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Ordering of the Unicyclic Signed Graphs With Perfect Matchings by Their Minimal Energies

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Abstract. Let $\mathcal{U}_{2n}^{\sigma}$ be the set of unicyclic signed graphs with perfect matchings having $2n$ vertices, where σ is a signing function from the edge set of the graphs considered to $\{-1, 1\}$. The increasing order of the signed graphs among $\mathcal{U}_{2n}^{\sigma}$ according to their minimal energies is considered. A relationship between the energies of a unicyclic graph and of its signed graphs is derived. A new integral formula for comparing the energies of two signed graph is introduced. In $\mathcal{U}_{2n}^{\sigma}$ with $n \geq 721$, the first 18 signed graphs in the increasing order by their minimal energies are obtained.

1. Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph of order n, where $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. A signed graph, denoted by $S = (G, \sigma)$, is obtained from G by assigning a positive or negative label on the edge of $E(G)$, where $\sigma : E(G) \to \{-1, 1\}$ is the signing function. Namely, for an edge v_iv_j of G, $\sigma(v_iv_j) = -1$ or 1, where $1 \leq i, j \leq n$. We say that G is the underlying graph of S and S is the signed graph of G. If each edge of $E(G)$ has a positive label, then S becomes G. The signed graph is of theoretical interest due to both their applications in modeling a variety of physical and socio-psychological processes and their interesting connections with many classical mathematical systems [6].

The sign of a signed cycle is defined to be the product of signs of its edges. A signed cycle is said to be positive/negative if its sign is positive/negative. Namely, a positive/negative signed cycle contains an even/odd number of negative edges. A signed graph is said to be balanced if each of its cycles is positive, and otherwise to be unbalanced.

The adjacency matrix of S is an $n \times n$ matrix $\mathbb{A}(S) = (a_{ij})$, where $a_{ij} = \sigma(v_iv_j)$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. The characteristic polynomial of S is

$$\phi_S(x) = \det[x\mathbb{I} - \mathbb{A}(S)] = x^n + a_1(S)x^{n-1} + \cdots + a_{n-1}(S)x + a_n(S), \tag{1}$$

where $\mathbb{I}$ is the unit matrix of order n and $a_1(S), \cdots, a_n(S)$ are the coefficients of $\phi_S(x)$. The n roots of $\phi_S(x) = 0$ are denoted by $\lambda_1, \cdots, \lambda_n$, which are called the eigenvalues of the corresponding signed graph S. Since $\mathbb{A}(S)$ is a real symmetric matrix, all λ_i with $1 \leq i \leq n$ are real.

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We denote the adjacency matrix of G by $\mathbb{A}(G) = (a_{ij})$, where $a_{ij} = a_{ji} = 1$ if i and j are adjacent, else $a_{ij} = a_{ji} = 0$ for $1 \leq i, j \leq n$. The energy of G, denoted by $\mathcal{E}(G)$, as introduced by Gutman [8], is defined as the sum of the absolute values of all the eigenvalues of $\mathbb{A}(G)$.

Let $\mathbb{M}$ be an $m \times n$ complex matrix. The singular values of $\mathbb{M}$ are the positive square roots of the eigenvalues of $\mathbb{M}\mathbb{M}^*$, where $\mathbb{M}^*$ is the conjugate transpose of $\mathbb{M}$. Nikiforov [16] first defined that the energy of a matrix $\mathbb{M}$ is the sum of its singular values. For undirected graphs, there are various generalizations of the graph energy, for example, the matching energy by Gutman and Wagner [10], Laplacian energy by Gutman and Zhou [11], the incidence energy by Jooyandeh et al. [14], the distance energy by Indulal et al. [13] and Ramane et al. [19], and so on. For directed graphs, Peña and Rada [17] introduced the energy of a digraph, and Adiga et al. [2] proposed the skew energy of an oriented graph. The energy for the signed digraphs was defined by Bhat and Pirzada [3, 18].

The energy of the signed graph S was first introduced by Germina et al. [6] and it can be reduced to

$$\mathcal{E}(S) = \sum_{i=1}^{n} |\lambda_i|. \tag{2}$$

Bhat and Pirzada [4] expressed $\mathcal{E}(S)$ as the following Coulson integral formula

$$\mathcal{E}(S) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \log\Big[\Big(\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} b_{2j}(S)x^{2j}\Big)^2 + \Big(\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} b_{2j+1}(S)x^{2j+1}\Big)^2\Big]\mathrm{d}x, \tag{3}$$

where $b_i(S) = |a_i(S)|$ for $0 \leq i \leq n$. Note that $b_0(S) = 1$, $b_1(S) = 0$ and $b_2(S)$ equals the number of edges in S. It can be seen from (3) that $\mathcal{E}(S)$ is a strictly monotonously increasing function of $b_i(S)$, where $0 \leq i \leq n$. Let S_1 and S_2 be two unicyclic signed graphs. We get

$$b_i(S_1) \geq b_i(S_2) \Longrightarrow \mathcal{E}(S_1) \geq \mathcal{E}(S_2), \tag{4}$$

where $\mathcal{E}(S_1) = \mathcal{E}(S_2)$ if and only if $b_i(S_1) = b_i(S_2)$ for all $0 \leq i \leq n$. We will refer to the relation (4) as the method of coefficient comparison. For the sake of conciseness, we introduce the symbols " $\rightharpoonup$ " and " $\rightleftharpoons$ ".

$$\mathcal{E}(S_1) < \mathcal{E}(S_2) \iff S_1 \rightharpoonup S_2, \qquad \mathcal{E}(S_1) = \mathcal{E}(S_2) \iff S_1 \rightleftharpoons S_2. \tag{5}$$

The Coulson integral formula for $\mathcal{E}(G)$, as introduced in [9], can readily be written by replacing S in (3) with G, where $b_i(G) = |a_i(G)|$ and $a_i(G)$ is the coefficients of the characteristic polynomial of G for $0 \leq i \leq n$.

