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Complete Moment Convergence for Weighted Sums of Pairwise Negatively Quadrant Dependent Random Variables

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Abstract. In this article, the complete moment convergence for weighted sums of pairwise negatively quadrant dependent (NQD, for short) random variables is studied. Several sufficient conditions to prove the complete moment convergence for weighted sums of NQD random variables are presented. The results obtained in the paper extend some corresponding ones in the literature. The simulation is also presented which can verify the validity of the theoretical result.

1. Introduction

As is known to all, complete convergence plays an important role in the probability limit theory and mathematical statistics, especially in establishing the strong convergence rate for partial sums of random variables.

The concept of complete convergence was first introduced by Hsu and Robbins [1] as follows: A sequence $\{X_n,\ n \geq 1\}$ of random variables converges completely to a constant C, if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty.$$

Gut [2] extended and generalized some recent results of complete convergence for an array of rowwise independent random variables. Liang and Su [3] obtained the complete convergence for weighted sums of negatively associated (NA, for short) sequences and discussed its necessity.

Chow [4] first investigated the complete moment convergence, which is stronger than the complete convergence. The concept of complete moment convergence is as follows : Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables and $a_n > 0,\ b_n > 0,\ q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty$$

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for all $\varepsilon > 0$, then $\{X_n,\ n \geq 1\}$ is said to be complete moment convergence. As is known to all, complete moment convergence implies complete convergence. For more details about the complete moment convergence, we refer the readers to Chen and Wang [5] and Qiu and Chen [6]. Recently, Wu et al. [7] obtained the following complete moment convergence for ρ^*-mixing random variables.

Theorem 1.1. *Let $\nu > 0$, $\alpha > \frac{1}{2}$, $\alpha p > 1$, $q > (p \vee \nu)$. Let $\{X, X_n,\ n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX = 0$ if $p \vee \nu \geq 1$. Assume that $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ is an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^q \ll n. \tag{1.1}$$

If

$$\begin{cases} E|X|^p < \infty, & \nu < p, \\ E|X|^p \log(1 + |X|) < \infty, & \nu = p, \\ E|X|^\nu < \infty, & \nu > p, \end{cases} \tag{1.2}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha\nu - 2} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^\alpha\right)_+^\nu < \infty, \tag{1.3}$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^\alpha\right) < \infty. \tag{1.4}$$

Qiu and Xiao [8] generalized the result of Theorem 1.1 for ρ^*-mixing random variables to the case of extended negatively dependent (END, for short) random variables and obtained the following result.

Theorem 1.2. *Let $\nu > 0$, $\alpha > \frac{1}{2}$, $\alpha p > 1$, $q > (p \vee \nu) \geq 1$. Let $\{X, X_n,\ n \geq 1\}$ be a sequence of identically distributed END random variables with EX=0. Assume that $\{a_{ni},\ 1 \leq i \leq n,\ n \geq 1\}$ is an array of constants satisfying (1.1). If (1.2) holds, then for any $\varepsilon > 0$, (1.3) holds, and thus (1.4) holds.*

The main purpose of this paper is to extend Theorem 1.1 and Theorem 1.2 from ρ^*-mixing, respectively and END random variables to the case of NQD random variables.

The concept of NQD random variables was introduced by Lehmann [9] as follows: two random variables X and Y are said to be negative quadrant dependent (NQD, for short), if for any $x, y \in \mathbb{R}$,

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y).$$

The sequence $\{X_n, n \geq 1\}$ is said be pairwise NQD, if X_i and X_j are NQD for any $i \neq j$. It's known that NQD contains NA as a special case. It's not difficult to see that NQD has many applications. For example, Matula [10] extended the classical strong law of large numbers for independent and identically distributed random variables and three series theorem to the case of negatively associated random variables, especially generalized to the case of pairwise NQD random variables. Wang et al. [11] established the Marcinkiewicz's weak law of large numbers and the strong stability of Jamison's weighted sum for pairwise NQD sequences. For more details about NQD, we refer the readers to Su and Wang [12] and Wu [13] among others.

The layout of this paper is as follows. Some preliminary lemmas are provided in Section 2. Our main results and their proofs are stated in Section 3. The simulation study is presented in Section 4.

Throughout the paper, C represents some positive constant whose value may vary in different places. Let $\log x = \ln \max(x, e)$. Denote $x_+ = xI(x \geq 0)$. $I(A)$ stands for the indicator function of the set A. $a \ll b$ implies that there exists some positive constant c, such that $a \leq cb$. $a \vee b$ stands for $\max(a, b)$ and $a \wedge b$ means $\min(a, b)$.

2. Preliminary Lemmas

To prove our main result of this paper, we need the following lemmas. The first one comes from Lehmann [9].

