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On a Sampling Expansion With Partial Derivatives for Functions of Several Variables

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Abstract. Let B_σ^p, $1 \le p < \infty$, $\sigma > 0$, denote the space of all $f \in L^p(\mathbb{R})$ such that the Fourier transform of f (in the sense of distributions) vanishes outside $[-\sigma, \sigma]$. The classical sampling theorem states that each $f \in B_\sigma^p$ may be reconstructed exactly from its sample values at equispaced sampling points $\{\pi m/\sigma\}_{m\in\mathbb{Z}}$ spaced by π/σ. Reconstruction is also possible from sample values at sampling points $\{\pi\theta m/\sigma\}_m$ with certain $1 < \theta \le 2$ if we know $f(\theta\pi m/\sigma)$ and $f'(\theta\pi m/\sigma)$, $m \in \mathbb{Z}$. In this paper we present sampling series for functions of several variables. These series involves samples of functions and their partial derivatives.

1. Introduction

We start with some notation and definitions. Let $\mathbb{Z}^n$, $\mathbb{R}^n$ and $\mathbb{C}^n$ be the n-dimensional integer lattice, the real Euclidean space and the complex Euclidean space, respectively. For any $\tau \in \mathbb{C}$ and each $a, b \in \mathbb{C}^n$ we write

$$\tau a = (\tau a_1, \dots, \tau a_n), \quad ab = (a_1 b_1, \dots, a_n b_n).$$

If, in addition, $b_j \neq 0$, $j = 1, \dots, n$, then a/b denotes the vector of fractions $\big(a_1/b_1, \dots, a_n/b_n\big)$. For $\sigma \in \mathbb{R}^n$ such that $\sigma_j > 0$, $j = 1, \dots, n$, let us denote by $\sigma\mathbb{Z}^n$ the lattice $\oplus_{j=1}^n \big(\sigma_j \mathbb{Z}\big)$. Also if $A, B \subset \mathbb{C}^n$, then $A + B = \{a + b : a \in A, b \in B\}$.

For $f \in L^1(\mathbb{R}^n)$, we define the Fourier of f transform by

$$\widehat{f}(t) = \int_{\mathbb{R}^n} e^{-i\langle x,t\rangle} f(x)\, dx,$$

$x \in \mathbb{R}^n$, where $\langle x, t\rangle = \sum_{k=1}^n x_k t_k$ is the scalar product on $\mathbb{R}^n$. If $f \notin L^1(\mathbb{R}^n)$, then we understand $\widehat{f}$ in a distributional sense of tempered distributions $S'(\mathbb{R}^n)$. Given a closed subset $\Omega \subset \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{C}$ is called bandlimited to Ω if $\widehat{f}$ vanishes outside Ω.

For $1 \le p \le \infty$ and $\sigma \in \mathbb{R}^n$ such that $\sigma_j > 0$, $j = 1, \dots, n$, let

$$Q_\sigma^n = \{x \in \mathbb{R}^n : |x_j| \le \sigma_j, j = 1, \dots, n\} \quad \text{and} \quad B_{Q_\sigma^n}^p = \{f \in L^p(\mathbb{R}^n) : \ supp\ \widehat{f} \subset Q_\sigma^n\}.$$

The space $B_{Q_\sigma^n}^p$ is a Banach space in $L^p(\mathbb{R}^n)$-norm. In the case of functions of one variable, let us write B_σ^p instead of $B_{Q_\sigma^1}^p$.

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By the Paley-Wiener-Schwartz theorem (see [9, p. 181]), each $f \in B^p_{Q^n_\sigma}$, $1 \le p \le \infty$, is infinitely differentiable on $\mathbb{R}^n$. Moreover, f has an extension onto $\mathbb{C}^n$ to an entire function. Note that we shall identify any $f \in B^p_{Q^n_\sigma}$ with a $L^p(\mathbb{R}^n)$-function $f(x)$ defined on $\mathbb{R}^n$ and in other cases we consider the same f as entire function $f(z)$ defined on the whole $\mathbb{C}^n$. In the sequel, we shall frequently use the following functions defined on $\mathbb{C}^n$

$$\operatorname{sinc}_n(z) = \prod_{j=1}^n \frac{\sin \pi z_j}{\pi z_j} \quad \text{and} \quad \operatorname{sic}_n(z) = \prod_{j=1}^n \sin \frac{z_j}{2}.$$

Of course, $\operatorname{sic}_n \in B^\infty_{Q^n_\sigma}$ with $\sigma = (1/2, \dots, 1/2)$ and $\operatorname{sinc}_n \in B^p_{Q^n_\pi}$ for each $1 < p \le \infty$.

