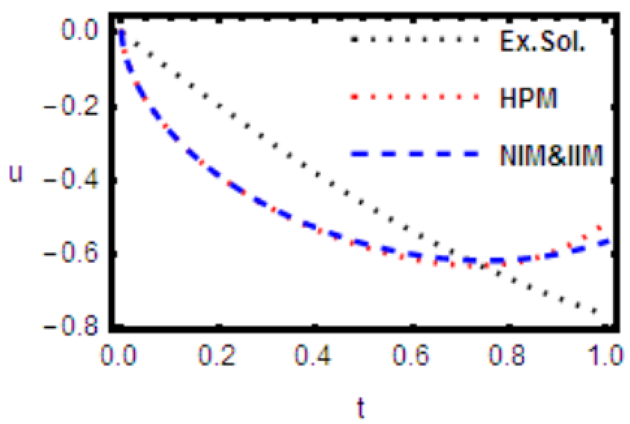

Figure 1: Approximate solution for (39) obtained by HPM, NIM, IIM and the exact solution in case $\alpha = 0.6$

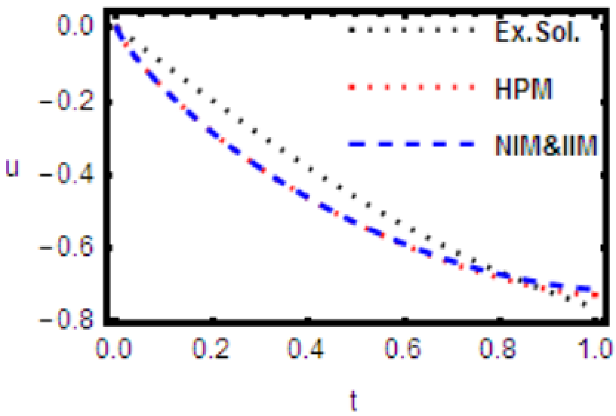

Figure 2: Approximate solution for (39) obtained by HPM, NIM, IIM and the exact solution in case $\alpha = 0.8$

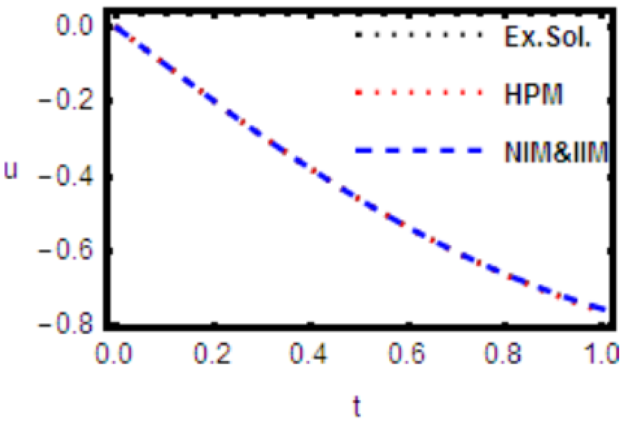

Figure 3: Approximate solution for (39) obtained by HPM, NIM, IIM andthe exact solution in case $\alpha = 1.0$

Problem 4.2. *The equation governing the unsteady flow of a polytropic gas with time-fractional derivatives in (2+1)-dimensions is given by [33–35]:*

$$D_t^\alpha u(x,y,t) + uu_x + vu_y + \frac{k_x}{\rho} = 0,$$
$$D_t^\alpha v(x,y,t) + uv_x + vv_y + \frac{k_y}{\rho} = 0,$$
$$D_t^\alpha \rho(x,y,t) + u\rho_x + v\rho_y + \rho(u_x + v_y) = 0,$$

$$D_t^\alpha k(x,y,t) + uk_x + vk_y + \gamma k(u_x + v_y) = 0, \qquad 0 < \alpha \leq 1, \tag{46}$$

where ρ is the density, k the pressure, u and v the velocity components in the x and y directions, respectively and the adiabatic index γ is the ratio of the specific heats. With the initial values:

$$u(x,y,0) = e^{x+y}, \quad v(x,y,0) = -1 - e^{x+y}, \quad \rho(x,y,0) = e^{x+y}, \quad k(x,y,0) = c. \tag{47}$$

Note that the selection of equations (46) that are obtained from [33] the fluid is incompressible and inviscid (no viscose).