The relation (4) has successfully been employed in the study on the extremal values of energy for signed graphs. Among all the unicyclic signed graphs with n vertices, Bhat and Pirzada [4] characterized the unicyclic signed graph with the minimal energy. Among all the bicyclic signed graphs with n vertices, Bhat et al. [5] determined the bicyclic signed graphs with the first and the second minimal energies.

We denote by $\mathcal{U}_{2n}$ the set of unicyclic graphs with perfect matchings having $2n$ vertices. Among $\mathcal{U}_{2n}$, for the increasing order of graphs according to their minimal energies, Wang [20] obtained the first 7 graphs for $l = 2r + 1$ and $l = 4j + 2$ when $n \geq 45$, where l is the girth of the graphs considered and r, j are positive integers; and Zhu [25] derived the first 7 graphs when $n \geq 191$.

Let $\mathcal{U}_{2n}^{\sigma}$ be the set of unicyclic signed graphs with perfect matchings having $2n$ vertices, where σ is a signing function from the edge set of the graphs considered to $\{-1, 1\}$. In this paper, we will study the increasing order of the signed graphs among $\mathcal{U}_{2n}^{\sigma}$ according to their minimal energies.

This paper is organized as follows. In Section 2, some graphs are introduced and necessary lemmas are presented. The results of Section 3 are divided into five parts. In Subsection 3.1, a relationship between the energies of a unicyclic graph and of its signed graphs is derived. In Subsection 3.2, a new integral formula for comparing the energies of two signed graph is introduced. The capped graphs, in which the number of 2-matchings is less than $2n - 3$, of the graphs among $\mathcal{U}_{2n}$ are characterized in Subsection 3.3. In Subsection 3.4, by using the method of coefficient comparison in (4), the theorem of zero points and the new integral formula (presented in Theorem 3.3 in Subsection 3.2), we compare energies for the signed graphs whose capped graphs have the number of 2-matchings being less than $2n-3$. In Subsection 3.5, the preceding 18 signed graphs in the increasing order by their minimal energies are obtained among $\mathcal{U}^{\sigma}_{2n}$ with $n \geq 721$; and the preceding 12 graphs in the increasing order by their minimal energies are derived among $\mathcal{U}_{2n}$ with $n \geq 721$.

2. Preliminaries

A basic figure is a graph whose components are cycles or edges or both. In 1980, Acharya [1] studied the characteristic polynomial of a signed graph and got Lemma 2.1 as follows.

Lemma 2.1. *[1] If S is a signed graph with characteristic polynomial $\phi_S(x) = x^n + a_1(S)x^{n-1} + \cdots + a_{n-1}(S)x + a_n(S)$, then for all $j = 1, 2, \ldots, n$, we have*

$$a_j(S) = \sum_{L \in \mathcal{L}_j} (-1)^{p(L)} 2^{|c(L)|} \prod_{Z \in c(L)} s(Z), \tag{6}$$

where $\mathcal{L}_j$ is the set of all basic figures L of order j in S, $p(L)$ the number of components of L, $c(L)$ the set of all cycles in L, and $s(Z)$ the sign of cycle Z.

For $u \in V(S)$, let $S-u$ be the graph obtained from S by deleting u and all the edges in S which are incident with u. For $e \in E(S)$, let $S-e$ be the graph obtained from S by deleting e.

Lemma 2.2. *[7] Let S be a unicyclic signed graph and uv a pendent edge of S with a pendent vertex v. Then*

$$\phi_S(x) = x\phi_{(S-v)}(x) - \phi_{(S-v-u)}(x). \tag{7}$$

A k-matching of G is a union of k independent edges in G. Let $m(G,k)$ be the number of k-matchings in G, where $0 \leq k \leq n$. For G and its signed graph S, it is obvious that $m(G,k) = m(S,k)$ for $0 \leq k \leq n$.

Lemma 2.3. *[9] Let $e = uv$ be an edge of a graph G. Then we have*

$$m(G,k) = m(G-e,k) + m(G-u-v,k-1). \tag{8}$$

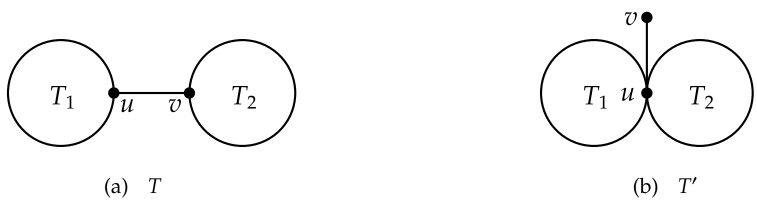

Figure 1: T and T' in Lemma 2.4.

Let T be a tree as shown in Fig. 1(a). Namely, T is obtained from an edge uv by attaching trees T_1 and T_2 at u and v, respectively. Let T' be the tree obtained from T by first identifying u and v and then attaching a pendent edge at u. T' is the tree as shown in Fig. 1(b). For the 2-matchings of T and T', by using Lemma 2.3, we can get Lemma 2.4 as follows. We omit the proof of Lemma 2.4 since it is straighforward.

Lemma 2.4. *Let T and T' be two trees as shown in Fig. 1. We have $m(T,2) > m(T',2)$.*

Lemma 2.5. *[4] If S is a signed graph on n vertices, then*

$$\mathcal{E}(S) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{x^2}\log\left|x^n\phi_S(\frac{i}{x})\right| dx, \tag{9}$$

where $i^2 = -1$.