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables. If $\{f_n(x), n \geq 1\}$ are all nondecreasing (or all nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ are still pairwise NQD.*

The next one is the Marcinkiewicz-Zygmund type moment inequality for NQD random variables, which can be found in Chen et al. [14].

Lemma 2.2. *Let $1 < \mu \leq 2$ and $\{X_n,\ n \geq 1\}$ be a sequence of pairwise NQD random variables with $EX_n = 0$ for each $n \geq 1$. Then there exists a positive constant C_μ depending only on μ such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^{\mu} \leq C_\mu \sum_{i=1}^{n} E|X_i|^{\mu},$$

and

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^{\mu}\right) \leq C_\mu (\log n)^{\mu} \sum_{i=1}^{n} E|X_i|^{\mu}.$$

The following one comes from Wu et al. [7].

Lemma 2.3. *Let Y and Z be two random variables. Then for any $\mu > \nu > 0, a > 0$ and $\varepsilon > 0$, we have that*

$$E(|Y+Z| - \varepsilon a)_+^{\nu} \leq C_\nu \left(\varepsilon^{-\mu} + \frac{\nu}{\mu - \nu}\right) a^{\nu-\mu} E|Y|^{\mu} + C_\nu E|Z|^{\nu},$$

where $C_\nu = 1$ if $0 < \nu \leq 1$, or $C_\nu = 2^{\nu-1}$ if $\nu > 1$.

The next one comes from Qiu and Xiao [8].

Lemma 2.4. *Let $\alpha > 0$, $p > 0$ and X be a random variable.*

(i) For any $\nu > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha\nu - 1} E|X|^{\nu} I(|X| > n^{\alpha}) \ll \begin{cases} E|X|^p, & \nu < p, \\ E|X|^p \log|X|, & \nu = p, \\ E|X|^{\nu}, & \nu > p. \end{cases}$$

(ii) For any $\mu > p$,

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha\mu - 1} E|X|^{\mu} I(|X| \leq n^{\alpha}) \ll E|X|^p.$$

Inspired by Lemma 2.4, we have the following lemma.

Lemma 2.5. *Let $\alpha > 0$, $p > 0$, $t \geq 0$ and X be a random variable.*

(i) For any $\nu > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha\nu - 1} \log^t n E|X|^{\nu} I(|X| > n^{\alpha}) \ll \begin{cases} E|X|^p \log^t |X|, & \nu < p, \\ E|X|^p \log^{t+1} |X|, & \nu = p, \\ E|X|^{\nu}, & \nu > p. \end{cases}$$

(ii) For any $\mu > p$,

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha\mu - 1} \log^t n E|X|^{\mu} I(|X| \leq n^{\alpha}) \ll E|X|^p \log^t |X|.$$

Proof. By some standard computation, we obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-1}\log^t nE|X|^\nu I(|X|>n^\alpha)\\
= &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-1}\log^t n\sum_{m=n}^{\infty} E|X|^\nu I(m^\alpha<|X|\le(m+1)^\alpha)\\
= &\sum_{m=1}^{\infty} E|X|^\nu I(m^\alpha<|X|\le(m+1)^\alpha)\sum_{n=1}^{m} n^{\alpha p-\alpha\nu-1}\log^t n\\
\ll &\begin{cases}\sum_{m=1}^{\infty} m^{\alpha p-\alpha\nu}\log^t mE|X|^\nu I(m^\alpha<|X|\le(m+1)^\alpha)\ll E|X|^p\log^t|X|, & \text{if } \nu<p,\\ \sum_{m=1}^{\infty}\log^{t+1} mE|X|^\nu I(m^\alpha<|X|\le(m+1)^\alpha)\ll E|X|^p\log^{1+t}|X|, & \text{if } \nu=p,\\ \sum_{m=1}^{\infty} E|X|^\nu I(m^\alpha<|X|\le(m+1)^\alpha)\ll E|X|^\nu, & \text{if } \nu>p,\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-1}\log^t nE|X|^\mu I(|X|\le n^\alpha)\\
= &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-1}\log^t n\sum_{m=1}^{n} E|X|^\mu I((m-1)^\alpha<|X|\le m^\alpha)\\
= &\sum_{m=1}^{\infty} E|X|^\mu I((m-1)^\alpha<|X|\le m^\alpha)\sum_{n=m}^{\infty} n^{\alpha p-\alpha\mu-1}\log^t n\\
\ll &\sum_{m=1}^{\infty} m^{\alpha p-\alpha\mu}\log^t mE|X|^\mu I((m-1)^\alpha<|X|\le m^\alpha)\\
\ll &E|X|^p\log^t|X|.
\end{aligned}$$

The proof of the lemma is completed. □

3. Main results and their proofs

With preliminaries accounted for, we can give our main results.