In view of the HPM, the homotopy for Eqs. (46)-(47) takes the form:

$$\begin{aligned}
D_t^\alpha u(x,y,t) &= -p\Big[uu_x + vu_y + \frac{k_x}{\rho}\Big],\\
D_t^\alpha v(x,y,t) &= -p\Big[uv_x + vv_y + \frac{k_y}{\rho}\Big],\\
D_t^\alpha \rho(x,y,t) &= -p[u\rho_x + v\rho_y + \rho(u_x+v_y)],
\end{aligned}$$

$$D_t^\alpha k(x,y,t) = -p[uk_x + vk_y + \gamma k(u_x+v_y)]. \tag{48}$$

Substituting: $u = u_0 + pu_1 + p^2u_2 + ...$, $v = v_0 + pv_1 + p^2v_2 + ...$, $\rho = \rho_0 + p\rho_1 + p^2\rho_2 + ...$, $k = k_0 + pk_1 + p^2k_2 + ...$, and the initial values (47) into (48) and equating the terms of equal powers of p, we obtain the following set of fractional differential equations:

$$\begin{aligned}
&p^0 : D_t^\alpha u_0 = 0, && u_0(x,y,0) = e^{x+y},\\
&D_t^\alpha v_0 = 0, && v_0(x,y,0) = -1 - e^{x+y},\\
&D_t^\alpha \rho_0 = 0, && \rho_0(x,y,0) = e^{x+y},\\
&D_t^\alpha k_0 = 0, && k_0(x,y,0) = c,\\
&p^1 : D_t^\alpha u_1 = -\Big(u_0u_{0x} + v_0u_{0y} + \frac{k_{0x}}{\rho_0}\Big), && u_1(x,y,0) = 0,\\
&D_t^\alpha v_1 = -\Big(u_0v_{0x} + v_0v_{0y} + \frac{k_{0y}}{\rho_0}\Big), && v_1(x,y,0) = 0,\\
&D_t^\alpha \rho_1 = -(u_0\rho_{0x} + v_0\rho_{0y} + u_{0x}\rho_0 + v_{0y}\rho_0), && \rho_1(x,y,0) = 0,\\
&D_t^\alpha k_1 = -(u_0k_{0x} + v_0k_{0y} + \gamma(u_{0x}k_0 + v_{0y}k_0), && k_1(x,y,0) = 0,\\
&p^2 : D_t^\alpha u_2 = -\Big(u_0u_{1x} + u_1u_{0x} + v_0u_{1y} + v_1u_{0y} + \frac{k_{1x}}{\rho_1}\Big), && u_2(x,y,0) = 0,\\
&D_t^\alpha v_2 = -\Big(u_0v_{1x} + u_1v_{0x} + v_0v_{1y} + v_1v_{0y} + \frac{k_{1y}}{\rho_1}\Big), && v_2(x,y,0) = 0,\\
&D_t^\alpha \rho_2 = -(u_0\rho_{1x} + u_1\rho_{0x} + v_0\rho_{1y} + v_1\rho_{0y} + u_{0x}\rho_1 + u_{1x}\rho_0 + v_{0y}\rho_1 + v_{1y}\rho_0), && \rho_2(x,y,0) = 0,\\
&D_t^\alpha k_2 = -(u_0k_{1x} + u_1k_{0x} + v_0k_{1y} + v_1k_{0y} + \gamma(u_{0x}k_1 + u_{1x}k_0 + v_{0y}k_1 + v_{1y}k_0)) && k_2(x,y,0) = 0,\\
&p^3 : D_t^\alpha u_3 = -\Big(u_0u_{2x} + u_1u_{1x} + u_2u_{0x} + v_0u_{2y} + v_1u_{1y} + v_2u_{0y} + \frac{k_{2x}}{\rho_2}\Big), && u_3(x,y,0) = 0,\\
&D_t^\alpha v_3 = -\Big(u_0v_{2x} + u_1v_{1x} + u_2v_{0x} + v_0v_{2y} + v_1v_{1y} + v_2v_{0y} + \frac{k_{2y}}{\rho_2}\Big), && v_3(x,y,0) = 0,\\
&D_t^\alpha \rho_3 = -(u_0\rho_{2x} + u_1\rho_{1x} + u_2\rho_{0x} + v_0\rho_{2y} + v_1\rho_{1y} + v_2\rho_{0y} + u_{0x}\rho_2 + u_{1x}\rho_1 + u_{2x}\rho_0 &&\\
&\qquad + v_{0y}\rho_2 + v_{1y}\rho_1 + v_{2y}\rho_0), && \rho_3(x,y,0) = 0,\\
&D_t^\alpha k_3 = -(u_0k_{2x} + u_1k_{1x} + u_2k_{0x} + v_0k_{2y} + v_1k_{1y} + v_2k_{0y} + \gamma(u_{0x}k_2 + u_{1x}k_1 + u_{2x}k_0 &&\\
&\qquad + v_{0y}k_2 + v_{1y}k_1 + v_{2y}k_0)), && k_3(x,y,0) = 0,\\
&\vdots
\end{aligned}$$