For $G \in \mathcal{U}_{2n}$, the relationship between $b_i(G)$ and $m(G, i)$ is shown in Theorem 4 in [12] and Lemma 2 in [22].

Next, we will give a formula to calculate $m(G, i)$, where $G \in \mathcal{U}_{2n}$.

It is consistent to define $m(G, 0) = 1$. Obviously, $m(G, 1) = 2n$. Let $\widehat{G} = G - M(G) - S_0$, where $M(G)$ is the perfect matching of G and S_0 the set of isolated vertices in $G - M(G)$. It is clear that $|M(G)| = n$, where $|M(G)|$ is the number of edges in $M(G)$. We call $\widehat{G}$ the capped graph of G and G the original graph of $\widehat{G}$. Each k-matching Ω of G can be partitioned into two parts: $\Omega = \Phi \bigcup \Psi$, where Φ is a matching in $\widehat{G}$ and $\Psi \subset M(G)$. Thus, for $G \in \mathcal{U}_{2n}$, we have [24]

$$m(G, k) = \sum_{i=0}^{k} m(\widehat{G}, i)\binom{n-j}{k-i} = p + \sum_{i=2}^{k} m(\widehat{G}, i)\binom{n-j}{k-i}, \tag{10}$$

where

$$p = \binom{n}{k} + n\binom{n-2}{k-1} \tag{11}$$

and j is the number of edges in $M(G)$ which are adjacent to the i-matching Φ.

For $n \geq 2$, P_n is a path with n vertices, and the vertices of P_n are labeled consecutively by $v_1, v_2, \ldots, v_n$. For $l \geq 3$, C_l is a cycle with l vertices, and the vertices of C_l are labeled consecutively by $u_1, u_2, \ldots, u_l$. For $n \geq 3$, X_n is a star with n vertices.

For $l = 3$, some graphs are introduced as follows.

For $n \geq 4$, C_n^a is the graph obtained from C_3 by attaching a and $n - a - 3$ pendent edges to u_1 and u_2 respectively, where $0 \leq a \leq [\frac{n-3}{2}]$.

For $n \geq 5$, Q_n is the graph obtained by identifying u_1 of C_3 with a pendent vertex of X_{n-2}.

For $n \geq 6$, let H_n, S_n and T_n be the graphs obtained by identifying v_2, v_3 and v_4 of P_4 of I_{n-2} with u_1 of C_3 respectively, where I_{n-2} with $n \geq 6$ is the graph obtained from $P_4 = v_1v_2v_3v_4$ by attaching $n - 6$ pendent edges to v_2 of P_4.

For $n \geq 6$, R_n is the graph obtained from C_3 by attaching $n - 5$ pendent edges to u_1 and a path of length 2 to u_2.

For $n \geq 6$, W_n is the graph obtained from C_3 by attaching one pendent edge to u_1 and identifying u_2 of C_3 with a pendent vertex of X_{n-3}.

For $n \geq 6$, Y_n is the graph obtained from C_3 by attaching one pendent edge to u_1 and u_2 and $n-5$ pendent edges to u_3.

Denote by $\mathcal{K} = \{C_n^a, Q_n, H_n, S_n, T_n, R_n, W_n, Y_n\}$, where $0 \leq a \leq [\frac{n-3}{2}]$.

For $n \geq 7$, O_n is the graph obtained from P_6 by attaching $n - 6$ pendent edges to v_2. For $n \geq 6$, Z_n is the graph obtained from C_5 by attaching $n - 5$ pendent edges to u_1.

Let $\mathcal{H}_n$ be the set of unicyclic graphs with n vertices. Let dia(G) be the diameter of a graph G.

Lemma 2.6. *[21, 23] Let $G \in \mathcal{H}_n$ with $n \geq 7$ and C_l the cycle contained in G. If one of the three conditions holds: (i) dia$(G - e) \geq 5$ and $G - e \neq O_n$, where e is an edge of C_l; or (ii) $l = 3$ and $G \notin \mathcal{K}$; or (iii) $l \geq 5$ and $G \neq Z_n$, then we have $m(G, 2) \geq 3n - 12$ and $m(G, 3) \geq 2n - 12$, where the two equalities do not hold simultaneously.*

3. Main results

3.1. Comparing the energies of a unicyclic graph and of its signed graphs

Bhat and Pirzada [4] obtained that for a unicyclic graph G with an odd girth, any two signed graphs on G have the same energy. In Theorem 3.2, we will extend this result to the relationship between the energies of a unicyclic graph and of its signed graphs. To obtain Theorem 3.2, Lemma 3.1 is needed. By Lemma 2.1, we can get Lemma 3.1 as follows.

Lemma 3.1. *Let $S \in \mathcal{U}_{2n}^{\sigma}$, l the length of the cycle contained in S and r and h integers with $r, h \geq 1$. Then*

$$b_{2i+1}(S) = \begin{cases} 0, & l = 2r, l = 2r+1 \;\&\; 2i+1 < l, \\ 2m(S - C_l, i - \frac{l-1}{2}), & l = 2r+1 \;\&\; 2i+1 \geq l; \end{cases} \tag{12}$$

$$b_{2i}(S) = \begin{cases} m(S,i), & l = 2r+1, \\ m(S,i) + 2m(S - C_l, i - \frac{l}{2}), & \text{if } S \text{ is balanced and } l = 4h+2 \text{ or} \\ & \quad S \text{ is unbalanced and } l = 4h, \\ m(S,i) - 2m(S - C_l, i - \frac{l}{2}), & \text{if } S \text{ is balanced and } l = 4h \text{ or} \\ & \quad S \text{ is unbalanced and } l = 4h+2. \end{cases} \tag{13}$$

For a unicyclic graph G, by Lemma 3.1 and (3), we can conclude that any two balanced graphs of G have the same energy and any two unbalanced graphs of G also have the same energy. If G is a unicyclic graph, we use, for simplicity, G^+ to denote its balanced graph and G^- its unbalanced graph throughout this paper. We have a relationship for $\mathcal{E}(G^+)$, $\mathcal{E}(G^-)$ and $\mathcal{E}(G)$, as shown in Theorem 3.2.