Theorem 3.1. *Let $\nu>0$, $\alpha>\frac{1}{2}$, $0<p<2$, $0<(p\vee\nu)\le 2$, $\alpha(p\vee\nu)>1$, and $\{X, X_n, n\ge 1\}$ be a sequence of identically distributed pairwise NQD random variables with $EX=0$. Assume that $\{a_{ni}, 1\le i\le n, n\ge 1\}$ is an array of constants satisfying*

$$\sum_{i=1}^{n}|a_{ni}|^q\ll n, \text{ for some } q>(p\vee\nu). \tag{3.1}$$

If

$$\begin{cases} E|X|^p<\infty, & \nu<p,\\ E|X|^p\log|X|<\infty, & \nu=p,\\ E|X|^\nu<\infty, & \nu>p,\end{cases} \tag{3.2}$$

then for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|-\varepsilon n^\alpha\right)_+^\nu<\infty, \tag{3.3}$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{3.4}$$

Proof. Without loss of generality, we assume that $\sum_{i=1}^{n} |a_{ni}|^q \le n$. By Hölder's inequality, we obtain

$$\sum_{i=1}^{n} |a_{ni}|^{\mu} \le \left(\sum_{i=1}^{n} |a_{ni}|^{q}\right)^{\frac{\mu}{q}} \left(\sum_{i=1}^{n} 1\right)^{1-\frac{\mu}{q}} \le n, \tag{3.5}$$

for any $0 < \mu < q$. The proof will be conducted under the following two cases.

Case 1. $0 < p \vee \nu < 1$.

Denote for $1 \le i \le n$ that

$$X_{ni}^{(1)} = a_{ni}X_i I(|X_i| \le n^{\alpha}),$$
$$X_{ni}^{(2)} = a_{ni}X_i - X_{ni}^{(1)} = a_{ni}X_i I(|X_i| > n^{\alpha}).$$

Take $\mu = q \wedge 1$. We have by Lemma 2.3, Lemma 2.4 and C_r-inequality that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)_+^{\nu} \\
= &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} (X_{ni}^{(1)} + X_{ni}^{(2)})\right| - \varepsilon n^{\alpha}\right)_+^{\nu} \\
\ll &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}^{(1)}\right|\right)^{\mu} + \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}^{(2)}\right|\right)^{\nu} \\
\ll &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} \sum_{i=1}^{n} |a_{ni}|^{\mu} E|X|^{\mu} I(|X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \sum_{i=1}^{n} |a_{ni}|^{\nu} E|X|^{\nu} I(|X| > n^{\alpha}) \\
< &\ \infty,
\end{aligned}$$

which implies (3.3).

Case 2. $1 \le p \vee \nu \le 2$.

For any fixed $n \ge 1$ and $\varepsilon > 0$, we obtain by C_r-inequality, (3.1) and (3.2) that

$$E\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}_+^{\nu} < \infty.$$

Take $\theta \in \left(\frac{1}{\alpha(p \vee \nu)}, 1\right)$. For $1 \le i \le n$ and $n \ge 1$, let

$$\begin{aligned}
X_{ni}^{(1)} &= -n^{\alpha\theta} I(a_{ni}X_i < -n^{\alpha\theta}) + a_{ni}X_i I(|a_{ni}X_i| \le n^{\alpha\theta}) + n^{\alpha\theta} I(a_{ni}X_i > n^{\alpha\theta}), \\
X_{ni}^{(2)} &= (a_{ni}X_i - n^{\alpha\theta}) I(n^{\alpha\theta} < a_{ni}X_i \le n^{\alpha\theta} + n^{\alpha}) + n^{\alpha} I(a_{ni}X_i > n^{\alpha\theta} + n^{\alpha}), \\
X_{ni}^{(3)} &= (a_{ni}X_i + n^{\alpha\theta}) I(-n^{\alpha\theta} - n^{\alpha} \le a_{ni}X_i < -n^{\alpha\theta}) - n^{\alpha} I(a_{ni}X_i < -n^{\alpha\theta} - n^{\alpha}), \\
X_{ni}^{(4)} &= (a_{ni}X_i - n^{\alpha\theta} - n^{\alpha}) I(a_{ni}X_i > n^{\alpha\theta} + n^{\alpha}), \\
X_{ni}^{(5)} &= (a_{ni}X_i + n^{\alpha\theta} + n^{\alpha}) I(a_{ni}X_i < -n^{\alpha\theta} - n^{\alpha}).
\end{aligned}$$

Then $a_{ni}X_i = \sum_{l=1}^{5} X_{ni}^{(l)}$.

