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Schur Convexity of Mixed Mean of n Variables Involving Three Parameters

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Abstract. In this paper, we discuss the Schur convexity, the Schur geometric convexity and Schur harmonic convexity of the mixed mean of n variables involving three parameters. As an application, we have established some inequalities of the Ky Fan type related to the mixed mean of n variables, and the lower bound inequality of Gini mean for n variables is given.

1. Introduction

Throughout the paper we assume that the set of n-dimensional row vector on the real number field by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \cdots, n\}.$$

In particular, $\mathbb{R}^1$ and $\mathbb{R}^1_+$ denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

In 2009, Kuang [1] defined the mixed mean of two variables with three parameters as follows:

$$K_2(w_1, w_2, p) = \left(\frac{w_1 A(x^p, y^p) + w_2 G(x^p, y^p)}{w_1 + w_2}\right)^{\frac{1}{p}} \tag{1}$$

where $A(a, b) = \frac{a+b}{2}$ is arithmetic mean, $G(a, b) = \sqrt{ab}$ is geometric mean, $p \neq 0$, $w_1, w_2 \geq 0$, $w_1 + w_2 \neq 0$.

In recent years, the research on Schur convexity of all kinds of means with two variables is moer and more active and fruiful(see references[5]-[8],[10]-[30]).

Fu et al.(see[8]) studied the Schur convexity, Schur geometric convexity and Schur harmonic convexity of $K_2(w_1, w_2, p)$.

Obviously, for $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$, $K_2(w_1, w_2, p)$ can be generalized as follows:

$$K_n(w_1, w_2, p) = \left(\frac{w_1 A_n(\mathbf{x}^p) + w_2 G_n(\mathbf{x}^p)}{w_1 + w_2}\right)^{\frac{1}{p}} \tag{2}$$

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where $A_n(\mathbf{x}^p) = \frac{1}{n}\sum_{i=1}^{n} x_i^p$, $G_n(\mathbf{x}^p) = (\prod_{i=1}^{n} x_i^p)^{\frac{1}{n}}$ is arithmetic mean and geometric mean of $\mathbf{x}^p = (x_1^p, \cdots, x_n^p)$.

Wang et al.(see[9]) studied the Schur convexity, Schur geometric convexity and Schur harmonic convexity of $K_n(w_1, w_2, p)$.

Related geometric mean and harmonic mean, we define following the mixed mean of n variables involving three parameters.

Definition 1.1.

$$W_n(\boldsymbol{x}, w_1, w_2, p) = \begin{cases} \left(\frac{w_1 H_n(\boldsymbol{x}^p) + w_2 G_n(\boldsymbol{x}^p)}{w_1 + w_2}\right)^{\frac{1}{p}}, & 0 \le w_1, w_2 < +\infty; \\ (H_n(\boldsymbol{x}^p))^{\frac{1}{p}}, & w_1 = +\infty; \\ (G_n(\boldsymbol{x}^p))^{\frac{1}{p}}, & w_2 = +\infty. \end{cases} \quad (3)$$

Where $H_n(\boldsymbol{x}^p) = \frac{n}{\sum_{i=1}^{n} x_i^{-p}}$, $G_n(\boldsymbol{x}^p) = \prod_{i=1}^{n} x_i^{\frac{p}{n}}$, $p \ne 0$, $w_1 \ge 0$, $w_2 \ge 0$, $w_1 + w_2 \ne 0$, $\boldsymbol{x} \in \mathbb{R}_+^n$.

In this paper, Schur convexity, Schur geometric convexity, Schur harmonic convexity of $W_n(\mathbf{x}, w_1, w_2, p)$ are discussed. As applications some interesting inequalities are obtained.

Our main results are as follows:

Theorem 1.2. *Let* $p \ne 0, w_1 \ge 0, w_2 \ge 0, w_1 + w_2 \ne 0$.

(*i*) *If* $p \ge -1$, *then* $W_n(\boldsymbol{x}, w_1, w_2, p)$ *is Schur-concave with* $\boldsymbol{x} \in \mathbb{R}_+^n$.
(*ii*) *If* $p \ge 0$, *then* $W_n(\boldsymbol{x}, w_1, w_2, p)$ *is Schur-geomertrically concave with* $\boldsymbol{x} \in \mathbb{R}_+^n$.
If $p < 0$, *then* $W_n(\boldsymbol{x}, w_1, w_2, p)$ *is Schur-geomertrically convex with* $\boldsymbol{x} \in \mathbb{R}_+^n$.
(*iii*) *If* $p \le 1$, *then* $W_n(\boldsymbol{x}, w_1, w_2, p)$ *is Schur-harmonically convex with* $\boldsymbol{x} \in \mathbb{R}_+^n$.

Theorem 1.3. *The function* $W_n(\boldsymbol{x}, w_1, w_2, p)$ *is decreasing with* $w_1 \in [0, +\infty)$, $W_n(\boldsymbol{x}, w_1, w_2, p)$ *is increasing with* $w_2 \in [0, +\infty)$.

Theorem 1.4. *Let* $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}_+^n$, $0 \le w_2^* \le w_2$.

(*i*) *If* $p \le -1$, *then* $\frac{W_n(\boldsymbol{x}, w_1, w_2, p)}{W_n(\boldsymbol{x}, w_1, w_2^*, p)}$ *is Schur-concave with* $\boldsymbol{x} \in \mathbb{R}_+^n$.
(*ii*) *If* $p \le 0$, *then* $\frac{W_n(\boldsymbol{x}, w_1, w_2, p)}{W_n(\boldsymbol{x}, w_1, w_2^*, p)}$ *is Schur-geomertrically concave with* $\boldsymbol{x} \in \mathbb{R}_+^n$.
If $p \ge 0$, *then* $\frac{W_n(\boldsymbol{x}, w_1, w_2, p)}{W_n(\boldsymbol{x}, w_1, w_2^*, p)}$ *is Schur-geomertrically convex with* $\boldsymbol{x} \in \mathbb{R}_+^n$.
(*iii*) *If* $p \ge 1$, *then* $\frac{W_n(\boldsymbol{x}, w_1, w_2, p)}{W_n(\boldsymbol{x}, w_1, w_2^*, p)}$ *is Schur -harmonically convex with* $\boldsymbol{x} \in \mathbb{R}_+^n$.