Solving the above set of equations, we obtain the following first few components of the homotopy perturbation solution for Eqs. (46)-(47):

$$
\begin{aligned}
&u_0(x,y,t)=e^{x+y}, && v_0(x,y,t)=-1-e^{x+y}, && \rho_0(x,y,t)=e^{x+y}, && k_0(x,y,t)=c,\\
&u_1(x,y,t)=e^{x+y}\cdot\frac{t^{\alpha}}{\Gamma(1+\alpha)}, && v_1(x,y,t)=-e^{x+y}\cdot\frac{t^{\alpha}}{\Gamma(1+\alpha)}, && \rho_1(x,y,t)=e^{x+y}\cdot\frac{t^{\alpha}}{\Gamma(1+\alpha)}, && k_1(x,y,t)=0,\\
&u_2(x,y,t)=e^{x+y}\cdot\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, && v_2(x,y,t)=-e^{x+y}\cdot\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, && \rho_2(x,y,t)=e^{x+y}\cdot\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, && k_2(x,y,t)=0,\\
&u_3(x,y,t)=e^{x+y}\cdot\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, && v_3(x,y,t)=-e^{x+y}\cdot\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, && \rho_3(x,y,t)=e^{x+y}\cdot\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, && k_3(x,y,t)=0,\\
&\vdots\\
&u_r(x,y,t)=e^{x+y}\cdot\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}, && v_r(x,y,t)=-e^{x+y}\cdot\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}, && \rho_r(x,y,t)=e^{x+y}\cdot\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}, && k_r(x,y,t)=0,
\end{aligned}
$$

and so on. The $(r+1)$-term approximate solution for (46)-(47) by HPM, in series form, is given by:

$$
\begin{aligned}
u(x,y,t)&=\sum_{i=0}^{r}u_i(x,y,t)=e^{x+y}\cdot\left(1+\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+...+\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\right),\\
v(x,y,t)&=\sum_{i=0}^{r}v_i(x,y,t)=-1-e^{x+y}\cdot\left(1+\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+...+\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\right),\\
\rho(x,y,t)&=\sum_{i=0}^{r}\rho_i(x,y,t)=e^{x+y}\cdot\left(1+\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+...+\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\right),\\
k(x,y,t)&=\sum_{i=0}^{r}k_i(x,y,t)=c. \qquad (49)
\end{aligned}
$$