By the definition of $X_{ni}^{(2)}$ and (3.5), we have that

$$\begin{aligned} n^{-\alpha}\max_{1\le k\le n}\left|\sum_{i=1}^{k} EX_{ni}^{(2)}\right| &= n^{-\alpha}\sum_{i=1}^{n} EX_{ni}^{(2)} \\ &\le n^{-\alpha}\sum_{i=1}^{n} E|a_{ni}X_i|I(|a_{ni}X_i|>n^{\alpha\theta}) \\ &\le n^{-\alpha}\sum_{i=1}^{n} E|a_{ni}X_i|\left(\frac{|a_{ni}X_i|}{n^{\alpha\theta}}\right)^{p\vee\nu-1} I(|a_{ni}X_i|>n^{\alpha\theta}) \\ &\ll n^{1-\alpha\theta(p\vee\nu)}E|X|^{p\vee\nu}\to 0,\ \text{as } n\to\infty. \end{aligned}$$

By the definition of $X_{ni}^{(4)}$ and the proof above, we have

$$n^{-\alpha}\max_{1\le k\le n}\left|\sum_{i=1}^{k} EX_{ni}^{(4)}\right| = n^{-\alpha}\sum_{i=1}^{n} EX_{ni}^{(4)} \le n^{-\alpha}\sum_{i=1}^{n} E|a_{ni}X_i|I(|a_{ni}X_i|>n^{\alpha\theta})\to 0,\ \text{as } n\to\infty.$$

Similarly, we have

$$\begin{aligned} &\lim_{n\to\infty} n^{-\alpha}\max_{1\le k\le n}\left|\sum_{i=1}^{k} EX_{ni}^{(3)}\right| = \lim_{n\to\infty}\left(-n^{-\alpha}\sum_{i=1}^{n} EX_{ni}^{(3)}\right) = 0, \\ &\lim_{n\to\infty} n^{-\alpha}\max_{1\le k\le n}\left|\sum_{i=1}^{k} EX_{ni}^{(5)}\right| = \lim_{n\to\infty}\left(-n^{-\alpha}\sum_{i=1}^{n} EX_{ni}^{(5)}\right) = 0. \end{aligned}$$

If $0<\nu<1$, by $EX_i=0$, Lemma 2.3 and C_r-inequality, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)_+^{\nu} \\ =\ &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\sum_{l=1}^{5}(X_{ni}^{(l)}-EX_{ni}^{(l)})\right| - \varepsilon n^{\alpha}\right)_+^{\nu} \\ \le\ &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}E\left(\sum_{l=1}^{5}\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_{ni}^{(l)}-EX_{ni}^{(l)})\right| - \varepsilon n^{\alpha}\right)_+^{\nu} \\ \ll\ &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_{ni}^{(1)}-EX_{ni}^{(1)})\right| + \sum_{l=2}^{5}\left|\sum_{i=1}^{n} X_{ni}^{(l)}\right| - \frac{\varepsilon n^{\alpha}}{2}\right)_+^{\nu} \\ \ll\ &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_{ni}^{(1)}-EX_{ni}^{(1)})\right| + \sum_{l=2}^{3}\left|\sum_{i=1}^{n}(X_{ni}^{(l)}-EX_{ni}^{(l)})\right| + \sum_{l=4}^{5}\left|\sum_{i=1}^{n} X_{ni}^{(l)}\right| - \varepsilon n^{\alpha}/3\right)_+^{\nu} \\ \ll\ &\sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}^{(1)}-EX_{ni}^{(1)}\right|^{\mu}\right) + \sum_{l=2}^{3}\sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}E\left(\left|\sum_{i=1}^{n}(X_{ni}^{(l)}-EX_{ni}^{(l)})\right|^{\mu}\right) \\ &+\sum_{l=4}^{5}\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2}E\left(\left|\sum_{i=1}^{k} X_{ni}^{(l)}\right|^{\nu}\right) \\ =:\ &I_1+I_2+I_3+I_4+I_5. \end{aligned}$$

If $1 \le \nu \le 2$, similarly we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)_+^{\nu} \\
\ll\ & \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} (X_{ni}^{(1)} - EX_{ni}^{(1)})\right| + \sum_{l=2}^{5}\left|\sum_{i=1}^{n} X_{ni}^{(l)}\right| - \frac{\varepsilon n^{\alpha}}{2}\right)_+^{\nu} \\
\ll\ & \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} (X_{ni}^{(1)} - EX_{ni}^{(1)})\right| + \sum_{l=2}^{5}\left|\sum_{i=1}^{n} (X_{ni}^{(l)} - EX_{ni}^{(l)})\right| - \frac{\varepsilon n^{\alpha}}{3}\right)_+^{\nu} \\
\ll\ & \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} \left(X_{ni}^{(1)} - EX_{ni}^{(1)}\right)\right|^{\mu}\right) \\
&+\sum_{l=2}^{3}\sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} E\left|\sum_{i=1}^{n} \left(X_{ni}^{(l)} - EX_{ni}^{(l)}\right)\right|^{\mu} + \sum_{l=4}^{5}\sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} E\left|\sum_{i=1}^{n} \left(X_{ni}^{(l)} - EX_{ni}^{(l)}\right)\right|^{\nu} \\
=:\ & I_1 + I_2 + I_3 + I_4^{'} + I_5^{'}.
\end{aligned}$$