Theorem 1.5. *If* $0 < x_i \le \frac{1}{2} (i = 1, \cdots, n)$, *then* $\frac{W_n(\boldsymbol{x}, w_1, w_2, 1)}{W_n((1-\boldsymbol{x}), w_1, w_2, 1)}$ *is increasing with* w_2.

2. Definitions and Lemmas

We introduce some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

Definition 2.1 ([2, 3]). *Let* $\boldsymbol{x} = (x_1, \cdots, x_n)$ *and* $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}_+^n$,

(*i*) $\boldsymbol{x}$ *is said to be majorized by* $\boldsymbol{y}$ *(in symbols* $\boldsymbol{x} \prec \boldsymbol{y}$*) if* $\sum_{i=1}^{n} x_{[i]} \le \sum_{i=1}^{n} y_{[i]}$ *for* $k = 1, \cdots, n-1$ *and* $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, *where* $x_{[1]} \ge \cdots \ge x_{[n]}$ *are rearrangements of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in a descending order.*
(*ii*) $\Omega \subset \mathbb{R}^n$ *is called a convex set if* $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$ *for any* $\boldsymbol{x}$ *and* $\boldsymbol{y} \in \Omega$, *where* α *and* $\beta \in [0, 1]$ *with* $\alpha + \beta = 1$.
(*iii*) *Let* $\Omega \in \mathbb{R}_n$,$\varphi : \Omega \to \mathbb{R}$ *is said to be a Schur convex function on* Ω *implies* $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$.$\varphi$ *is said to be a Schur concave function on* Ω *if* $-\varphi$ *is Schur convex function.*

Definition 2.2 ([4, 5]). *Let* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *and* $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$.

(*i*) *A set* $\Omega \subset \mathbb{R}_+^n$ *is called a geometrically convex set if* $(x_1^{\alpha} y_1^{\beta}, \ldots, x_n^{\alpha} y_n^{\beta}) \in \Omega$ *for any* $\boldsymbol{x}$ *and* $\boldsymbol{y} \in \Omega$, *where* α *and* $\beta \in [0,1]$ *with* $\alpha + \beta = 1$.

(*ii*) *Let* $\Omega \subset \mathbb{R}_+^n$, φ: $\Omega \to \mathbb{R}_+$ *is said to be a Schur geometrically convex function on* Ω *if* $(\log x_1, \ldots, \log x_n) \prec (\log y_1, \ldots, \log y_n)$ *on* Ω *implies* $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ *is said to be a Schur geometrically concave function on* Ω *if and only if* $-\varphi$ *is Schur geometrically convex function.*

Definition 2.3 ([5, 6]). *Let* $\Omega \subset \mathbb{R}_+^n$.

(*i*) *A set* Ω *is said to be a harmonically convex set if* $\frac{\boldsymbol{xy}}{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ *and* $\lambda \in [0,1]$, *where* $\boldsymbol{xy} = \sum_{i=1}^n x_i y_i$ *and* $\frac{1}{\boldsymbol{x}} = \left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right)$.

(*ii*) *A function* $\varphi : \Omega \to \mathbb{R}_+$ *is said to be a Schur harmonically convex function on* Ω *if* $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ *implies* $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. *A function* φ *is said to be a Schur harmonically concave function on* Ω *if and only if* $-\varphi$ *is a Schur harmonically convex function.*

Lemma 2.4 ([2, 3]). *Let* $\Omega \subset \mathbb{R}^n$ *is convex set, and has a nonempty interior set* Ω^0 . *Let* $\varphi : \Omega \to \mathbb{R}$ *is continuous on* Ω *and differentiable in* Ω^0. *Then* φ *is the* $Schur-convex(Schur-concave) function$, *if and only if it is symmetric on* Ω *and if*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (or \leq 0, respectively)$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 2.5 ([4, 5]). *Let* $\Omega \subset \mathbb{R}_+^n$ *be a symmetric geometrically convex set with a nonempty interior* Ω^0. *Let* $\varphi : \Omega \to \mathbb{R}_+$ *be continuous on* Ω *and differentiable on* Ω^0. *Then* φ *is a Schur geometrically convex* (*Schur geometrically concave*) *function if and only if* φ *is symmetric on* Ω *and*

$$(x_1 - x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (or \leq 0, respectively)$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 2.6 ([6, 7]). *Let* $\Omega \subset \mathbb{R}_+^n$ *be a symmetric harmonically convex set with a nonempty interior* Ω^0. *Let* $\varphi : \Omega \to \mathbb{R}_+$ *be continuous on* Ω *and differentiable on* Ω^0. *Then* φ *is a Schur harmonically convex* (*Schur harmonically concave*) *function if and only if* φ *is symmetric on* Ω *and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial \varphi}{\partial x_1} - x_2^2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (or \leq 0, respectively)$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 2.7. *[3] Let* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ *and* $G_n(\boldsymbol{x}) = \prod_{i=1}^n x_i^{\frac{1}{n}}$. *Then*

(*i*)

$$\Bigg(\underbrace{\log G_n(\boldsymbol{x}), \log G_n(\boldsymbol{x}), \cdots, \log G_n(\boldsymbol{x})}_{n}\Bigg) \prec (\log x_1, \log x_2, \cdots, \log x_n). \tag{4}$$

(*ii*) *If* $\sum_{i=1}^n x_i = s$, *then*

$$\left(\frac{s - x_1}{n-1}, \frac{s - x_2}{n-1}, \cdots, \frac{s - x_n}{n-1}\right) \prec (x_1, x_2, \cdots, x_n). \tag{5}$$