Theorem 3.2. *Let G be a unicyclic graph with girth l. For positive integers $r, h \geq 1$, we have*
(i) $\mathcal{E}(G^+) = \mathcal{E}(G^-) = \mathcal{E}(G)$ for $l = 2r+1$;
(ii) $\mathcal{E}(G^+) = \mathcal{E}(G) < \mathcal{E}(G^-)$ for $l = 2r$ with $r = 2h$;
(iii) $\mathcal{E}(G^+) = \mathcal{E}(G) > \mathcal{E}(G^-)$ for $l = 2r$ with $r = 2h+1$.

Proof. Let $l = 2r+1$. By Lemma 2 in [22] and Lemma 3.1, we have $b_{2i}(G^-) = b_{2i}(G^+) = b_{2i}(G) = m(G,i)$, $b_{2i+1}(G^-) = b_{2i+1}(G^+) = b_{2i+1}(G) = 0$ for $2i+1 < l$, and $b_{2i+1}(G^-) = b_{2i+1}(G^+) = b_{2i+1}(G) = 2m(G - C_l, i - r)$ for $2i+1 \geq l$, where $0 \leq i \leq [n/2]$. Furthermore, by comparing $\mathcal{E}(G)$, $\mathcal{E}(G^+)$ and $\mathcal{E}(G^-)$ in terms of their Coulson integral formula in (3), we get Theorem 3.2(i).

By the method similar to that for Theorem 3.2(i), we can get Theorem 3.2(ii) and (iii). □

3.2. A new integral formula for comparing the energies of two signed graphs

From Lemma 2.5, we obtain a new integral formula for comparing the energies between two signed graphs by directly using their characteristic polynomials, which are shown in Theorem 3.3.

Theorem 3.3. *Let $\phi_{S_1}(x)$ and $\phi_{S_2}(x)$ be the characteristic polynomials of signed graphs S_1 and S_2 respectively, where S_1 and S_2 have the same number of vertices. Then*

$$\mathcal{E}(S_1) - \mathcal{E}(S_2) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \log \left| \frac{\phi_{S_1}(ix)}{\phi_{S_2}(ix)} \right| dx. \tag{14}$$

Proof. From Lemma 2.5, we have

$$\mathcal{E}(S_1) - \mathcal{E}(S_2) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \log \left| \frac{x^n \phi_{S_1}(\frac{i}{x})}{x^n \phi_{S_2}(\frac{i}{x})} \right| \mathrm{d}x. \tag{15}$$

Obviously, we get

$$\mathcal{E}(S_1) - \mathcal{E}(S_2) = -\frac{1}{\pi} \Big[\int_{-\infty}^{0} \log \left| \frac{\phi_{S_1}(\frac{i}{x})}{\phi_{S_2}(\frac{i}{x})} \right| \mathrm{d}(\frac{1}{x}) + \int_{0}^{+\infty} \log \left| \frac{\phi_{S_1}(\frac{i}{x})}{\phi_{S_2}(\frac{i}{x})} \right| \mathrm{d}(\frac{1}{x}) \Big]. \tag{16}$$

Since 0 is a flaw, we obtain

$$\int_{-\infty}^{0}\log\left|\frac{\phi_{S_1}(\frac{\mathrm{i}}{x})}{\phi_{S_2}(\frac{\mathrm{i}}{x})}\right|\mathrm{d}(\frac{1}{x})=\int_{-\infty}^{c}\log\left|\frac{\phi_{S_1}(\frac{\mathrm{i}}{x})}{\phi_{S_2}(\frac{\mathrm{i}}{x})}\right|\mathrm{d}(\frac{1}{x})+\int_{c}^{0}\log\left|\frac{\phi_{S_1}(\frac{\mathrm{i}}{x})}{\phi_{S_2}(\frac{\mathrm{i}}{x})}\right|\mathrm{d}(\frac{1}{x})$$
$$=\int_{0}^{\frac{1}{c}}\log\left|\frac{\phi_{S_1}(\mathrm{i}t)}{\phi_{S_2}(\mathrm{i}t)}\right|\mathrm{d}t+\int_{\frac{1}{c}}^{-\infty}\log\left|\frac{\phi_{S_1}(\mathrm{i}t)}{\phi_{S_2}(\mathrm{i}t)}\right|\mathrm{d}t=\int_{0}^{-\infty}\log\left|\frac{\phi_{S_1}(\mathrm{i}t)}{\phi_{2}(\mathrm{i}t)}\right|\mathrm{d}t, \tag{17}$$

where $\frac{1}{x}=t$. Similarly, we have

$$\int_{0}^{+\infty}\log\left|\frac{\phi_{S_1}(\frac{\mathrm{i}}{x})}{\phi_{S_2}(\frac{\mathrm{i}}{x})}\right|\mathrm{d}(\frac{1}{x})=\int_{+\infty}^{0}\log\left|\frac{\phi_{S_1}(\mathrm{i}t)}{\phi_{S_2}(\mathrm{i}t)}\right|\mathrm{d}t. \tag{18}$$

Substituting (17) and (18) into (16), we obtain (14). □

3.3. *Finding the graphs whose capped graphs have 2-matching less than* $2n-3$

In this section, among $\mathcal{U}_{2n}$, the graphs whose capped graphs have 2-matching less than $2n-3$ are characterized in Lemma 3.8. To obtain Lemma 3.8, we need to introduce Lemmas 3.4–3.7 first.