Take $\mu = q \wedge 2$. By Lemmas 2.1 and 2.2, C_r-inequality, Jensen's inequality, (3.2) and the definition of $X_{ni}^{(1)}$, we have

$$\begin{aligned}
I_1 &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}(\log n)^{\mu}\left(\sum_{i=1}^{n} E\left|X_{ni}^{(1)}\right|^{\mu}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}(\log n)^{\mu}\left(\sum_{i=1}^{n} E\left|a_{ni}X_i\right|^{p} n^{(\mu-p)\alpha\theta}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}(\log n)^{\mu}\left(E|X|^{p} n^{1+(\mu-p)\alpha\theta}\right) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2+1+(\mu-p)\alpha\theta}(\log n)^{\mu} E|X|^{p} \\
&\ll \sum_{n=1}^{\infty} n^{-\alpha(\mu-p)(1-\theta)-1}(\log n)^{\mu} < \infty.
\end{aligned}$$

By Lemmas 2.1 and 2.2, C_r-inequality, Jensen's inequality, we also have that

$$\begin{aligned}
I_2 &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\sum_{i=1}^{n} E|X_{ni}^{(2)}|^{\mu} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}\sum_{i=1}^{n} \left[E|a_{ni}X_i|^{\mu} I(|a_{ni}X_i| \le 2n^{\alpha}) + n^{\alpha\mu}P(|a_{ni}X_i| > n^{\alpha})\right] \\
&=: B_1 + B_2.
\end{aligned}$$

Take $t \in (p, \mu)$ and $\lambda = \frac{p}{2}$. Note that

$$\begin{aligned}
|a_{ni}X|^{\mu} I(|a_{ni}X| \le 2n^{\alpha}) &= |a_{ni}X|^{\mu}[I(|a_{ni}X| \le 2n^{\alpha}, |X| \le n^{\alpha}) + I(|a_{ni}X| \le 2n^{\alpha}, |X| > n^{\alpha})] \\
&\le (2n^{\alpha})^{\mu-t}|a_{ni}X|^{t} I(|X| \le n^{\alpha}) + (2n^{\alpha})^{\mu-\lambda}|a_{ni}X|^{\lambda} I(|X| > n^{\alpha}).
\end{aligned}$$

Then we have by Lemma 2.4 that

$$\begin{aligned} B_1 &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} \sum_{i=1}^{n} E|a_{ni}X|^{\mu} I(|a_{ni}X| \le 2n^{\alpha}) \\ &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha t-2} \sum_{i=1}^{n} E|a_{ni}X|^{t} I(|X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{-2+\frac{\alpha p}{2}} \sum_{i=1}^{n} E|a_{ni}X|^{\frac{p}{2}} I(|X| > n^{\alpha}) \\ &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha t-1} E|X|^{t} I(|X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{-1+\frac{\alpha p}{2}} E|X|^{\frac{p}{2}} I(|X| > n^{\alpha}) \\ &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha t-1} E|X|^{t} I(|X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{-1+\frac{\alpha p}{2}} \sum_{i=n}^{\infty} E|X|^{\frac{p}{2}} I(i^{\alpha} < |X| < (i+1)^{\alpha}) \\ &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha t-1} E|X|^{t} I(|X| \le n^{\alpha}) + \sum_{i=1}^{\infty} E|X|^{\frac{p}{2}} I(i^{\alpha} < |X| < (i+1)^{\alpha}) \sum_{n=1}^{i} n^{-1+\frac{\alpha p}{2}} \\ &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha t-1} E|X|^{t} I(|X| \le n^{\alpha}) + \sum_{i=1}^{\infty} E|X|^{\frac{p}{2}} I(i^{\alpha} < |X| < (i+1)^{\alpha}) i^{\frac{\alpha p}{2}} \\ &\ll E|X|^{p} \\ &< \infty. \end{aligned}$$

For any $s \ge 0$, we have

$$\begin{aligned} I(|a_{ni}X| > n^{\alpha}) &= I(|a_{ni}X| > n^{\alpha}, |X| \le n^{\alpha}) + I(|a_{ni}X| > n^{\alpha}, |X| > n^{\alpha}) \\ &\le (\frac{|a_{ni}X|}{n^{\alpha}})^{s} I(a_{ni}X| > n^{\alpha}, |X| \le n^{\alpha}) + I(|X| > n^{\alpha}). \end{aligned}$$