(*iii*) *If* $0 < r \leq s$, *then*

$$\left(\frac{x_1^r}{\sum_{i=1}^n x_i^r}, \cdots, \frac{x_n^r}{\sum_{i=1}^n x_i^r}\right) \prec \left(\frac{x_1^s}{\sum_{i=1}^n x_i^s}, \cdots, \frac{x_n^s}{\sum_{i=1}^n x_i^s}\right). \tag{6}$$

(*iv*) *Let* $\sum_{i=1}^n x_i = s$. *For any* $c > 0$, *we have*

$$\left(\frac{c+x_1}{nc+s}, \cdots, \frac{c+x_n}{nc+s}\right) \prec \left(\frac{x_1}{s}, \cdots, \frac{x_n}{s}\right). \tag{7}$$

3. Proofs of Main results

Proof. [Proof of Theorem 1.2] Write

$$w(w_1, w_2, p) = \frac{w_1 H_n(\mathbf{x}^p) + w_2 G_n(\mathbf{x}^p)}{w_1 + w_2}.$$

It is clear that $W_n(\mathbf{x}, w_1, w_2, p)$ is symmetric with $(x_1, \cdots, x_n) \in \mathbb{R}_+^n$, without loss generality, we may assume that $x_1 \geq x_2 > 0$. We have

$$\frac{\partial H_n}{\partial x_1} = \frac{p}{n}[H_n(\mathbf{x}^p)]^2 \frac{1}{x_1^{p+1}}, \frac{\partial H_n}{\partial x_2} = \frac{p}{n}[H_n(\mathbf{x}^p)]^2 \frac{1}{x_2^{p+1}}.$$

$$\frac{\partial G_n}{\partial x_1} = \frac{p}{n} G_n(\mathbf{x}^p) \frac{1}{x_1}, \frac{\partial G_n}{\partial x_2} = \frac{p}{n} G_n(\mathbf{x}^p) \frac{1}{x_2}.$$

And then

$$\begin{aligned}
\frac{\partial W_n}{\partial x_1} &= \frac{1}{p}[w(w_1, w_2, p]^{\frac{1}{p}-1} \frac{1}{w_1 + w_2}\left(w_1 \frac{\partial H_n}{\partial x_1} + w_2 \frac{\partial G_n}{\partial x_1}\right) \\
&= \frac{1}{n} \frac{1}{w_1 + w_2}[w(w_1, w_2, p)]^{\frac{1}{p}-1}\left(w_1 (H_n(\mathbf{x}^p))^2 \frac{1}{x_1^{p+1}} + w_2 G_n(\mathbf{x}^p) \frac{1}{x_1}\right),
\end{aligned}$$

$$\frac{\partial W_n}{\partial x_2} = \frac{1}{n} \frac{1}{w_1 + w_2}[w(w_1, w_2, p)]^{\frac{1}{p}-1}\left(w_1 (H_n(\mathbf{x}^p))^2 \frac{1}{x_2^{p+1}} + w_2 G_n(\mathbf{x}^p) \frac{1}{x_2}\right),$$

$$x_1 \frac{\partial W_n}{\partial x_1} = \frac{1}{n} \frac{1}{w_1 + w_2}[w(w_1, w_2, p)]^{\frac{1}{p}-1}\left(w_1 (H_n(\mathbf{x}^p))^2 \frac{1}{x_1^p} + w_2 G_n(\mathbf{x}^p)\right),$$

$$x_2 \frac{\partial W_n}{\partial x_2} = \frac{1}{n} \frac{1}{w_1 + w_2}[w(w_1, w_2, p)]^{\frac{1}{p}-1}\left(w_1 (H_n(\mathbf{x}^p))^2 \frac{1}{x_2^p} + w_2 G_n(\mathbf{x}^p)\right),$$

$$x_1^2 \frac{\partial W_n}{\partial x_1} = \frac{1}{n} \frac{1}{w_1 + w_2}[w(w_1, w_2, p)]^{\frac{1}{p}-1}\left(w_1 (H_n(\mathbf{x}^p))^2 \frac{1}{x_1^{p-1}} + w_2 G_n(\mathbf{x}^p) x_1\right),$$

$$x_2^2 \frac{\partial W_n}{\partial x_2} = \frac{1}{n} \frac{1}{w_1 + w_2}[w(w_1, w_2, p)]^{\frac{1}{p}-1}\left(w_1 (H_n(\mathbf{x}^p))^2 \frac{1}{x_2^{p-1}} + w_2 G_n(\mathbf{x}^p) x_2\right).$$

Therefore

(*i*)

$$\begin{aligned}\Delta_1 :=&(x_1 - x_2)\left(\frac{\partial W_n}{\partial x_1} - \frac{\partial W_n}{\partial x_2}\right)\\ =&(x_1 - x_2)\frac{1}{n}\frac{1}{w_1 + w_2}[w(w_1, w_2, p)]^{\frac{1}{p}-1}\\ &\times\left(w_1(H_n(\mathbf{x}^p))^2(\frac{1}{x_1^{p+1}} - \frac{1}{x_2^{p+1}}) + w_2G_n(\mathbf{x}^p)(\frac{1}{x_1} - \frac{1}{x_2})\right).\end{aligned}$$

If $p \geq -1$, then $\Delta_1 \leq 0$. By Lemma 2.4, it follows that $W_n(\mathbf{x}, w_1, w_2, p)$ is Schur-concave with $\mathbf{x} \in \mathbb{R}^n_+$.