Let $\mathcal{T}_{n+1}$ be the set of trees with $n+1$ vertices. Let $E_{n+1}^{a,b}$ be the tree obtained from P_4 by attaching a and b pendent edges at v_2 and v_3 respectively, where $0\le a\le[\frac{n-3}{2}]$ and $b=n-a-3$. Let $F_{n+1}^{a,b,c}$ be the tree obtained from P_5 by attaching a, b and c pendent edges at v_2, v_3 and v_4 respectively, where $0\le a,b,c\le n-4$ and $a+b+c=n-4$.

Lemma 3.4. *Let* $G\in\mathcal{U}_{2n}$ *and* $n\ge 9$. *If* $\widehat{G}\in\mathcal{T}_{n+1}$ *and* $\widehat{G}\ne X_{n+1},E_{n+1}^{0,n-3},E_{n+1}^{1,n-4},F_{n+1}^{0,0,n-4},F_{n+1}^{0,n-4,0}$, *then* $m(\widehat{G},2)\ge 2n-3$.

Proof. Let $G\in\mathcal{U}_{2n}$ with $n\ge 9$ and $\widehat{G}\in\mathcal{T}_{n+1}$. We suppose $\widehat{G}\ne X_{n+1}$. Thus, $\mathrm{dia}(\widehat{G})\ge 3$. Four cases are considered according to the value of $\mathrm{dia}(\widehat{G})$.

Case (i). $\mathrm{dia}(\widehat{G})=3$.

In this case, $\widehat{G}=E_{n+1}^{a,b}$. By (8), we have $m(E_{n+1}^{a,b},2)=(a+1)(n-2-a)\triangleq f_1(a)$. Therefore, $m(E_{n+1}^{a,b},2)$ is an increasing function of a since $f_1'(a)=-2a+n-3\ge 0$, where $0\le a\le[\frac{n-3}{2}]$. We have $m(E_{n+1}^{0,n-3},2)=f_1(0)=n-2<2n-3$, $m(E_{n+1}^{1,n-4},2)=f_1(1)=2(n-3)<2n-3$, and $m(E_{n+1}^{a,b},2)=f_1(a)\ge f_1(2)=3(n-4)\ge 2n-3$ for $2\le a\le[\frac{n-3}{2}]$. Namely, in Case (i), if $G\ne E_{n+1}^{0,n-3},E_{n+1}^{1,n-4}$, then we have $m(\widehat{G},2)\ge 2n-3$.

Case (ii). $\mathrm{dia}(\widehat{G})=4$.

In this case, $\widehat{G}$ may be viewed as a tree obtained from P_5 by attaching trees T_1, T_2 and T_3 at v_2, v_3 and v_4, respectively. Let $|E(T_1)|=a$, $|E(T_2)|=b$ and $|E(T_3)|=c$, where $0\le a,b,c\le n-4$ and $a+b+c=n-4$. By using Lemma 2.4 repeatedly, we get $m(\widehat{G},2)\ge m(F_{n+1}^{a,b,c},2)$. Three subcases are considered as follows.

Subcase (ii.i). $a=0$, $0\le b,c\le n-4$ and $c=n-4-b$.

By (8), we have

$$m(F_{n+1}^{0,b,c},2)=(b+2)(n-3-b)+b+1\triangleq f_2(b). \tag{19}$$

We get $f_2'(b)=-2b+n-4$. Thus, $m(F_{n+1}^{0,b,c},2)$ is an increasing function of b with $0\le b\le\lfloor\frac{n-4}{2}\rfloor$ and a decreasing function of b with $\lceil\frac{n-4}{2}\rceil\le b\le n-4$. We have $m(F_{n+1}^{0,0,n-4},2)=f_2(0)=2n-5<2n-3$, $m(F_{n+1}^{0,n-4,0},2)=f_2(n-4)=2n-5<2n-3$ and $m(F_{n+1}^{0,b,c},2)=f_2(b)\ge\min\{f_2(1),f_2(n-5)\}=3n-10\ge 2n-3$, where $1\le b\le n-5$.

Next, we use the same analysis as those for Case (i) or Subcase (ii.i) to derive the property of the 2-matching of the graphs considered.

Subcase (ii.ii). $b=0$, $1\le a,c\le[\frac{n-4}{2}]$ and $c=n-4-a$.

By (8), we have $m(F_{n+1}^{a,0,c},2)=(a+2)(n-3-a)+a+1\triangleq f_3(a)\ge f_3(1)=3n-10\ge 2n-3$.

Subcase (ii.iii). $c = 0$, $0 \leq a, b \leq n-4$ and $a = n-4-b$.

Subcase (ii.iii) is the same as that for Subcase (ii.i).

Subcase (ii.iv). $1 \leq a, c \leq [\frac{n-5}{2}]$, $1 \leq b \leq n-6$ and $c = n-4-a-b$.

If $a = 1$, then by (8), we obtain $m(F^{1,b,c}_{n+1}, 2) = (b+1)(n-4-b) + 2(n-3) \triangleq f_4(b) \geq \min\{f_4(1), f_4(n-6)\} = 4n-16 \geq 2n-3$. If $2 \leq a \leq [\frac{n-5}{2}]$, then by (8), we get

$$\begin{aligned} m(F^{a,b,c}_{n+1}, 2) &= (a+1)(n-a-2) + b(n-3-a-b) + n-3-a-b \\ &\geq (a+1)(n-a-2) \triangleq f_5(a) \geq f_5(2) = 3(n-4) \geq 2n-3. \end{aligned}$$

By the proofs of Subcases (ii.i)–(ii.iv), we have $m(\widehat{G}, 2) \geq m(F^{a,b,c}_{n+1}, 2) \geq 2n-3$ if $\widehat{G} \neq F^{0,0,n-4}_{n+1}, F^{0,n-4,0}_{n+1}$.