The classical Whittaker-Shanon-Kotelnikov theorem states that for any $f \in B^p_\sigma$, $1 \le p < \infty$, the following sampling expansion holds (see, e.g., [8, p. 51])

$$f(z) = \sum_{m \in \mathbb{Z}} f\Big(\frac{\pi m}{\sigma}\Big) \operatorname{sinc}_1\Big(\frac{\sigma}{\pi}\Big(z - \frac{\pi m}{\sigma}\Big)\Big) = \sum_{u \in (\pi/\sigma)\mathbb{Z}} f(u) \operatorname{sinc}_1\Big(\frac{\sigma}{\pi}(z-u)\Big). \tag{1}$$

In particular, this series converges absolutely for $z \in \mathbb{C}$, uniformly on $\mathbb{R}$ and also on compact subsets of $\mathbb{C}$. Hence, (1) shows that each $f \in B^p_\sigma$, $1 \le p < \infty$, may be reconstructed exactly from sample values $f(\pi m/\sigma)$ at equispaced sampling points $\{\pi m/\sigma\}_{m \in \mathbb{Z}}$ spaced by π/σ. Note, that this spaced value π/σ is exact, i.e., for any $\theta > 1$ there exist two $f_j \in B^p_\sigma$, $j = 1, 2$ such that $f_1 \not\equiv f_2$, but $f_1(\theta\pi m/\sigma) = f_2(\theta\pi m/\sigma)$ for all $m \in \mathbb{Z}$. On the other hand, if we know sample values $f(\theta\pi m/\sigma)$ with certain $1 < \theta \le 2$, then the reconstruction of f is possible in the case if we also use its derivative values $f'(\theta\pi m/\sigma)$. In particular, if $f \in B^p_\sigma$ with $1 \le p < \infty$, then (see e.g., [10, p. 145])

$$f(z) = \sum_{u \in (2\pi/\sigma)\mathbb{Z}} \Big(f(u) + f'(u)(z-u)\Big) \operatorname{sinc}_1^2\Big(\frac{\sigma}{2\pi}(z-u)\Big). \tag{2}$$

Derivative sampling formula (2) is well known in classical sampling theory. The general formula, which uses samples of f and samples of its derivatives $\{f^{(j)}\}_{j=1}^k$ was first given in [12]. Reconstruction formulas of such a type are also called the uniform sampled formulas because the nodes (sampling sequences) $\{2\pi m/\sigma\}_m$ and $\{k\pi m/\sigma\}_m$ are equidistantly spaced. On the other hand, if $\{f^{(j)}(\cdot)\}_{j=0}^k$ are sampled nonuniformly, then much less is known about reconstruction of f (see, for example, [13] and the references therein).

The following n-dimensional sampling theorem is a standard extension of (1) in $B^p_{Q^n_\sigma}$, $1 \le p < \infty$ (see, e.g., [6, p. 172])

$$f(z) = \sum_{u \in (\pi/\sigma)\mathbb{Z}^n} f(u) \operatorname{sinc}_n\Big(\frac{\sigma}{\pi}(z-u)\Big). \tag{3}$$

The first multidimensional sampling series using values of the function and its partial derivatives was introduced by Montgomery [14]. The aim of this paper is to prove a multidimensional version of (2). Note that in [3] (see also a tutorial review [11, p. 40]) was given the following expression

$$f(z) = \sum_{u \in \mathbb{Z}^n} \left\{ \left[f\Big(2\pi \frac{u}{\sigma}\Big) + \sum_{k=1}^n \Big(z_k - \frac{2\pi u_k}{\sigma_k}\Big) \cdot \frac{\partial f}{\partial z_k}\Big(2\pi \frac{u}{\sigma}\Big) \right] {\operatorname{sinc}_n}^2\Big(\frac{1}{2\pi}\sigma(z-u)\Big) \right\} \tag{4}$$

for $f \in B^p_{Q^n_\sigma}$ with $1 \le p < \infty$. We say that such a sampling theorem fails in general. Indeed, let χ be any function of two variables from the Schwartz space $S(\mathbb{R}^2)$ such that $\chi \not\equiv 0$ and $\operatorname{supp} \chi \subset Q^2_{\sigma/2}$. Then the function

$$f(z) = \widehat{\chi}(z) \operatorname{sic}_2(\sigma z) \tag{5}$$

is in $B^p_{Q^2_\sigma}$ for each $1 \le p \le \infty$. Moreover,

$$f\Big(2\pi \frac{u}{\sigma}\Big) = \frac{\partial f}{\partial z_1}\Big(2\pi \frac{u}{\sigma}\Big) = \frac{\partial f}{\partial z_2}\Big(2\pi \frac{u}{\sigma}\Big) = 0$$

for all $u \in \mathbb{Z}^2$. Hence, in this case the series (4) generates the zero function, but not the function (5). Even more, (5) shows that the same still true if we added to (4) an arbitrary number of the following sample values

$$\frac{\partial^m f}{\partial x_j^m}\Big(2\pi\frac{u}{\sigma}\Big)$$

with $u \in \mathbb{Z}^2$, $j = 1,2$ and $m = 2,3,\ldots$. Therefore, any multidimensional version of (2) must necessarily contains also mixed partial derivatives of f.