(*ii*)

$$\begin{aligned}\Delta_2 :&= (x_1 - x_2)\left(x_1\frac{\partial W_n}{\partial x_1} - x_2\frac{\partial W_n}{\partial x_2}\right)\\ &= (x_1 - x_2)\frac{1}{n}\frac{1}{w_1 + w_2}[w(w_1, w_2, p)]^{\frac{1}{p}-1}w_1(H_n(\mathbf{x}^p))^2(\frac{1}{x_1^p} - \frac{1}{x_2^p}).\end{aligned}$$

If $p \geq 0$, then $\Delta_2 \leq 0$. By Lemma 2.5, it follows that $W_n(\mathbf{x}, w_1, w_2, p)$ is Schur-geometrically concave with $\mathbf{x} = (x_1, ...x_n) \in \mathbb{R}^n_+$. If $p < 0$, then $\Delta_2 \geq 0$. By Lemma 2.5, it follows that $W_n(\mathbf{x}, w_1, w_2, p)$ is Schur-geometrically convex with $\mathbf{x} \in \mathbb{R}^n_+$.

(*iii*)

$$\begin{aligned}\Delta_3 :=&(x_1 - x_2)\left(x_1^2\frac{\partial W_n}{\partial x_1} - x_2^2\frac{\partial W_n}{\partial x_2}\right)\\ =&(x_1 - x_2)\frac{1}{n}\frac{1}{w_1 + w_2}[w(w_1, w_2, p)]^{\frac{1}{p}-1}\\ &\times\left(w_1(H_n(\mathbf{x}^p))^2(\frac{1}{x_1^{p-1}} - \frac{1}{x_2^{p-1}}) + w_2G_n(\mathbf{x}^p)(x_1 - x_2)\right).\end{aligned}$$

If $p \leq 1$, then $\Delta_3 \geq 0$. By Lemma 2.6, it follows that $W_n(\mathbf{x}, w_1, w_2, p)$ is Schur harmonically convex with $\mathbf{x} \in \mathbb{R}^n_+$.

The proof of Theorem 1.2 is complete. □

Proof. [Proof of Theorem 1.3] Because

$$\begin{aligned}\frac{\partial W_n}{\partial w_1} &= \frac{1}{(w_1 + w_2)^2}[H_n(\mathbf{x}^p)(w_1 + w_2) - (w_1H_n(\mathbf{x}^p) + w_2G_n(\mathbf{x}^p))]\\ &= \frac{1}{(w_1 + w_2)^2}w_2[H_n(\mathbf{x}^p) - G_n(\mathbf{x}^p)] \leq 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial W_n}{\partial w_2} &= \frac{1}{(w_1 + w_2)^2}[G_n(\mathbf{x}^p)(w_1 + w_2) - (w_1H_n(\mathbf{x}^p) + w_2G_n(\mathbf{x}^p))]\\ &= \frac{1}{(w_1 + w_2)^2}w_1[G_n(\mathbf{x}^p) - H_n(\mathbf{x}^p)] \geq 0.\end{aligned}$$

So that, $W_n(\mathbf{x}, w_1, w_2, p)$ is decreasing with w_1 on $[0, +\infty)$,$W_n(\mathbf{x}, w_1, w_2, p)$ is increasing with w_2 on $[0, +\infty)$.

The proof of Theorem 1.3 is complete. □

Proof. [Proof of Theorem 1.3] Write

$$\overline{W}_n(\mathbf{x}) = \frac{W_n(\mathbf{x}, w_1, w_2, p)}{W_n(\mathbf{x}, w_1, w_2^*, p)},$$

$$w_n(\mathbf{x}) = \frac{w_1 H_n(\mathbf{x}^p) + w_2 G_n(\mathbf{x}^p)}{w_1 + w_2},$$

$$w_n^*(\mathbf{x}) = \frac{w_1 H_n(\mathbf{x}^p) + w_2^* G_n(\mathbf{x}^p)}{w_1 + w_2^*}.$$

We have

$$\begin{aligned}
\frac{\partial \overline{W}_n}{\partial x_1} =& \frac{1}{W_n^2(\mathbf{x}, w_1, w_2^*, p)} \{ \frac{1}{p} (w_n(\mathbf{x}))^{\frac{1}{p}-1} (w_n^*(\mathbf{x}))^{-\frac{1}{p}} \left[\frac{\frac{w_1 p}{n} (H_n(\mathbf{x}^p))^2 \frac{1}{x_1^{p+1}} + \frac{w_2 p}{n} G_n(\mathbf{x}^p) \frac{1}{x_1}}{w_1 + w_2} \right] \\
& - \frac{1}{p} (w_n(\mathbf{x}))^{\frac{1}{p}} (w_n^*(\mathbf{x}))^{-\frac{1}{p}-1} \left[\frac{\frac{w_1 p}{n} (H_n(\mathbf{x}^p))^2 \frac{1}{x_1^{p+1}} + \frac{w_2^* p}{n} G_n(\mathbf{x}^p) \frac{1}{x_1}}{w_1 + w_2^*} \right] \} \\
=& \frac{1}{W_n^2(\mathbf{x}, w_1, w_2^*, p)} \{ \frac{1}{n} (w_n(\mathbf{x}))^{\frac{1}{p}-1} (w_n^*(\mathbf{x}))^{-\frac{1}{p}-1} \left[\frac{w_1 (H_n(\mathbf{x}^p))^2 \frac{1}{x_1^{p+1}} + w_2 G_n(\mathbf{x}^p) \frac{1}{x_1}}{w_1 + w_2} \right] \\
& \times \left[\frac{w_1 H_n(\mathbf{x}^p) + w_2^* G_n(\mathbf{x}^p)}{w_1 + w_2^*} \right] - [\frac{1}{n} (w_n(\mathbf{x}))^{\frac{1}{p}-1} (w_n^*(\mathbf{x}))^{-\frac{1}{p}-1}] \\
& \times \left[\frac{w_1 (H_n(\mathbf{x}^p))^2 \frac{1}{x_1^{p+1}} + w_2^* G_n(\mathbf{x}^p) \frac{1}{x_1}}{w_1 + w_2^*} \right] \left[\frac{w_1 H_n(\mathbf{x}^p) + w_2 G_n(\mathbf{x}^p)}{w_1 + w_2} \right] \} \\
=& \frac{1}{W_n^2(\mathbf{x}, w_1, w_2^*, p)} \frac{1}{n} (w_n(\mathbf{x}))^{\frac{1}{p}-1} (w_n^*(\mathbf{x}))^{-\frac{1}{p}-1} \frac{1}{(w_1 + w_2^*)(w_1 + w_2)} [w_1 (H_n(\mathbf{x}^p))^2 \\
& \times G_n(\mathbf{x}^p)(w_2^* - w_2) \frac{1}{x_1^{p+1}} + w_1 H_n(\mathbf{x}^p) G_n(\mathbf{x}^p)(w_2 - w_2^*) \frac{1}{x_1}] \\
=& \frac{1}{W_n^2(\mathbf{x}, w_1, w_2^*, p)} \frac{1}{n} (w_n(\mathbf{x}))^{\frac{1}{p}-1} (w_n^*(\mathbf{x}))^{-\frac{1}{p}-1} \frac{w_1}{(w_1 + w_2^*)(w_1 + w_2)} H_n(\mathbf{x}^p) G_n(\mathbf{x}^p) \\
& \times (w_2^* - w_2)(H_n(\mathbf{x}^p) \frac{1}{x_1^{p+1}} - \frac{1}{x_1}),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \overline{W}_n}{\partial x_2} =& \frac{1}{W_n^2(\mathbf{x}, w_1, w_2^*, p)} \frac{1}{n} (w_n(\mathbf{x}))^{\frac{1}{p}-1} (w_n^*(\mathbf{x}))^{-\frac{1}{p}-1} \frac{w_1}{(w_1 + w_2^*)(w_1 + w_2)} H_n(\mathbf{x}^p) G_n(\mathbf{x}^p) \\
& \times (w_2^* - w_2)(H_n(\mathbf{x}^p) \frac{1}{x_2^{p+1}} - \frac{1}{x_2}),
\end{aligned}$$