Case (iii). $\mathrm{dia}(\widehat{G}) = 5$.

In this case, $\widehat{G}$ may be viewed as a tree obtained from P_6 by attaching trees T_1, T_2, T_3, and T_4 at v_2, v_3, v_4, and v_5 respectively. Let $|E(T_1)| = a$, $|E(T_2)| = b$, $|E(T_3)| = c$, and $|E(T_4)| = d$, where $0 \leq a, b, c, d \leq n-5$ and $a+b+c+d = n-5$. By using Lemma 2.4 repeatedly, we have

$$m(\widehat{G}, 2) \geq m(G^{a,b,c,d}_{n+1}, 2), \tag{20}$$

where $G^{a,b,c,d}_{n+1}$ is the graph obtained from P_6 by attaching a, b, c, and d pendent edges to v_2, v_3, v_4, and v_5 respectively. By the methods similar to those for Case (ii), we can obtain $m(\widehat{G}, 2) \geq m(G^{a,b,c,d}_{n+1}, 2) \geq 2n-3$.

Case (iv). $\mathrm{dia}(\widehat{G}) \geq 6$.

In this case, $\widehat{G}$ can be viewed as a connected tree obtained from P_{d+1} by attaching trees, say $T_1, \ldots, T_{d-1}$, at $v_2, \ldots, v_d$ respectively. By using Lemma 2.4 repeatedly and by the proof of Case (iii), we obtain $m(\widehat{G}, 2) \geq m(G^{a,b,c,d}_{n+1}, 2) \geq 2n-3$.

By combining the proofs of Cases (i)–(iv), we get Lemma 3.4. □

For $l = 4$, some graphs are introduced as follows.

For $n \geq 6$, A_n is the graph obtained from C_4 by attaching one and $n-5$ pendent edges to u_1 and u_2 respectively.

For $n \geq 7$, B_n is the graph obtained from C_4 by attaching a path P_3 to u_1 and $n-6$ pendent edges to u_3.

For $n \geq 6$, C_n is the graph obtained from C_4 by identifying u_1 of C_4 with a pendent vertex of X_{n-3}.

For $n \geq 5$, D^a_n is the graph obtained from C_4 by attaching a and $n-a-4$ pendent edges to u_1 and u_3 respectively, where $0 \leq a \leq [\frac{n-4}{2}]$.

Lemma 3.5. *Let $G \in \mathcal{U}_{2n}$ and $n \geq 10$. If $\widehat{G} \in \mathcal{H}_n$ and $\widehat{G} \neq D^0_n, H_n, C^0_n, C^1_n$, then $m(\widehat{G}, 2) \geq 2n-3$.*

Proof. Let $\widehat{G} \in \mathcal{H}_n$ and $n \geq 10$, where $G \in \mathcal{U}_{2n}$. Suppose that the cycle contained in $\widehat{G}$ is C_l. Three cases are considered as follows.

Case (i). $l \geq 5$.

If $\widehat{G} = Z_n$, then $m(\widehat{G}, 2) = 3n-10 > 2n-3$. If $\widehat{G} \neq Z_n$, then by Lemma 2.6, we have $m(\widehat{G}, 2) \geq 3n-12 \geq 2n-3$.

Case (ii). $l = 4$.

If $\widehat{G} \neq A_n, B_n, C_n, D^a_n$, then we can choose an edge e on C_4 of $\widehat{G}$ such that $\widehat{G} - e \neq O_n$ and $\mathrm{dia}(\widehat{G} - e) \geq 5$. Therefore, by Lemma 2.6, we have $m(\widehat{G}, 2) \geq 3n-12 \geq 2n-3$. By direct calculation, we get $m(A_n, 2) = 3n-11 \geq 2n-3$ and $m(B_n, 2) = m(C_n, 2) = 4n-16 > 2n-3$. By (8), we have

$$m(D^a_n, 2) = (n-3-a)(a+2) + a \triangleq f_6(a). \tag{21}$$

Thus, $f_6(a)$ is an increasing function of a since $f'_6(a) = -2a+n-4 \geq 0$ for $0 \leq a \leq [\frac{n-4}{2}]$. If $\widehat{G} = D^0_n$, then $m(D^0_n, 2) = f_6(0) = 2n-6 < 2n-3$. If $\widehat{G} = D^a_n$ with $1 \leq a \leq [\frac{n-4}{2}]$, then $m(D^a_n, 2) = f_6(a) \geq f_6(1) = 3n-11 \geq 2n-3$.

Case (iii). $l = 3$.