Now we shall provide some more notation and formulate our main theorem. For $k \in \mathbb{Z}^n$ with $k_j \geq 0$, $j = 1,\ldots,n$, and $z \in \mathbb{C}^n$, here and subsequently, we denote the operator

$$\frac{\partial^{|k|}}{\partial^{k_1} z_1 \ldots \partial^{k_n} z_n}, \quad |k| = k_1 + \cdots + k_n$$

by ∂_z^k for short. Note that if $k_j = 0$ for all $j = 1,\ldots,n$, then $\partial_z^k f(z)$ is simply $f(z)$. Set

$$E^n = \big\{(t_1,\ldots,t_n):\ t_j \in \{0;1\}, j = 1,\ldots,n\big\}.$$

For fixed $k \in E^n$, $v \in \mathbb{Z}^n$ and $f \in B^p_{Q^n_\sigma}$, let us define the following polynomial in $\lambda \in \mathbb{C}^n$ by

$$P_{f,k,v}(\lambda) = \Big(\partial_z^k f\Big)(v) \cdot \prod_{j=1}^{n} \lambda_j^{k_j}. \tag{6}$$

Theorem 1.1. . *Let $f \in B^p_{Q^n_\sigma}$ with $1 \leq p < \infty$. Then*

$$f(z) = \sum_{u \in (2\pi/\sigma)\mathbb{Z}^n} \Bigg(\sum_{k \in E^n} P_{f,k,u}(z-u)\Bigg) \operatorname{sinc}_n{}^2\Big(\frac{1}{2\pi}\sigma(z-u)\Big). \tag{7}$$

The series (7) converges absolutely and uniformly on $\mathbb{R}^n$ and also on any compact subset of $\mathbb{C}^n$.

For fixed $u \in (2\pi/\sigma)\mathbb{Z}^n$, the representation (7) contains 2^n values of partial derivatives $\partial_z^k f(u)$ when k obtain all possible values from E^n. To demonstrate this, let us give two special cases of (7) for $n = 2$ and $n = 3$. If $f \in B^p_{Q^2_\sigma}$, then the following formula was given in [1, Corollary 3.6]

$$\begin{aligned} f(z) = \sum_{u \in \mathbb{Z}^2} \Bigg[& f\Big(2\pi\frac{u}{\sigma}\Big) + \Big(z_1 - \frac{2\pi u_1}{\sigma_1}\Big)\cdot\frac{\partial f}{\partial z_1}\Big(2\pi\frac{u}{\sigma}\Big) + \Big(z_2 - \frac{2\pi u_2}{\sigma_2}\Big)\cdot\frac{\partial f}{\partial z_2}\Big(2\pi\frac{u}{\sigma}\Big) \\ & + \Big(z_1 - \frac{2\pi u_1}{\sigma_1}\Big)\Big(z_2 - \frac{2\pi u_2}{\sigma_2}\Big)\cdot\frac{\partial^2 f}{\partial z_1 \partial z_2}\Bigg] \operatorname{sinc}_2{}^2\Big(\frac{1}{2\pi}\sigma(z-u).\Big) \end{aligned} \tag{8}$$

(compare representation (4)). Also we have that

$$\begin{aligned} f(z) = \sum_{u \in \mathbb{Z}^3} \Bigg[& f\Big(2\pi\frac{u}{\sigma}\Big) + \Big(z_1 - \frac{2\pi u_1}{\sigma_1}\Big)\cdot\frac{\partial f}{\partial z_1}\Big(2\pi\frac{u}{\sigma}\Big) + \Big(z_2 - \frac{2\pi u_2}{\sigma_2}\Big)\cdot\frac{\partial f}{\partial z_2}\Big(2\pi\frac{u}{\sigma}\Big) \\ & + \Big(z_3 - \frac{2\pi u_3}{\sigma_3}\Big)\cdot\frac{\partial f}{\partial z_3}\Big(2\pi\frac{u}{\sigma}\Big) + \Big(z_1 - \frac{2\pi u_1}{\sigma_1}\Big)\Big(z_2 - \frac{2\pi u_2}{\sigma_2}\Big)\cdot\frac{\partial^2 f}{\partial z_1 \partial z_2} \\ & + \Big(z_1 - \frac{2\pi u_1}{\sigma_1}\Big)\Big(z_3 - \frac{2\pi u_3}{\sigma_3}\Big)\cdot\frac{\partial^2 f}{\partial z_1 \partial z_3} + \Big(z_2 - \frac{2\pi u_2}{\sigma_2}\Big)\Big(z_3 - \frac{2\pi u_3}{\sigma_3}\Big)\cdot\frac{\partial^2 f}{\partial z_2 \partial z_3} \\ & + \Big(z_1 - \frac{2\pi u_1}{\sigma_1}\Big)\Big(z_2 - \frac{2\pi u_2}{\sigma_2}\Big)\Big(z_3 - \frac{2\pi u_3}{\sigma_3}\Big)\cdot\frac{\partial^3 f}{\partial z_1 \partial z_2 \partial z_3}\Bigg] \operatorname{sinc}_3{}^2\Big(\frac{1}{2\pi}\sigma(z-u)\Big). \end{aligned}$$

for any $f \in B^p_{Q^3_\sigma}$. Note that representation (7) is exact in some sense. More precisely, if we eliminate in (7) an arbitrary polynomial $P_{f,k,u}$ with certain $k = \widetilde{k} \in E^n$, then such a formula will be false. Indeed, let

$$\widetilde{f}(z) = \widehat{\chi}(z) \frac{\partial^{n-|\widetilde{k}|} \mathrm{sic}_n(\sigma z)}{\partial^{1-\widetilde{k}_1} z_1 \dots \partial^{1-\widetilde{k}_n} z_n}$$

where $\widehat{\chi}(z) \in S(\mathbb{R}^n)$ is such that $\chi \not\equiv 0$ and $supp\, \chi \subset Q^n_{\sigma/2}$. Then an easy computation shows that

$$\partial^k_z \widetilde{f}\Big(\frac{2\pi u}{\sigma}\Big) = 0$$

for all $u \in \mathbb{Z}^n$ and each $k \in E^n$ such that $k \neq \widetilde{k}$. Therefore, in such a case $\widetilde{f}$ will generate by (7) the zero function, but not $\widetilde{f}$.