Then we have by Lemma 2.4 and (3.5) that

$$\begin{aligned} B_2 &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \sum_{i=1}^{n} E|a_{ni}X|^{\nu} I(|a_{ni}X| > n^{\alpha}) \\ &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \sum_{i=1}^{n} E|a_{ni}X|^{\nu} (\frac{|a_{ni}X|}{n^{\alpha}})^{\mu-\nu} I(|a_{ni}X| > n^{\alpha}, |X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \sum_{i=1}^{n} E|a_{ni}X|^{\nu} I(|X| > n^{\alpha}) \\ &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2} \sum_{i=1}^{n} E|a_{ni}X|^{\mu} I(|X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \sum_{i=1}^{n} E|a_{ni}X|^{\nu} I(|X| > n^{\alpha}) \\ &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-1} E|X|^{\mu} I(|X| \le n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-1} E|X|^{\nu} I(|X| > n^{\alpha}) \\ &< \infty. \end{aligned}$$

Hence, $I_2 < \infty$ follows from $B_1 < \infty$ and $B_2 < \infty$.

When $0 < \nu < 1$, similar to the proof of $B_2 < \infty$, we have by the definition of $X_{ni}^{(4)}$ and C_r-inequality that

$$\begin{aligned} I_4 &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \sum_{i=1}^{n} E|X_{ni}^{(4)}|^{\nu} \\ &\le \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \sum_{i=1}^{n} E|a_{ni}X_i|^{\nu} I(|a_{ni}X_i| > n^{\alpha}) < \infty. \end{aligned}$$

When $1 \le \nu \le 2$, by the definition of $X_{ni}^{(4)}$, Lemma 2.2, C_r-inequality and Jensen's inequality, we also have

$$I_4' \ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \sum_{i=1}^{n} E|X_{ni}^{(4)}|^{\nu} < \infty.$$

Similar to the proofs of $I_2 < \infty$, $I_4 < \infty$ and $I_4' < \infty$, we can obtain $I_3 < \infty$, $I_5 < \infty$ and $I_5' < \infty$. This completes the proof of the theorem. □

For $\alpha(p \vee \nu) = 1$, we have the following result.

Theorem 3.2. *Let $\nu > 0$, $\alpha > \frac{1}{2}$, $p > 0$, $0 < (p \vee \nu) < 2$, $\alpha(p \vee \nu) = 1$, and $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed pairwise NQD random variables with $EX = 0$ if $p \vee \nu \geq 1$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^q \ll n, \text{ for some } q > (p \vee \nu). \tag{3.6}$$

If

$$\begin{cases} E|X|^p \log^{\mu} |X| < \infty, & \nu < p, \\ E|X|^p \log^{(1+\mu)} |X| < \infty, & \nu = p, \quad \text{where } \mu = q \wedge 2, \\ E|X|^{\nu} < \infty, & \nu > p, \end{cases} \tag{3.7}$$

then for any $\varepsilon > 0$, (3.3) holds, and thus (3.4) holds.

Proof. We also assume without loss of generality that $\sum_{i=1}^{n} |a_{ni}|^q \leq n$. Note that if $0 < p \vee \nu < 1$, the proof is the same as that of Theorem 3.1. Now we only consider the case $\alpha(p \vee \nu) = 1$ for $1 \leq p \vee \nu < 2$. Denote

$$\begin{aligned} Y_{ni} &= a_{ni} \left[-n^{\alpha} I(X_i < -n^{\alpha}) + X_i I(|X_i| \leq n^{\alpha}) + n^{\alpha} I(|X_i| > n^{\alpha}) \right], \\ Z_{ni} &= a_{ni} X_i - Y_{ni} = a_{ni} \left[(X_i + n^{\alpha}) I(X_i < -n^{\alpha}) + (X_i - n^{\alpha}) I(X_i > n^{\alpha}) \right]. \end{aligned}$$

By the definition of Y_{ni}, $|a_{ni}| \leq n^{\frac{1}{q}}$ and $\alpha q > 1$, we have that

$$\begin{aligned} n^{-\alpha} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} EY_{ni} \right| &\leq n^{-\alpha} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|X_i| > n^{\alpha}) \\ &\leq n^{1-\alpha} E|X| I(|X| > n^{\alpha}) \\ &\leq n^{1-\alpha} E|X| \left(\frac{|X|}{n^{\alpha}} \right)^{p \vee \nu - 1} I(|X| > n^{\alpha}) \\ &= E|X|^{p \vee \nu} I(|X| > n^{\alpha}) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Take $\mu = q \wedge 2$. If $0 < \nu < 1$, we have by Lemma 2.3 that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p - \alpha \nu - 2} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| - \varepsilon n^{\alpha} \right)_{+}^{\nu} \\ \ll & \sum_{n=1}^{\infty} n^{\alpha p - \alpha \mu - 2} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (Y_{ni} - EY_{ni} + Z_{ni}) \right| - \frac{\varepsilon n^{\alpha}}{2} \right)_{+}^{\nu} \\ \ll & \sum_{n=1}^{\infty} n^{\alpha p - \alpha \mu - 2} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (Y_{ni} - EY_{ni}) \right|^{\mu} \right) + \sum_{n=1}^{\infty} n^{\alpha p - \alpha \nu - 2} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} Z_{ni} \right|^{\nu} \right) \\ =: & \; C_1 + C_2. \end{aligned}$$