and then

(*i*)

$$\begin{aligned}\triangle_4 :=&(x_1 - x_2)\left(\frac{\partial \overline{W}_n}{\partial x_1} - \frac{\partial \overline{W}_n}{\partial x_2}\right)\\ =&\frac{1}{W_n^2(\mathbf{x}, w_1, w_2^*, p)}(x_1 - x_2)\frac{1}{n}(w_n(\mathbf{x}))^{\frac{1}{p}-1}(w_n^*(\mathbf{x}))^{-\frac{1}{p}-1}\frac{w_1}{(w_1 + w_2^*)(w_1 + w_2)}\\ &\times H_n(\mathbf{x}^p)G_n(\mathbf{x}^p)(w_2^* - w_2)[H_n(\mathbf{x}^p)(x_1^{-(p+1)} - x_2^{-(p+1)}) + (x_2^{-1} - x_1^{-1})],\end{aligned}$$

so, if $w_2^* \le w_2$ and $p \le -1$, then $\Delta_4 \le 0$. By Lemma 2.4, it follows that $\overline{W}_n(\mathbf{x}) = \frac{W_n(\mathbf{x}, w_1, w_2, p)}{W_n(\mathbf{x}, w_1, w_2^*, p)}$ is Schur concave with $\mathbf{x} \in \mathbb{R}_+^n$.

(*ii*)

$$\begin{aligned}\triangle_5 :=&(x_1 - x_2)\left(x_1\frac{\partial \overline{W}_n}{\partial x_1} - x_2\frac{\partial \overline{W}_n}{\partial x_2}\right)\\ =&\frac{1}{W_n^2(\mathbf{x}, w_1, w_2^*, p)}(x_1 - x_2)\frac{1}{n}(w_n(\mathbf{x}))^{\frac{1}{p}-1}(w_n^*(\mathbf{x}))^{-\frac{1}{p}-1}\frac{w_1}{(w_1 + w_2^*)(w_1 + w_2)}\\ &\times H_n^2(\mathbf{x}^p)G_n(\mathbf{x}^p)(w_2^* - w_2)(x_1^{-p} - x_2^{-p}),\end{aligned}$$

so, if $w_2^* \le w_2$ and $p \le 0$, then $\Delta_5 \le 0$. By Lemma 2.5, it follows that $\overline{W}_n(\mathbf{x}) = \frac{W_n(\mathbf{x}, w_1, w_2, p)}{W_n(\mathbf{x}, w_1, w_2^*, p)}$ is Schur geometrically concave with $\mathbf{x} \in \mathbb{R}_+^n$. If $p \ge 0$, then $\Delta_5 \ge 0$. By Lemma 2.5, it follows that $\overline{W}_n(\mathbf{x}) = \frac{W_n(\mathbf{x}, w_1, w_2, p)}{W_n(\mathbf{x}, w_1, w_2^*, p)}$ is Schur geometrically convex with $\mathbf{x} \in \mathbb{R}_+^n$.

(*iii*)

$$\begin{aligned}\triangle_6 :=&(x_1 - x_2)\left(x_1^2\frac{\partial \overline{W}_n}{\partial x_1} - x_1^2\frac{\partial \overline{W}_n}{\partial x_2}\right)\\ =&\frac{1}{W_n^2(\mathbf{x}, w_1, w_2^*, p)}(x_1 - x_2)\frac{1}{n}(w_n(\mathbf{x}))^{\frac{1}{p}-1}(w_n^*(\mathbf{x}))^{-\frac{1}{p}-1}\frac{w_1}{(w_1 + w_2^*)(w_1 + w_2)}\\ &\times H_n(\mathbf{x}^p)G_n(\mathbf{x}^p)(w_2^* - w_2)[H_n(\mathbf{x}^p)(x_1^{-(p-1)} - x_2^{-(p-1)}) + (x_2 - x_1)],\end{aligned}$$

so, if $w_2^* \le w_2$ and $p \ge 1$, then $\Delta_6 \ge 0$. By Lemma 2.6, it follows that $\overline{W}_n(\mathbf{x}) = \frac{W_n(\mathbf{x}, w_1, w_2, p)}{W_n(\mathbf{x}, w_1, w_2^*, p)}$ is Schur harmonically convex with $\mathbf{x} \in \mathbb{R}_+^n$.