Finally, note that relatively loss papers have investigated nonuniform sampling with derivatives for functions of several variables. For example, for functions from $L^2(\mathbb{R}^n)$ bandlimited to a compact subset Ω in [7] certain necessary density conditions on sampling nodes for stable reconstruction are given. On the other hand, in [16] a numerical analysis of the truncation error for (8) was given.

2. Preliminaries and proofs

In the sequel, we consider only the case $Q^n_\sigma = Q^n_\pi = \{x \in \mathbb{R}^n : \max_{1\le j\le n} |x_j| \le \pi\}$. This involves no loss of generality, since the operator

$$T_\sigma f(z) = \theta \cdot f\Big(\frac{\pi z}{\sigma}\Big) \quad \text{with} \quad \theta = \bigg(\frac{\prod_{j=1}^n \sigma_j}{\pi^n}\bigg)^{1/p},$$

$z \in \mathbb{C}^n$, is an isometric isomorphism between $B^p_{Q^n_\sigma}$ and $B^p_{Q^n_\pi}$ for all $1 \le p \le \infty$.

For technical reasons, let us define the following Banach space

$$B_{Q^n_\pi} = \{f \in C_0(\mathbb{R}^n) : \; supp\, \widehat{f} \subset Q^n_\pi\},$$

where $C_0(\mathbb{R}^n)$ is the usual space of continuous functions on $\mathbb{R}^n$ that vanish at infinity. If $1 \le p < \infty$, then any $f \in B^p_{Q^n_\pi}$ satisfies $\lim_{|x|\to\infty} f(x) = 0$ (see [15, p. 118]). Hence $B^p_{Q^n_\pi} \subset B_{Q^n_\pi}$ for all $1 \le p < \infty$. Moreover,

$$B^1_{Q^n_\sigma} \subset B^p_{Q^n_\pi} \subset B^q_{Q^n_\pi} \subset B_{Q^n_\pi} \subset B^\infty_{Q^n_\pi} \tag{9}$$

for $1 \le p < q < \infty$. Given $m \in \{1, 2, \dots, n\}$, set

$$H^n_m = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : z_m \in 2\mathbb{Z}\}.$$

It is clear that

$$2k + H^n_m = H^n_m \tag{10}$$

for each $k \in \mathbb{Z}^n$. Let

$$H^n = \bigcup_{m=1}^n H^n_m.$$

If ρ is a permutation of the set $\{1, 2, \dots, n\}$, then

$$w \in H^n \quad \text{if and only if} \quad (w_{\rho(1)}, w_{\rho(2)}, \dots, w_{\rho(n)}) \in H^n. \tag{11}$$

Assume that $f : \mathbb{C}^n \to \mathbb{C}$ is analytic in a neighbourhood U_a of $a \in \mathbb{C}^n$ and $f(a) = 0$. Let $f(z) = \sum_{m=0}^\infty P_m(z-a)$ be the expansion of f into homogeneous polynomials in $(z-a)$-powers. Recall that the minimal value of m such that $P_m \not\equiv 0$ on U_a is called the order of the zero a for f. We denote this order by $\mathrm{ord}_a(f)$. Note that if $f(a) \neq 0$, then say that $\mathrm{ord}_a(f) = 0$.

Lemma 2.1. *Let $f \in B_{Q_\pi^n}$. If $f(z) = 0$ for all $z \in H^n$, then there is an entire function $g : \mathbb{C}^n \to \mathbb{C}$ such that*

$$f(z) = \mathrm{sic}_n(\pi z)g(z),$$

$z \in \mathbb{C}^n$.

Proof. It is easy to see that the zeros set (all complex zeros) of $\mathrm{sic}_n(\pi z)$ coincides with H^n. Therefore, the statement of our lema follows immediately from application to f of the following fact (see [2, p. 12]): if F and H are entire functions on $\mathbb{C}^n$ such that $\mathrm{ord}_z(H) \le \mathrm{ord}_z(F)$ for all $z \in \mathbb{C}^n$, then there is an entire function G such that $F \equiv G \cdot H$. The proof is complete.