and if $1 \leq \nu \leq 2$, we have that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha p - \alpha \nu - 2} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| - \varepsilon n^{\alpha} \right)_{+}^{\nu} \\ \ll & \sum_{n=1}^{\infty} n^{\alpha p - \alpha \mu - 2} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (Y_{ni} - EY_{ni}) \right|^{\mu} \right) + \sum_{n=1}^{\infty} n^{\alpha p - \alpha \nu - 2} E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (Z_{ni} - EZ_{ni}) \right|^{\nu} \right) \\ =: & \; C_1 + C_2'. \end{aligned}$$

We have by Lemmas 2.2 and 2.5, C_r-inequality and Jensen's inequality that

$$\begin{aligned}
C_1 &\leq \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-2}(\log n)^{\mu} \sum_{i=1}^{n} E|Y_{ni}|^{\mu} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\mu-1}(\log n)^{\mu} E|X|^{\mu} I(|X| \leq n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-1}(\log n)^{\mu} E|X|^{\nu} I(|X| > n^{\alpha}) \\
&< \infty.
\end{aligned}$$

By Lemmas 2.2, 2.4 and 2.5, C_r-inequality and Jensen's inequality, we also have that

$$\begin{aligned}
C_2 &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \sum_{i=1}^{n} |a_{ni}|^{\nu} E|X|^{\nu} I(|X| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-1} E|X|^{\nu} I(|X| > n^{\alpha}) \\
&< \infty,
\end{aligned}$$

and

$$\begin{aligned}
C_2' &\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-2} \log^{\nu} n \sum_{i=1}^{n} |a_{ni}|^{\nu} E|X|^{\nu} I(|X| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-\alpha\nu-1} \log^{\nu} n E|X|^{\nu} I(|X| > n^{\alpha}) \\
&< \infty.
\end{aligned}$$

The proof is completed. □

From Theorem 3.2, we can obtain the following Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums of pairwise NQD random variables.

Corollary 3.1. *Let $0 < p < 2$. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed pairwise NQD random variables with $EX = 0$ if $1 \leq p < 2$. Assume further that $\{a_n, n \geq 1\}$ is a sequence of constants satisfying $\sum_{i=1}^{n} |a_i|^q \ll n$ for some $q > p$. If $E|X|^p \log^{q \wedge 2} |X| < \infty$, then*

$$n^{-\frac{1}{p}} \sum_{i=1}^{n} a_i X_i \to 0 \text{ a.s., as } n \to \infty. \tag{3.8}$$

Proof Taking $0 < \nu < p, \alpha = \frac{1}{p}$ and $a_{ni} = a_i$ for each $1 \leq i \leq n$ and $n \geq 1$ in Theorem 3.2, we can obtained that

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n} \left|\sum_{i=1}^{k} a_i X_i\right| > \varepsilon n^{\frac{1}{p}}\right) < \infty, \tag{3.9}$$

which can derive (3.8) by some standard computation. The proof is completed. □

4. Simulation

In this section, take $a_n = 1$ for each $n \geq 1$ in Corollary 3.1. The data generation process is shown as follows. First, we generate the data. Let $X = (X_1, X_2, \cdots, X_n)' \sim N_n(\mathbf{0}, \mathbf{\Sigma})$, in which $\mathbf{0}$ represents n-dimensional zero column vector, and

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & -\sigma & 0 & \cdots & 0 & 0 & 0 \\ -\sigma & 1 & -\sigma & \cdots & 0 & 0 & 0 \\ 0 & -\sigma & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\sigma & 0 \\ 0 & 0 & 0 & \cdots & -\sigma & 1 & -\sigma \\ 0 & 0 & 0 & \cdots & 0 & -\sigma & 1 \end{pmatrix}_{n\times n}, \quad 0 < \sigma < 1.$$

According to Joag-Dev and Proschan [15], the X generated by the above method is proved to be a vector of NA for each $n \geq 3$ with finite moment of any order, which is a special case of pairwise NQD. Taking $\sigma = 0.06, 0.12, 0.24, 0.48$ and $n = 200, 400, 600, 800$ respectively, we calculate $n^{-1}S_n$ for 100 times with the help of MATLAB software. Then, we get some different boxplots of $n^{-1}S_n$ with $\sigma = 0.06, 0.12, 0.24, 0.48$ in Figure 1 as follows.