If $\widehat{G} \notin \mathcal{K}$, then by Lemma 2.6, we have $m(\widehat{G}, 2) \geq 3n - 12 \geq 2n - 3$. When $n \geq 10$, we obtain $m(Q_n, 2) = m(R_n, 2) = 3n - 11 > 2n - 3$, $m(S_n, 2) = 4n - 18 > 2n - 3$, $m(T_n, 2) = 4n - 16 > 2n - 3$, $m(W_n, 2) = 4n - 17 > 2n - 3$, $m(Y_n, 2) = 3n - 12 > 2n - 3$, and $m(H_n, 2) = 2n - 6 < 2n - 3$. If $\widehat{G} = C_n^a$, then by (8), we have

$$m(C_n^a, 2) = (n - 3 - a)(a + 1) + a \triangleq f_7(a). \tag{22}$$

Thus, $f_7(a)$ is an increasing function of a since $f_7'(a) = -2a + n - 3 \geq 0$ for $0 \leq a \leq [\frac{n-3}{2}]$. If $\widehat{G} = C_n^0$, then $m(C_n^0, 2) = f_7(0) = n - 3 < 2n - 3$. If $\widehat{G} = C_n^1$, then $m(C_n^1, 2) = f_7(1) = 2n - 7 < 2n - 3$. If $\widehat{G} = C_n^a$ with $2 \leq a \leq [\frac{n-3}{2}]$, then $f_7(a) \geq f_7(2) = 3n - 13 \geq 2n - 3$.

By the proofs of Cases (i)–(iii), we obtain that if $\widehat{G} \neq D_n^0, H_n, C_n^0, C_n^1$, then $m(\widehat{G}, 2) \geq 2n - 3$ for $n \geq 10$. □

Let the number of the components of $\widehat{G}$ be $o(\widehat{G})$.

Lemma 3.6. *Let $G \in \mathcal{U}_{2n}$ and $n \geq 10$. If $o(\widehat{G}) \geq 3$ and each component of $\widehat{G}$ is a tree, then $m(\widehat{G}, 2) \geq 2n - 3$.*

Proof. If $o(\widehat{G}) \geq 4$, then we concatenate $\widehat{G}$ together into a graph (denoted by $\tilde{G}$) in such a way that $o(\tilde{G}) = 3$. Obviously, $m(\widehat{G}, 2) > m(\tilde{G}, 2)$. We suppose that the numbers of vertices for the three components of $\tilde{G}$ are $a + 1$, $b + 1$ and $n - a - b + 1$. Using Lemma 2.4 repeatedly, we obtain $m(\tilde{G}, 2) \geq m(X_{a+1} \cup X_{b+1} \cup X_{n-a-b+1}, 2)$. We suppose $1 \leq a \leq b$ and $a + b \leq n - 1$.

If $a = 1$, then we suppose $1 \leq b \leq [\frac{n-1}{2}]$. By (8), we have $m(X_2 \cup X_{b+1} \cup X_{n-b}, 2) = b(n - 1 - b) + n - 1 \triangleq f_8(b) \geq f_8(1) = 2n - 3$ since $f_8(b)$ is an increasing function of b for $1 \leq b \leq [\frac{n-1}{2}]$. If $a \geq 2$, then by (8), we have

$$\begin{aligned} m(X_{a+1} \cup X_{b+1} \cup X_{n-a-b+1}, 2) &= m(X_{a+1} \cup X_{n-a+1}, 2) + b(n - a - b) \\ &\geq m(X_{a+1} \cup X_{n-a+1}, 2) + 1 \geq 2n - 3. \end{aligned} \tag{23}$$

Therefore, we obtain Lemma 3.6. □

Lemma 3.7. *Let $G \in \mathcal{U}_{2n}$ with $n \geq 10$. If $\widehat{G}$ is unconnected and $\widehat{G} \neq X_2 \cup C_{n-1}^0, X_2 \cup X_n, X_3 \cup X_{n-1}, X_2 \cup E_n^{0,n-4}$, then $m(\widehat{G}, 2) \geq 2n - 3$.*

Proof. Let $G \in \mathcal{U}_{2n}$ and $n \geq 10$. Two cases are considered according to the types of the capped graph $\widehat{G}$.

Case (i). $\widehat{G}$ is an unconnected graph whose components are trees and a cycle.

If $\widehat{G}$ is composed of trees and a cycle, then we might concatenate them together into a connected unicyclic graph (denoted by $\widehat{G}_1$). If $\widehat{G}_1 \notin \{D_n^0, H_n, C_n^0, C_n^1\}$, then by Lemma 3.5, $m(\widehat{G}, 2) > m(\widehat{G}_1, 2) \geq 2n - 3$. Otherwise, if $\widehat{G}_1 \in \{D_n^0, H_n, C_n^0, C_n^1\}$, then four subcases are considered as follows.

Subcase (i.i). $\widehat{G}_1 = D_n^0$.

If $o(\widehat{G}) = 2$, then $\widehat{G} = X_{a+1} \cup D_{n-a}^0$ with $1 \leq a \leq n - 4$. By (8), we have

$$\begin{aligned} m(X_{a+1} \cup D_{n-a}^0, 2) &= (a + 2)(n - a) - 6 \triangleq f_9(a) \\ &\geq \min\{f_9(1), f_9(n - 4)\} = 3n - 9 \geq 2n - 3. \end{aligned} \tag{24}$$

If $o(\widehat{G}) \geq 3$, then we can concatenate $\widehat{G}$ into a graph with two components $X_{a+1} \cup D_{n-a}^0$, where $2 \leq a \leq n-4$. Thus, by (24), we get $m(\widehat{G}, 2) \geq m(X_{a+1} \cup D_{n-a}^0, 2) \geq 2n - 3$.

Subcase (i.ii). $\widehat{G}_1 = C_n^0$.