Let $M(\mathbb{R}^n)$ denote the Banach algebra of bounded regular Borel measures on $\mathbb{R}^n$ with the total variation norm $\|\mu\| = \|\mu\|_{M(\mathbb{R}^n)}$ and convolution as multiplication. The Fourier-Stieltjes transform of $\mu \in M(\mathbb{R}^n)$ is given by

$$\hat{\mu}(x) = \int_{\mathbb{R}^n} e^{-i\langle x,t\rangle}\, d\mu(t), \quad x \in \mathbb{R}^n.$$

We need certain facts about differential and convolution operators on $B_{Q_\pi^n}$. Bernstein's inequality (see 15, [p. 116]) states that each partial derivative operator acts on $B^p_{Q_\pi^n}$, $1 \le p \le \infty$, as bounded operator. We do not find the proof of the similar fact in the case of $B_{Q_\pi^n}$. For this reason, the proof of the following lemma is added here for completeness.

Lemma 2.2. *Let $f \in B_{Q_\pi^n}$. Then $\partial_z^k f \in B_{Q_\pi^n}$ for all $k \in \mathbb{Z}^n$ such that $k_1, \ldots, k_n \ge 0$.*

Proof. Let $\mu \in M(\mathbb{R}^n)$. Then

$$T_\mu f(x) = \int_{\mathbb{R}^n} f(x-y)\, d\mu(y) = f * \mu(x), \tag{12}$$

$x \in \mathbb{R}^n$, is well-defined linear bounded operator operator on $B^\infty_{Q_\pi^n}$ (see [4, p. 646]). Next, if $\nu \in M(\mathbb{R}^n)$ is such that $\widehat{\mu} = \widehat{\nu}$ on Q_π^n, then $T_\mu = T_\nu$ on $B^\infty_{Q_\pi^n}$ ([5, p. 90]). From (12) it follows that $\|T_\mu\|_{B^\infty_{Q_\pi^n}} \le \|\mu\|_{M(\mathbb{R}^n)}$. Moreover ([4, p. 646]),

$$\|T_\mu\|_{B^\infty_{Q_\pi^n}} = \inf\{\|\nu\| : \nu \in M(\mathbb{R}^n), \widehat{\nu} = \widehat{\mu} \text{ on } Q_\pi^n\}. \tag{13}$$

Next, if in a complex neighbourhood $U_c \subset \mathbb{C}^n$ of Q_π^n there exists an analytic function ζ such that $\zeta(x) = \widehat{\mu}(x)$ for all $x \in Q_\pi^n$, then T_μ coincides on $B^\infty_{Q_\pi^n}$ with the differential operator $\zeta(D)$, where

$$D = \Big(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}\Big),$$

(see [4, p. 646]). It is clearly that, given $m \in E^n$, there exists $\mu_m \in M(\mathbb{R}^n)$ such that

$$\widehat{\mu}_m(t) = \prod_{j=1}^n t_j^{m_j}$$

for all $t \in Q_\pi^n$. Therefore, if $f \in B^\infty_{Q_\pi^n}$, then

$$\partial_z^m f(x) = \widehat{\mu}_m(D) f(x) = \int_{\mathbb{R}^n} f(x-y)\, d\mu_m(y). \tag{14}$$

Assume now that $f \in B_{Q_\pi^n}$. Fix any $y \in \mathbb{R}^n$. Then the function $f_y(x) := f(x-y)$, $x \in \mathbb{R}^n$, is also in $B_{Q_\pi^n}$ Finally, using that $\|\mu_m\|_{M(\mathbb{R}^n)} < \infty$ and that $B_{Q_\pi^n}$ is a closed subspace of $B^\infty_{Q_\pi^n}$, we conclude from (14) that $\partial_z^m f \in B_{Q_\pi^n}$. The lemma is proved.

Recall that any $f \in B^\infty_{Q_\pi^n}$ satisfies

$$|f(z)| \le \sup_{x \in \mathbb{R}^n} |f(x)| e^{\pi \sum_{j=1}^n |y_j|}, \tag{15}$$

where $z = x + iy$, $x, y \in \mathbb{R}^n$ (see, e.g., [15, p. 117]). In addition, (9) implies that this estimate is also true for any $f \in B^p_{Q_\pi^n}$, $1 \le p < \infty$.

Proposition 2.3. *Let $f \in B_{Q_\pi^n}$. Suppose that*

$$\partial_z^k f(u) = 0 \tag{16}$$

for each $k \in E^n$ and all $u \in 2\mathbb{Z}^n$. Then $f \equiv 0$.

Proof. Our proof is by induction on the dimension n of $\mathbb{C}^n$. If n = 1, then (16) is equivalent to

$$f(2k) = f'(2k) = 0 \tag{17}$$

for each $f \in B_\pi$ and all $k \in \mathbb{Z}$. Set

$$g(z) = \frac{f(z)}{\sin^2(\pi z/2)}. \tag{18}$$

Then (17) implies that g is an entire function on $\mathbb{C}$. For each $\tau \in \mathbb{Z}$, $\tau > 0$, let us define

$$D_\tau = \{z \in \mathbb{C} : |\Re z| \leq 1 + 2\tau,\ |\Im z| \leq 2\}.$$

Then

$$\min_{z\in\partial D_\tau} \left|\sin \frac{\pi z}{2}\right| \geq 1 \tag{19}$$

for all $\tau = 1, 2, \ldots$. Combining (15) with (19) and using the maximum modules principe for analytic function in the domain D_τ, we see that $|g|$ is bounded on each D_τ by the same finite constant. Hence, $|g|$ is bounded on $D =: \cup_{p=1}^{\infty} D_\tau = \{z \in \mathbb{C} : |\Im z| \leq 2\}$. On the other hand, if $z \in \mathbb{C} \setminus D$, then it is easy to verify that

$$\left|\sin \frac{\pi z}{2}\right| \geq \frac{1}{2} e^{\pi|\Im z|/2}. \tag{20}$$

Combining this with (15), we conclude that $|g|$ is bounded on $\mathbb{C} \setminus D$. Hence g is a constant $c \in \mathbb{C}$. Then (18) gives that $f(z) = c\sin^2(\pi z/2)$. Finally, $c = 0$, since $\lim_{x\in\mathbb{R}; x\to\infty} f(x) = 0$.