In closed form, in the special case $\alpha=1$, (49) gives:

$$
\begin{aligned}
&u(x,y,t)=\sum_{i=0}^{\infty}u_i(x,y,t)=e^{x+y+t}, && v(x,y,t)=\sum_{i=0}^{\infty}v_i(x,y,t)=-1-e^{x+y+t},\\
&\rho(x,y,t)=\sum_{i=0}^{\infty}\rho_i(x,y,t)=e^{x+y+t}, && k(x,y,t)=\sum_{i=0}^{\infty}k_i(x,y,t)=c. \qquad (50)
\end{aligned}
$$

which is the exact solution for (46)-(47).

In view of the NIM, the initial value problem (46)-(47) is equivalent to the fractional integral equations:

$$
\begin{aligned}
u(x,y,t)&=e^{x+y}-I_t^{\alpha}\left[uu_x+vu_y+\frac{k_x}{\rho}\right],\\
v(x,y,t)&=-1-e^{x+y}-I_t^{\alpha}\left[uv_x+vv_y+\frac{k_y}{\rho}\right],\\
\rho(x,y,t)&=e^{x+y}-I_t^{\alpha}\left[u\rho_x+v\rho_y+\rho(u_x+v_y)\right],\\
k(x,y,t)&=c-I_t^{\alpha}[uk_x+vk_y+\gamma k(u_x+v_y)]. \qquad (51)
\end{aligned}
$$

Let $N(u) = -I_t^{\alpha}\Big[uu_x + vu_y + \frac{k_x}{\rho}\Big]$, $N(v) = -I_t^{\alpha}\Big[uv_x + vv_y + \frac{k_y}{\rho}\Big]$, $N(\rho) = -I_t^{\alpha}\Big[u\rho_x + v\rho_y + \rho(u_x + v_y)\Big]$, and $N(k) = -I_t^{\alpha}[uk_x + vk_y + \gamma k(u_x + v_y)]$. Therefore, in view of Eq. 25), we have the following first few components of the new iterative solution for (46)-(47):

$$
\begin{aligned}
&u_0(x,y,t) = e^{x+y}, && v_0(x,y,t) = -1 - e^{x+y},\\
&\rho_0(x,y,t) = e^{x+y}, && k_0(x,y,t) = c,\\
&u_1(x,y,t) = N(u_0) = e^{x+y}\cdot\frac{t^{\alpha}}{\Gamma(1+\alpha)}, && v_1(x,y,t) = N(v_0) = -e^{x+y}\cdot\frac{t^{\alpha}}{\Gamma(1+\alpha)},\\
&\rho_1(x,y,t) = N(\rho_0) = e^{x+y}\cdot\frac{t^{\alpha}}{\Gamma(1+\alpha)}, && k_1(x,y,t) = N(k_0) = 0,\\
&u_2(x,y,t) = N(u_0+u_1) - N(u_0) = e^{x+y}\cdot\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, && v_2(x,y,t) = N(v_0+v_1) - N(v_0) = -e^{x+y}\cdot\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\\
&\rho_2(x,y,t) = N(\rho_0+\rho_1) - N(\rho_0) = e^{x+y}\cdot\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, && k_2(x,y,t) = N(k_0+k_1) - N(k_0) = 0,\\
&u_3(x,y,t) = N\Big(\sum_{i=0}^{2}u_i\Big) - N\Big(\sum_{i=0}^{1}u_i\Big) = e^{x+y}\cdot\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, && v_3(x,y,t) = N\Big(\sum_{i=0}^{2}v_i\Big) - N\Big(\sum_{i=0}^{1}v_i\Big) = -e^{x+y}\cdot\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\rho_3(x,y,t) = N\Big(\sum_{i=0}^{2}\rho_i\Big) - N\Big(\sum_{i=0}^{1}\rho_i\Big) = e^{x+y}\cdot\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, && k_3(x,y,t) = N\Big(\sum_{i=0}^{2}k_i\Big) - N\Big(\sum_{i=0}^{1}k_i) = 0,\\
&\vdots\\
&u_r(x,y,t) = N\Big(\sum_{i=0}^{r-1}u_i\Big) - N\Big(\sum_{i=0}^{r-2}u_i\Big) = e^{x+y}\cdot\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}, && v_r(x,y,t) = N\Big(\sum_{i=0}^{r-1}v_i\Big) - N\Big(\sum_{i=0}^{r-2}v_i) = -e^{x+y}\cdot\frac{t^{r\alpha}}{\Gamma(1+r\alpha)},\\
&\rho_r(x,y,t) = N\Big(\sum_{i=0}^{r-1}\rho_i\Big) - N\Big(\sum_{i=0}^{r-2}\rho_i\Big) = e^{x+y}\cdot\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}, && k_r(x,y,t) = N\Big(\sum_{i=0}^{r-1}k_i\Big) - N\Big(\sum_{i=0}^{r-2}k_i\Big) = 0,
\end{aligned}
$$