The proof of Theorem 1.4 is complete. □

Proof. [Proof of Theorem 1.5] Let

$$\begin{aligned}W_{w_2,p} =& \frac{W(\mathbf{x}, w_1, w_2, p)}{W(1 - \mathbf{x}, w_1, w_2, p)}\\ =& \left[\frac{w_1 H_n(\mathbf{x}^p) + w_2 G_n(\mathbf{x}^p)}{w_1 H_n((1 - \mathbf{x})^p) + w_2 G_n((1 - \mathbf{x})^p)}\right]^{\frac{1}{p}},\end{aligned}$$

then

$$\begin{aligned}\frac{\partial W_{w_2,p}}{\partial w_2} =&\frac{1}{p}\left[\frac{w_1 H_n(\mathbf{x}^p) + w_2 G_n(\mathbf{x}^p)}{w_1 H_n((1 - \mathbf{x})^p) + w_2 G_n((1 - \mathbf{x})^p)}\right]^{\frac{1}{p}-1}\\ &\times \frac{w_1[G_n(\mathbf{x}^p)H_n((1 - \mathbf{x})^p) - H_n(\mathbf{x}^p)G_n((1 - \mathbf{x})^p)]}{[w_1 H_n((1 - \mathbf{x})^p) + w_2 G_n((1 - \mathbf{x})^p)]^2}.\end{aligned}$$

Because $0 < x_i \le \frac{1}{2}$, by Wang-Wang inequality([1]):

$$\frac{H_n(\mathbf{x})}{H_n(1-\mathbf{x})} \le \frac{G_n(\mathbf{x})}{G_n(1-\mathbf{x})},$$

we have

$$H_n(\mathbf{x})G_n(1-\mathbf{x}) - G_n(\mathbf{x})H_n(1-\mathbf{x}) \le 0.$$

and then $\frac{\partial W_{w_2,1}}{\partial w_2} \ge 0$. So that, $W_{w_2,1} = \frac{W(\mathbf{x},w_1,w_2,1)}{W(1-\mathbf{x},w_1,w_2,1)}$ is increasing with w_2.

The proof of Theorem 1.5 is complete. □

4. Applications

Theorem 4.1. *The inequalities*

$$[H(x_1^p, \cdots, x_n^p)]^{\frac{1}{p}} \le W_n(\boldsymbol{x}, w_1, w_2, p) \le [G(x_1^p, \cdots, x_n^p)]^{\frac{1}{p}} = G_n(\boldsymbol{x}) \tag{8}$$

hold.

Proof. Note that

$$W_n(\mathbf{x}, 0, w_2, p) = [G(x_1^p, \cdots, x_n^p)]^{\frac{1}{p}}, W_n(\mathbf{x}, +\infty, w_2, p) = [H(x_1^p, \cdots, x_n^p)]^{\frac{1}{p}},$$

by Theorem 1.3, we have

$$[H(x_1^p, \cdots, x_n^p)]^{\frac{1}{p}} \le W_n(\mathbf{x}, w_1, w_2, p) \le [G(x_1^p, \cdots, x_n^p)]^{\frac{1}{p}}.$$

The proof is complete. □

Remark 4.2. *Let $p = 1$, we get sharpening of $H - G$ inequality:*

$$H_n(\boldsymbol{x}) \le W_n(\boldsymbol{x}, w_1, w_2, 1) \le G_n(\boldsymbol{x}). \tag{9}$$

Theorem 4.3. *Let $x_i > 0$, $i = 1, 2, \cdots, n$, and $\sum_{i=1}^n x_i = s$. If $p \le -1$, $0 \le w_2^* < w_2$, then Ky-Fan type inequality:*

$$\frac{W_n(\boldsymbol{x}, w_1, w_2, p)}{W_n(s-\boldsymbol{x}, w_1, w_2, p)} \le \frac{W_n(\boldsymbol{x}, w_1, w_2^*, p)}{W_n(s-\boldsymbol{x}, w_1, w_2^*, p)} \tag{10}$$

holds.

Proof. From Lemma 2.7, we have

$$\left(\frac{s-x_1}{n-1}, \frac{s-x_2}{n-1}, \cdots, \frac{s-x_n}{n-1}\right) \prec (x_1, x_2, \cdots, x_n),$$

by Theorem 1.4(i), it is follows that

$$\frac{W_n(\mathbf{x}, w_1, w_2, p)}{W_n(\mathbf{x}, w_1, w_2^*, p)} \le \frac{W_n\left(\frac{s-\mathbf{x}}{n-1}, w_1, w_2, p\right)}{W_n\left(\frac{s-\mathbf{x}}{n-1}, w_1, w_2^*, p\right)} = \frac{W_n(s-\mathbf{x}, w_1, w_2, p)}{W_n(s-\mathbf{x}, w_1, w_2^*, p)}$$

$$\Rightarrow \frac{W_n(\mathbf{x}, w_1, w_2, p)}{W_n(s-\mathbf{x}, w_1, w_2, p)} \le \frac{W_n(\mathbf{x}, w_1, w_2^*, p)}{W_n(s-\mathbf{x}, w_1, w_2^*, p)}.$$

The proof is complete. □

Remark 4.4. *By Theorem 4.3 we know* $M_1(w_2) = \frac{W_n(\boldsymbol{x},w_1,w_2,p)}{W_n(s-\boldsymbol{x},w_1,w_2,p)}$ *is decreasing with* w_2. *Notice that* $W_n(\boldsymbol{x}, w_1, 0, p) = [H_n(\boldsymbol{x}^p)]^{\frac{1}{p}}$, $W_n(\boldsymbol{x}, w_1, +\infty, p) = [G_n(\boldsymbol{x}^p)]^{\frac{1}{p}} = G_n(\boldsymbol{x})$. *So that, for* $x_i > 0$, $i = 1, 2, \cdots, n$ *and* $\sum_{i=1}^n x_i = 1$, *if* $p \le -1$ *and* $w_1 > 0$, $0 \le w_2 < +\infty$, *then inequalities:*

$$\frac{G_n(\boldsymbol{x})}{G_n(1-\boldsymbol{x})} \le \frac{W_n(\boldsymbol{x}, w_1, w_2, p)}{W_n(1-\boldsymbol{x}, w_1, w_2, p)} \le \frac{[H_n(\boldsymbol{x}^p)]^{\frac{1}{p}}}{[H_n((1-\boldsymbol{x})^p)]^{\frac{1}{p}}}$$

holds.