Suppose that our proposition holds for dimension $m \geq 1$. First, we claim that if $f \in B_{Q_\pi^{m+1}}$ satisfies (16) with $n = m + 1$, then

$$\partial_z^s f(z) = 0 \tag{21}$$

for each $s \in E^{m+1}$ and all $z \in H^{m+1}$. Fix any $\widetilde{z} \in H^{m+1}$. According to (11), without loss of generality we can assume that $\widetilde{z} = (2u, \widetilde{z}_2, \ldots, \widetilde{z}_n)$ with $u \in \mathbb{Z}$ and $(\widetilde{z}_2, \ldots, \widetilde{z}_{m+1}) \in \mathbb{C}^m$. Let us define

$$F(z) = f(2u, z_1, \ldots, z_m)$$

for our fixed $u \in \mathbb{Z}$ and all $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$. Then $F \in B_{Q_\pi^m}$. By the induction hypothesis, the condition (16) with $n = m$ implies that

$$\partial_z^s F(z) = 0$$

for each $s \in E^m$ and all $z \in 2\mathbb{Z}^m$. Therefore, the induction hypothesis gives $F \equiv 0$ on $\mathbb{C}^m$, yielding our claim (21).

Now from lemma 2.1 it follows that there is an entire function g on $\mathbb{C}^{m+1}$ such that

$$f(z) = \mathrm{sic}_{m+1}(\pi z) g(z). \tag{22}$$

Second, we claim that

$$g(z) = 0 \tag{23}$$

for all $z \in H^{m+1}$. Fix any $\widetilde{z} \in H^{m+1}$. We need to show that $g(\widetilde{z}) = 0$. By (11), we can assume that there is $r \in \mathbb{Z}$, $1 \le r \le m+1$, such that

$$\widetilde{z} = (2u_1, \dots, 2u_r, \widetilde{z}_{r+1}, \dots, \widetilde{z}_{m+1})$$

for certain $u_1, \dots, u_r \in \mathbb{Z}$ and some $\widetilde{z}_{r+1}, \dots, \widetilde{z}_{m+1} \notin 2\mathbb{Z}$. Using our fixed numbers $u_1, \dots, u_r \in \mathbb{Z}$, let us define on $\mathbb{C}^{m+1-r}$ the function

$$F_{\widetilde{z}}(\lambda) = \frac{\partial^r}{\partial z_1 \cdots \partial z_r} f(2u_1, \dots, 2u_r, \lambda_1, \dots, \lambda_{m+1-r}), \tag{24}$$

$\lambda \in \mathbb{C}^{m+1-r}$. Then (16) implies that

$$\frac{\partial^{|\omega|}}{\partial^{\omega_1}\lambda_1 \cdots \partial^{\omega_{m+1-r}}\lambda_{m+1-r}} F_{\widetilde{z}}(\lambda) = 0 \tag{25}$$

for each $\omega \in E^{m+1-r}$ and all $\lambda \in 2\mathbb{Z}^{m+1-r}$. Next, Lemma 2.2 shows that $F_{\widetilde{z}} \in B_{Q_\pi^{m+1-r}}$. Therefore, using the induction hypothesis for dimension m and keeping in mind that $m+1-r \le m$, we conclude from (25) that $F_{\widetilde{z}} \equiv 0$ on $\mathbb{C}^{m+1-r}$. On the other hand, (22) implies that

$$F_{\widetilde{z}}(\lambda) = (-1)^{u_1+\cdots+u_r}\left(\frac{\pi}{2}\right)^r \prod_{j=1}^{m+1-r} \sin\left(\frac{\pi}{2}\lambda_j\right) g(2u_, \dots, 2u_r, \lambda_1, \dots, \lambda_{m+1-r}),$$

for all $\lambda \in \mathbb{C}^{m+1-r}$. Finally, if we take here $\lambda_j = \widetilde{z}_{r+j}$, $j = 1, \dots, m+1-r$, use the fact that $F_{\widetilde{z}} \equiv 0$ on $\mathbb{C}^{m+1-r}$, and keeping in mind that $\widetilde{z}_{r+j} \notin 2\mathbb{Z}$ for all $j = 1, \dots, m+1-r$, then we get $g(\widetilde{z}) = 0$. This proves (22).