and so on. The $(r+1)$-term approximate solution for (46)-(47), by NIM, is given by:

$$
\begin{aligned}
u(x,y,t) &= \sum_{i=0}^{r}u_i(x,y,t) = e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + ... + \frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\Big),\\
v(x,y,t) &= \sum_{i=0}^{r}v_i(x,y,t) = -1 - e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + ... + \frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\Big),\\
\rho(x,y,t) &= \sum_{i=0}^{r}\rho_i(x,y,t) = e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + ... + \frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\Big),\\
k(x,y,t) &= \sum_{i=0}^{r}k_i(x,y,t) = c. \qquad (51)
\end{aligned}
$$

In closed form, in the special case $\alpha = 1$, Eq. (51) gives:

$$
\begin{aligned}
&u(x,y,t) = \sum_{i=0}^{\infty}u_i(x,y,t) = e^{x+y+t}, && v(x,y,t) = \sum_{i=0}^{\infty}v_i(x,y,t) = -1 - e^{x+y+t},\\
&\rho(x,y,t) = \sum_{i=0}^{\infty}\rho_i(x,y,t) = e^{x+y+t}, && k(x,y,t) = \sum_{i=0}^{\infty}k_i(x,y,t) = c. \qquad (52)
\end{aligned}
$$

which is the same result as obtained by HPM in (50) and which is the exact solution for (46)- (47). In view of the IIM, the initial value problem (46)-(47) is equivalent to the fractional integral equations:

$$
\begin{aligned}
u_{r+1}(x,y,t) &= u_0 - I_t^{\alpha}\Big[u_r(u_r)_x + v_r(u_r)_y + \frac{(k_r)_x}{\rho_r}\Big], && u_0(x,y,t) = e^{x+y},\\
v_{r+1}(x,y,t) &= v_0 - I_t^{\alpha}\Big[u_r(v_r)_x + v_r(v_r)_y + \frac{(k_r)_y}{\rho_r}\Big], && v_0(x,y,t) = -1 - e^{x+y},\\
\rho_{r+1}(x,y,t) &= \rho_0 - I_t^{\alpha}[u_r(\rho_r)_x + v_r(\rho_r)_y + \rho_r((u_r)_x + (v_r)_y)], && \rho_0(x,y,t) = e^{x+y}, \qquad (53)\\
k_{r+1}(x,y,t) &= k_0 - I_t^{\alpha}[u_r(k_r)_x + v_r(k_r)_y + \gamma k_r((u_r)_x + (v_r)_y)], && k_0(x,y,t) = c, r = 0,1,2,\dots. \qquad (54)
\end{aligned}
$$