Let $p = -1$. *We get the sharpening of Ky Fen's inequality:*

$$\frac{G_n(\boldsymbol{x})}{G_n(1-\boldsymbol{x})} \le \frac{W_n(\boldsymbol{x}, w_1, w_2, -1)}{W_n(1-\boldsymbol{x}, w_1, w_2, -1)} \le \frac{A_n(\boldsymbol{x})}{A_n(1-\boldsymbol{x})}. \tag{11}$$

Theorem 4.5. *If* $x_i > 0$, $i = 1, 2, \cdots, n$, *and* $\sum_{i=1}^n x_i = 1$, *then*

$$\frac{H_n(\boldsymbol{x}^{-1})}{H_n((1-\boldsymbol{x})^{-1})} \le \frac{W_n(\boldsymbol{x}^{-1}, w_1, w_2, 1)}{W_n((1-\boldsymbol{x})^{-1}, w_1, w_2, 1)} \le \frac{G_n(\boldsymbol{x}^{-1})}{G_n((1-\boldsymbol{x})^{-1})}. \tag{12}$$

Proof. If $x_i > 0$, $i = 1, 2, \cdots, n$, $\sum_{i=1}^n x_i = 1$, when $0 \le w_2^* < w_2$, by Lemma 2.7 the majorization inequality:

$$\left(\frac{1}{\frac{n-1}{1-x_1}}, \cdots, \frac{1}{\frac{n-1}{1-x_n}}\right) \prec \left(\frac{1}{\frac{1}{x_1}}, \dots, \frac{1}{\frac{1}{x_n}}\right)$$

holds. By Theorem 1.4(iii), when $p \ge 1$, we get

$$\frac{W_n\left(\frac{n-1}{1-\mathbf{x}}, w_1, w_2, p\right)}{W_n\left(\frac{n-1}{1-\mathbf{x}}, w_1, w_2^*, p\right)} = \frac{W_n\left(\frac{1}{1-\mathbf{x}}, w_1, w_2, p\right)}{W_n\left(\frac{1}{1-\mathbf{x}}, w_1, w_2^*, p\right)} \le \frac{W_n\left(\frac{1}{\mathbf{x}}, w_1, w_2, p\right)}{W_n\left(\frac{1}{\mathbf{x}}, w_1, w_2^*, p\right)}$$

$$\Rightarrow \frac{W_n\left(\frac{1}{\mathbf{x}}, w_1, w_2^*, p\right)}{W_n\left(\frac{1}{1-\mathbf{x}}, w_1, w_2^*, p\right)} \le \frac{W_n\left(\frac{1}{\mathbf{x}}, w_1, w_2, p\right)}{W_n\left(\frac{1}{1-\mathbf{x}}, w_1, w_2, p\right)},$$

so $M_2(w_2) = \frac{W_n\left(\frac{1}{\mathbf{x}}, w_1, w_2, p\right)}{W_n\left(\frac{1}{1-\mathbf{x}}, w_1, w_2, p\right)}$ is increasing with w_2.

When $0 \le w_1 < +\infty$, we have

$$\frac{W_n\left(\frac{1}{\mathbf{x}}, w_1, 0, p\right)}{W_n\left(\frac{1}{1-\mathbf{x}}, w_1, 0, p\right)} \le \frac{W_n\left(\frac{1}{\mathbf{x}}, w_1, w_2, p\right)}{W_n\left(\frac{1}{1-\mathbf{x}}, w_1, w_2, p\right)} \le \frac{W_n\left(\frac{1}{\mathbf{x}}, w_1, +\infty, p\right)}{W_n\left(\frac{1}{1-\mathbf{x}}, w_1, +\infty, p\right)}.$$

By Definition 1.1, we get

$$\frac{\left(H_n(\frac{1}{\mathbf{x}})^p\right)^{\frac{1}{p}}}{\left(H_n(\frac{1}{1-\mathbf{x}})^p\right)^{\frac{1}{p}}} \le \frac{W_n\left(\frac{1}{\mathbf{x}}, w_1, w_2, p\right)}{W_n\left(\frac{1}{1-\mathbf{x}}, w_1, w_2, p\right)} \le \frac{\left(G_n(\frac{1}{\mathbf{x}})^p\right)^{\frac{1}{p}}}{\left(G_n(\frac{1}{1-\mathbf{x}})^p\right)^{\frac{1}{p}}}.$$

Let $p = 1$, we have

$$\frac{H_n(\mathbf{x}^{-1})}{H_n((1-\mathbf{x})^{-1})} \le \frac{W_n(\mathbf{x}^{-1}, w_1, w_2, 1)}{W_n((1-\mathbf{x})^{-1}, w_1, w_2, 1)} \le \frac{G_n(\mathbf{x}^{-1})}{G_n((1-\mathbf{x})^{-1})}.$$

The proof is complete. □

Theorem 4.6. *If* $0 < x_i \le \frac{1}{2}$, $i = 1, 2, \cdots, n$, *then*

$$\frac{H_n(\boldsymbol{x})}{H_n(1-\boldsymbol{x})} \le \frac{W_n(\boldsymbol{x}, w_1, w_2, 1)}{W_n(1-\boldsymbol{x}, w_1, w_2, 1)} \le \frac{G_n(\boldsymbol{x})}{G_n(1-\boldsymbol{x})}. \tag{13}$$