Now lemma 2.1 shows that there is an entire function h on $\mathbb{C}^{m+1}$ such that $g(z) = \mathrm{sic}_{m+1}(\pi z)h(z)$. Using (21), we get

$$h(z) = \frac{f(z)}{\mathrm{sic}^2_{m+1}(\pi z)}$$

for all $z \in \mathbb{C}^{m+1}$. Now combining (15) with (19) and (20), we conclude that h is bounded on $\mathbb{C}^{m+1}$. By Liouville's theorem, it follows that h is a constant $c \in \mathbb{C}$. Therefore, $f(z) = c \cdot \mathrm{sic}^2_{m+1}(\pi z)$, $z \in \mathbb{C}^{m+1}$. Using the fact that $f \in B_{Q_\pi^n}$, i.e., $\lim_{|x|\to\infty} f(x) = 0$, we see that $c = 0$, which completes the proof of Proposition 2.3.

For $1 \le p < \infty$, let l_n^p denote the usual Banach space of sequences of complex numbers $\{c_n\}_{u \in \mathbb{Z}^n}$ such that $\sum_{u \in \mathbb{Z}^n} |c_u|^p < \infty$. By Nikol'skii's inequality ([15, p. 123]), for any $0 < \theta < \infty$, there exists a finite constant $a = a(p, \sigma, \theta)$ such that

$$\left(\sum_{u \in \mathbb{Z}^n} |f(\theta u)|^p\right)^{1/p} \le a\|f\|_{B^p_{Q_\pi^n}}$$

for all $f \in B^p_{Q_\pi^n}$. Combining this with Bernstein's inequality in $B^p_{Q_\pi^n}$ (see [15, p. 116]), we deduce that a similar estimate holds also for all derivatives $\partial_z^k f$ with $k \in E^n$, i.e.,

$$\left(\sum_{u \in \mathbb{Z}^n} |\partial_z^k f(\theta u)|^p\right)^{1/p} \le A\|f\|_{B^p_{Q_\pi^n}}$$

for each $0 < \theta < \infty$, certain $A = A(k, p, \sigma, \theta) < \infty$ and all $f \in B^p_{Q_\pi^n}$. In particular, this this means that if $f \in B^p_{Q_\pi^n}$, then

$$\left\{\partial_z^k f(\theta u)\right\}_{u \in \mathbb{Z}^n} \in l_n^p \tag{26}$$

for each $0 < \theta < \infty$ and all $k \in E^n$.

Recall that if $1 < r < \infty$, then (see [17, p. 811])

$$\sum_{m\in\mathbb{Z}^n} \Big|\mathrm{sinc}_n\Big(ax-m\Big)\Big|^r \le \Big(1+\frac{1}{r-1}\Big)^n \tag{27}$$

for each $a > 0$ and all $x \in \mathbb{R}^n$.

Proof of Theorem 1.1. Let us first show that (7) converges absolutely and uniformly on $\mathbb{R}^n$. To this end, we divide (7) into 2^n series of the following type

$$S_k(x) = \sum_{u\in 2\mathbb{Z}^n} P_{f,k,u}(x-u)\mathrm{sinc}_n{}^2\Big(\frac{x-u}{2}\Big) = \sum_{u\in 2\mathbb{Z}^n}\Big(\partial_x^k f(u)\prod_{j=1}^n (x_j-u_j)^{k_j}\mathrm{sinc}_n{}^2\Big(\frac{x-u}{2}\Big)\Big), \tag{28}$$

$x \in \mathbb{R}^n$, $k \in E^n$. Now it remains to prove that for any $\varepsilon > 0$ there is a positive integer τ such that

$$|S_{k,\tau}(x)| = \Big|\sum_{\substack{u\in 2\mathbb{Z}^n\\ |u_1|,\dots|u_n|\ge\tau}} \partial_x^k f(u)\prod_{j=1}^n (x_j-u_j)^{k_j}\mathrm{sinc}_n{}^2\Big(\frac{x-u}{2}\Big)\Big| < \varepsilon \tag{29}$$

for all $x \in \mathbb{R}^n$ and each $k \in E^n$.

Fix an arbitrary number p_1 such that $p_1 \ge p$ and $1 < p_1 < \infty$. Let $q_1 = p_1/(p_1-1)$. Applying Hölder's inequality to (28) gives

$$|S_{k,\tau}(x)| \le \Big(\sum_{\substack{u\in 2\mathbb{Z}^n\\ |u_1|,\dots|u_n|\ge\tau}} \Big|\partial_x^k f(u)\Big|^{p_1}\Big)^{1/p_1}\Big(\sum_{\substack{u\in 2\mathbb{Z}^n\\ |u_1|,\dots|u_n|\ge\tau}}\prod_{j=1}^n |x_j-u_j|^{q_1k_j}\Big|\mathrm{sinc}_n\Big(\frac{x-u}{2}\Big)\Big|^{2q_1}\Big)^{1/q_1}. \tag{30}$$

The condition $p_1 \ge p$ implies that $l_n^p \subset l_n^{p_1}$. Hence, we conclude from (27) that there exists a positive integer τ_1 such that

$$\Big(\sum_{\substack{u\in 2\mathbb{Z}^n\\ |u_1|,\dots|u_n|\ge\tau_1}} \Big|\partial_x^k f(u)\Big|^{p_1}\Big)^{1/p_1} < \varepsilon. \tag{31}$$