Therefore, from Eq. (33), we obtain the following first few components of the integral iterative solution for (46)-(47):

$$
\begin{aligned}
&u_0(x,y,t) = e^{x+y}, && v_0(x,y,t) = -1 - e^{x+y},\\
&\rho_0(x,y,t) = e^{x+y}, && k_0(x,y,t) = c,\\
&u_1(x,y,t) = u_0 + N(u_0) = e^{x+y}\cdot(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}),\\
&v_1(x,y,t) = v_0 + N(v_0) = -1 - e^{x+y}\cdot(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}),\\
&\rho_1(x,y,t) = \rho_0 + N(\rho_0) = e^{x+y}\cdot(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}), && k_1(x,y,t) = k_0 + N(k_0) = c,\\
&u_2(x,y,t) = u_0 + N(u_1) = e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\Big),\\
&v_2(x,y,t) = v_0 + N(v_1) = -1 - e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\Big),\\
&\rho_2(x,y,t) = \rho_0 + N(\rho_1) = e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\Big), && k_2(x,y,t) = k_0 + N(k_1) = c,\\
&u_3(x,y,t) = u_0 + N(u_2) = e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\Big),\\
&v_3(x,y,t) = v_0 + N(v_2) = -1 - e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\Big),\\
&\rho_3(x,y,t) = \rho_0 + N(\rho_2) = e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\Big), && k_3(x,y,t) = k_0 + N(k_2) = c,\\
&\vdots\\
&u_r(x,y,t) = u_0 + N(u_{r-1}) = e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots + \frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\Big),\\
&v_r(x,y,t) = v_0 + N(v_{r-1}) = -1 - e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots + \frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\Big),\\
&\rho_r(x,y,t) = \rho_0 + N(\rho_{r-1}) = e^{x+y}\cdot\Big(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots + \frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\Big),\\
&k_r(x,y,t) = k_0 + N(k_{r-1}) = c, \qquad (55)
\end{aligned}
$$

and so on. In closed form, in the special case $\alpha = 1$, Eq. (55) gives:

$$
\begin{aligned}
u(x,y,t) &= \lim_{r\to\infty} = e^{x+y+t}, & v(x,y,t) &= \lim_{r\to\infty} = -1 - e^{x+y+t}\\
\rho(x,y,t) &= \lim_{r\to\infty} = e^{x+y+t}, & k(x,y,t) &= \lim_{r\to\infty} = c. \qquad (56)
\end{aligned}
$$

which is the same result as obtained by HPM in (50) and by NIM in (52) but without calculating the $r-$term approximate solution from the relation $u(t) = \sum_{i=0}^{r-1} u_i$ or equating the terms of equal powers of the impeding parameter p or calculating the values $N\Big(sum_{i=0}^{r} u_i\Big) - N\Big(\sum_{i=0}^{r-1} u_i\Big)$, $r = 1, 2, ...$ These advantages of IIM over HPM and NIM simplify and reduce the computational procedures and time and make IIM more suitable and convenient for solving fractional differential equations. Also, this result is the exact solution for (46)-(47).

5. Conclusion

In this work, the NIM, HPM, and IIM were used to solve exactly the equation governing the unsteady flow of a polytropic gas with time-fractional derivative. The prove of the existence and uniqueness of the solution and convergence of the IIM are made. The comparisons between these methods and also between their results were made and it was found that the results obtained by the IIM is the same as obtained by both the NIM and HPM but without calculating the r-term approximate solution from the relation $u(t) = \sum_{i=0}^{r-1} u_i$ as done in both NIM and HPM or calculating the values $N\Big(\sum_{i=0}^{r} u_i\Big) - N\Big(\sum_{i=0}^{r-1} u_i\Big)$,$r = 1, 2, ...,$ as done in NIM or equating the terms of equal powers of the impeding parameter p as done in HPM. These advantages shorten the time and procedures of calculations and make the IIM more effective and suitable technique in finding the exact solutions for wide classes of nonlinear fractional problems in applied sciences.

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