Proof. By Theorem 1.5 we have

$$\frac{W_n(\mathbf{x}, w_1, 0, 1)}{W_n(1-\mathbf{x}, w_1, 0, 1)} \le \frac{W_n(\mathbf{x}, w_1, w_2, 1)}{W_n(1-\mathbf{x}, w_1, w_2, 1)} \le \frac{W_n(\mathbf{x}, w_1, +\infty, 1)}{W_n(1-\mathbf{x}, w_1, +\infty, 1)},$$

and by Definition 1.1 we get

$$\frac{H_n(\mathbf{x})}{H_n(1-\mathbf{x})} \le \frac{W_n(\mathbf{x}, w_1, w_2, 1)}{W_n(1-\mathbf{x}, w_1, w_2, 1)} \le \frac{G_n(\mathbf{x})}{G_n(1-\mathbf{x})}.$$

The proof is complete. □

The following inequalities are introduced in reference [1](see[1],p52):
Let $x_i \in \mathbb{R}_+, i = 1, \cdots, n$. If $c > 0$, then

$$\frac{A_n(\mathbf{x}+c)}{G_n(\mathbf{x}+c)} \le \frac{A_n(\mathbf{x})}{G_n(\mathbf{x})}. \tag{14}$$

We obtain the following sharpening of inequality (14).

Theorem 4.7. *Let* $x_i \in \mathbb{R}_+, i = 1, \cdots, n$. *For any* $c > 0$, *we have*

$$\frac{G_n(\boldsymbol{x})}{G_n(c+\boldsymbol{x})} \le \frac{W_n(\boldsymbol{x}, w_1, w_2, -1)}{W_n(c+\boldsymbol{x}, w_1, w_2, -1)} \le \frac{A_n(\boldsymbol{x})}{A_n(c+\boldsymbol{x})}. \tag{15}$$

Proof. By Theorem 1.4 (i) and according to the majorization inequality in Lemma 2.7:

$$\left(\frac{c+x_1}{nc+s}, \cdots, \frac{c+x_n}{nc+s}\right) \prec \left(\frac{x_1}{s}, \cdots, \frac{x_n}{s}\right),$$

it is easy to prove inequality (15) is hold.
The proof is complete. □

Let $(x_1, \cdots, x_n) \in \mathbb{R}_+^n$,

$$G(r, s; \mathbf{x}) = \left(\frac{\sum_{i=1}^n x_i^s}{\sum_{i=1}^n x_i^r}\right)^{\frac{1}{s-r}}, (s \ne r)$$

is Gini mean of n variables.
For Gini mean of n variables, we have the following conclusions.

Theorem 4.8. *Let* $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}_+^n$. *If* $0 < r < s$, *then*

$$G(r, s; \boldsymbol{x}) \ge \left(\frac{W_n(\boldsymbol{x}^s, w_1, w_2, -1)}{W_n(\boldsymbol{x}^r, w_1, w_2, -1)}\right)^{\frac{1}{s-r}} \ge G_n(\boldsymbol{x}). \tag{16}$$

Proof. By Lemma 2.7(*iii*) the majorization inequality:

$$\left(\frac{x_1^r}{\sum_{i=1}^n x_i^r}, \cdots, \frac{x_n^r}{\sum_{i=1}^n x_i^r}\right) \prec \left(\frac{x_1^s}{\sum_{i=1}^n x_i^s}, \cdots, \frac{x_n^s}{\sum_{i=1}^n x_i^s}\right)$$

holds, when $0 \leq w_2^* < w_2$, by Theorem 1.4(i) we have

$$\frac{W_n(\frac{\mathbf{x}^s}{\sum_{i=1}^n x_i^s}, w_1, w_2, p)}{W_n(\frac{\mathbf{x}^s}{\sum_{i=1}^n x_i^s}, w_1, w_2^*, p)} \leq \frac{W_n(\frac{\mathbf{x}^r}{\sum_{i=1}^n x_i^r}, w_1, w_2, p)}{W_n(\frac{\mathbf{x}^r}{\sum_{i=1}^n x_i^r}, w_1, w_2^*, p)}$$

$$\Rightarrow \frac{W_n(\mathbf{x}^s, w_1, w_2, p)}{W_n(\mathbf{x}^r, w_1, w_2, p)} \leq \frac{W_n(\mathbf{x}^s, w_1, w_2^*, p)}{W_n(\mathbf{x}^r, w_1, w_2^*, p)}.$$

So, $M_3(w_2) = \frac{W_n(\mathbf{x}^s, w_1, w_2, p)}{W_n(\mathbf{x}^r, w_1, w_2, p)}$ is decreasing with w_2.

Notice that $W_n(\mathbf{x}^k, w_1, 0, p) = [H_n(\mathbf{x}^{kp})]^{\frac{1}{p}}$, $W_n(\mathbf{x}^k, w_1, +\infty, p) = [G_n(\mathbf{x}^{kp})]^{\frac{1}{p}} = G_n(\mathbf{x}^k)$, so that, for $x_i > 0$, $i = 1, 2, \cdots, n$ and $s > r > 0$, if $p \leq -1$ and $w_1 > 0$, $0 \leq w_2 < +\infty$, then inequality

$$\frac{G_n(\mathbf{x}^s)}{G_n(\mathbf{x}^r)} \leq \frac{W_n(\mathbf{x}^s, w_1, w_2, p)}{W_n(\mathbf{x}^r, w_1, w_2, p)} \leq \frac{[H_n(\mathbf{x}^{sp})]^{\frac{1}{p}}}{[H_n(\mathbf{x}^{rp})]^{\frac{1}{p}}}$$

holds.

Let $p = -1$, we get inequality

$$G(r, s; \mathbf{x}) \geq \left(\frac{W_n(\mathbf{x}^s, w_1, w_2, -1)}{W_n(\mathbf{x}^r, w_1, w_2, -1)} \right)^{\frac{1}{s-r}} \geq G_n(\mathbf{x}).$$

The proof is complete. □

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