Since $|\mathrm{sinc}_1 t| \le 1$ for $t \in \mathbb{R}$ and $k_1,\dots,k_n \in \{0;1\}$, we see that

$$\begin{aligned}&|x_j-u_j|^{q_1k_j}\Big|\mathrm{sinc}_1\Big(\frac{x_j-u_j}{2}\Big)\Big|^{2q_1} \le \Big(\frac{2}{\pi}\Big)^{q_1k_j}\Big|\mathrm{sinc}_1\Big(\frac{x_j-u_j}{2}\Big)\Big|^{q_1(2-k_j)}\\ &\le \Big(\frac{2}{\pi}\Big)^{q_1k_j}\Big|\mathrm{sinc}_1\Big(\frac{x_j-u_j}{2}\Big)\Big|^{q_1}\end{aligned}$$

for all $x_j \in \mathbb{R}$ and each $u_j \in \mathbb{Z}$, $j = 1,\dots,n$. Therefore, the second factor on the right of (30) is estimated by

$$\begin{aligned}&\sum_{\substack{u\in 2\mathbb{Z}^n\\ |u_1|,\dots|u_n|\ge\tau}}\prod_{j=1}^n |x_j-u_j|^{q_1k_j}\Big|\mathrm{sinc}_n\Big(\frac{x-u}{2}\Big)\Big|^{2q_1}\\ &\le \Big(\frac{2}{\pi}\Big)^{(k_1+\dots+k_n)q_1}\sum_{\substack{u\in 2\mathbb{Z}^n\\ |u_1|,\dots|u_n|\ge\tau}}\prod_{j=1}^n\Big|\mathrm{sinc}_1\Big(\frac{x_j-u_j}{2}\Big)\Big|^{q_1}\\ &= \Big(\frac{2}{\pi}\Big)^{(k_1+\dots+k_n)q_1}\sum_{\substack{u\in 2\mathbb{Z}^n\\ |u_1|,\dots|u_n|\ge\tau}}\Big|\mathrm{sinc}_n\Big(\frac{x-u}{2}\Big)\Big|^{q_1},\end{aligned} \tag{32}$$

$x \in \mathbb{R}^n$. Now the condition $1 < p_1 < \infty$ implies that $1 < q_1 < \infty$. Therefore, using (27), we conclude that there exists positive integer τ_2 such that

$$\left(\sum_{\substack{u \in 2\mathbb{Z}^n \\ |u_1|,\dots|u_n| \geq \tau_2}} \prod_{j=1}^{n} |x_j - u_j|^{q_1 k_j} \left| \operatorname{sinc}_n \left(\frac{x-u}{2} \right) \right|^{2q_1} \right)^{1/q_1} < \varepsilon \tag{33}$$

for all $x \in \mathbb{R}^n$. Finally, if we assume that $\varepsilon < 1$ and take $\tau = \max\{\tau_1, \tau_2\}$, then combining (30), (31) and (33), we obtain (29). This complete the proof that (7) converges absolutely and uniformly on $\mathbb{R}^n$.

If $f \in B^p_{Q^n_\pi}$, $1 < p < \infty$, then any partial sum of (7) is also in $B^p_{Q^n_\pi}$. For $f \in B^1_{Q^n_\pi}$, these partial sums is in $B^p_{Q^n_\pi}$ for each $1 < p < \infty$. Hence, by (9), these sums are also elements of $B_{Q^n_\pi}$ for any $1 \leq p < \infty$. Let F be the sum of (7). Since $B_{Q^n_\pi}$ is a Banach space and (7) converges absolutely and uniformly on $\mathbb{R}^n$, it follows that $F \in B_{Q^n_\pi}$. Let us denote $f_1 = f - F$. Then $f_1 \in B_{Q^n_\pi}$. Using the fact that (7) converges uniformly on $\mathbb{R}^n$ to F, we conclude that f_1 satisfies the all conditions (16). By Proposition 2.3, we have that $f_1 \equiv 0$. Hence, $F \equiv f$, i.e., equality (7) holds for all $x \in \mathbb{R}^n$.

Let K be a compact subset of $\mathbb{C}^n$. Then there exists $0 < a < \infty$ such that K is a subset of the strip $M_a = \{z \in \mathbb{C}^n : |\Im z_j| \leq a, j = 1, \dots, n\}$. Given $\omega \in \mathbb{Z}^n$, let us define the partial sum of (7) by

$$W_\omega(z) = \sum_{\substack{u \in 2\mathbb{Z}^n \\ |u_1| \geq |\omega_1|, \dots |u_n| \geq |\omega_n|}} \left(\sum_{k \in E^n} P_{f,k,u}(z-u) \right) \operatorname{sinc}_n{}^2 \left(\frac{1}{2\pi} \sigma(z-u) \right).$$

According to (9), we have from (15) that

$$\sup_{z \in K} |f(z) - W_\omega(z)| \leq \sup_{x \in \mathbb{R}^n} |f(x) - W_\omega(x)| e^{\pi a n}. \tag{34}$$

Since (7) converges uniformly on $\mathbb{R}^n$ to f, we see from (34) that (7) converges uniformly also on K to f. Theorem 1.1 is proved.

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