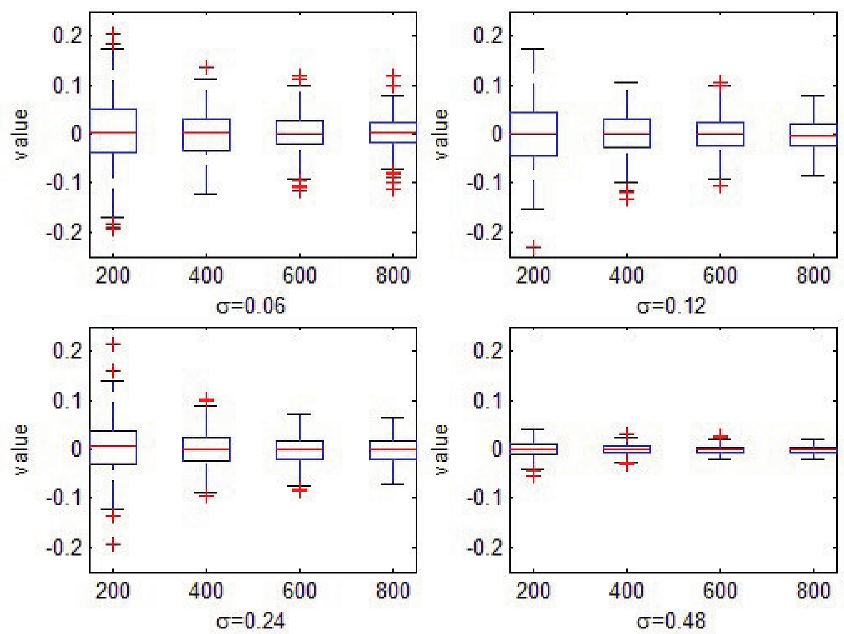

Figure 1: Boxplots of $n^{-1}S_n$ with σ=0.06, 0.12, 0.24, 0.48

We can see that the values of $n^{-1}S_n$ fluctuate zero, and the range of variation decreases as σ increases. These simulation results show good fits of the theoretical results.

To show the convergence behavior of $n^{-1}S_n$ in a more intuitive way, we use a scatter plot to show the convergence trend with $\sigma = 0.06, 0.12, 0.24, 0.48$ and $n = 1, 2, \cdots, 800$ respectively. Similar to the boxplots process shown above, the scatter plots of $n^{-1}S_n$ are shown below in Figure 2.

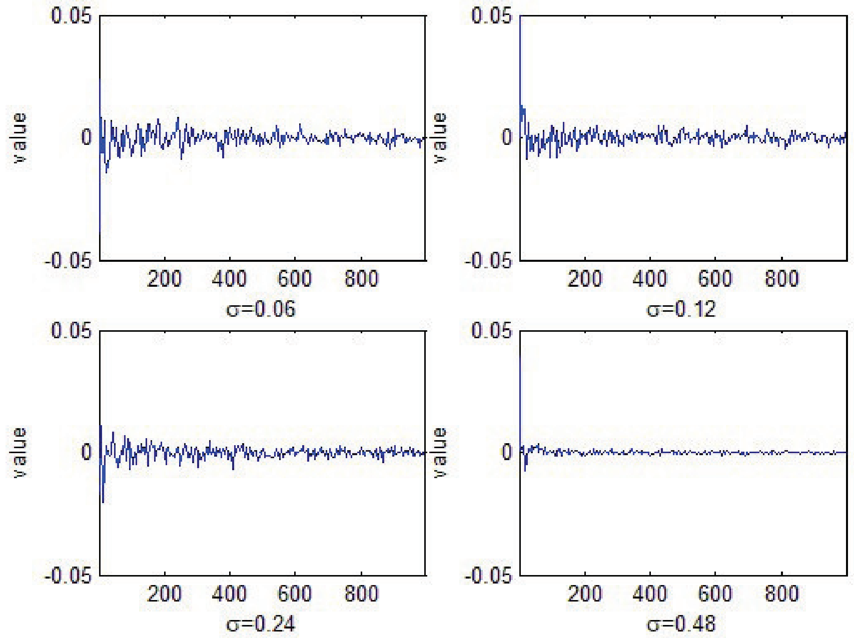

Figure 2: Scatter plot of $n^{-1}S_n$ with σ=0.06, 0.12, 0.24, 0.48

From the scatter plots in Figure 2, we confirm that $n^{-1}S_n$ converges to zero as n increases. Besides, with the increase of σ, $n^{-1}S_n$ accelerates its convergence trend to 0. These conclusions verify the validity of our theoretical result.

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