If $o(\widehat{G}) = 2$, then $\widehat{G} = X_{a+1} \cup C_{n-a}^0$ with $1 \leq a \leq n - 3$. For $a = 1$, we have

$$m(X_2 \cup C_{n-1}^0, 2) = 2n - 5 < 2n - 3. \tag{25}$$

For $2 \leq a \leq n - 3$, by (8), we get

$$\begin{aligned} m(X_{a+1} \cup C_{n-a}^0, 2) &= (a + 1)(n - a) - 3 \triangleq f_{10}(a) \\ &\geq \min\{f_{10}(2), f_{10}(n - 3)\} = 3n - 9 \geq 2n - 3. \end{aligned} \tag{26}$$

If $o(\widehat{G}) \geq 3$, then we can concatenate $\widehat{G}$ into a graph with two components $X_{a+1} \cup C^0_{n-a}$, where $2 \leq a \leq n-3$. Therefore, by (26), we get $m(\widehat{G}, 2) \geq m(X_{a+1} \cup C^0_{n-a}, 2) \geq 2n-3$.

Subcase (i.iii). $\widehat{G}_1 = H_n$.

If $o(\widehat{G}) = 2$, then $\widehat{G}$ may be one among the four types: $X_2 \cup C^0_{n-1}$, $X_3 \cup C^0_{n-2}$, $X_{a+1} \cup H_{n-a}$ with $1 \leq a \leq n-6$, and $E^{0,a-1}_{a+3} \cup C^0_{n-a-2}$ with $1 \leq a \leq n-5$. By (25) and (26), we have $m(X_2 \cup C^0_{n-1}, 2) < 2n-3$ and $m(X_3 \cup C^0_{n-2}) \geq 2n-3$ respectively. By (8), we get

$$\begin{aligned} m(X_{a+1} \cup H_{n-a}, 2) &= (a+2)(n-a) - 6 \triangleq f_{11}(a) \\ &\geq \min\{f_{11}(1), f_{11}(n-5)\} = 3n-9 \geq 2n-3, \\ m(E^{0,a-1}_{a+3} \cup C^0_{n-a-2}, 2) &= (a+2)(n-a-2) + n - 5 \triangleq f_{12}(a) \\ &\geq \min\{f_{12}(1), f_{12}(n-5)\} = 4n-14 \geq 2n-3. \end{aligned}$$

If $o(\widehat{G}) \geq 3$, then we concatenate $\widehat{G}$ into a graph with two components such that the graph is not $X_2 \cup C^0_{n-1}$. By the proof as above, we get $m(\widehat{G}, 2) \geq 2n-3$.

Subcase (i.iv). $\widehat{G}_1 = C^1_n$.

If $o(\widehat{G}) = 2$, then $\widehat{G}$ may be $X_2 \cup C^0_{n-1}$ or $X_{a+1} \cup C^1_{n-a}$ with $1 \leq a \leq n-4$. By (25), we have $m(X_2 \cup C^0_{n-1}, 2) < 2n-3$. By (8), for $1 \leq a \leq n-4$, we get

$$\begin{aligned} m(X_{a+1} \cup C^1_{n-a}, 2) &= (a+2)(n-a) - 7 \triangleq f_{13}(a) \\ &\geq \min\{f_{13}(1), f_{13}(n-4)\} = 3n-10 \geq 2n-3. \end{aligned} \tag{27}$$

If $o(\widehat{G}) \geq 3$, then we concatenate $\widehat{G}$ into a graph with two components such that it is not $X_2 \cup C^0_{n-1}$. Therefore, by (27), we get $m(\widehat{G}, 2) \geq 2n-3$.

By combining the proofs of Subcases (i.i)–(i.iv), we have $m(\widehat{G}, 2) \geq 2n-3$ if $\widehat{G} \neq X_2 \cup C^0_{n-1}$.

Case (ii). $\widehat{G}$ is an unconnected graph whose components are trees only.

If $o(\widehat{G}) \geq 3$, then by Lemma 3.6, we have $m(\widehat{G}, 2) \geq 2n-3$. Next, we always assume $o(\widehat{G}) = 2$. We concatenate $\widehat{G}$ together into a tree, denoted by $\widehat{G}_2$. Obviously, $\widehat{G}_2 \in \mathcal{T}_{n+1}$. If $\widehat{G}_2 \notin \{X_{n+1}, E^{0,n-3}_{n+1}, E^{1,n-4}_{n+1}, F^{0,0,n-4}_{n+1}, F^{0,n-4,0}_{n+1}\}$, then by Lemma 3.4, $m(\widehat{G}, 2) > m(\widehat{G}_2, 2) \geq 2n-3$. Otherwise, if $\widehat{G}_2 \in \{X_{n+1}, E^{0,n-3}_{n+1}, E^{1,n-4}_{n+1}, F^{0,0,n-4}_{n+1}, F^{0,n-4,0}_{n+1}\}$, then five subcases are considered as follows.

Subcase (ii.i). $\widehat{G}_2 = X_{n+1}$.

As $o(\widehat{G}) = 2$, $\widehat{G} = X_{a+1} \cup X_{n-a+1}$ with $1 \leq a \leq [\frac{n}{2}]$. By (8), we have $m(X_{a+1} \cup X_{n-a+1}, 2) = a(n-a) \triangleq f_{14}(a)$. Thus, $f_{14}(a)$ is an increasing function of a since $f'_{14}(a) \geq 0$ for $1 \leq a \leq [\frac{n}{2}]$. We have

$$m(X_2 \cup X_n, 2) = f_{14}(1) = n-1 < 2n-3, \tag{28}$$
$$m(X_3 \cup X_{n-1}, 2) = f_{14}(2) = 2n-4 < 2n-3, \tag{29}$$
$$m(X_{a+1} \cup X_{n-a+1}, 2) = f_{14}(a) \geq f_{14}(3) = 3(n-3) \geq 2n-3, \tag{30}$$

for $3 \leq a \leq [\frac{n}{2}]$.

Subcase (ii.ii). $\widehat{G}_2 = E^{0,n-3}_{n+1}$.

As $o(\widehat{G}) = 2$, $\widehat{G}$ may be one among the three types: $X_2 \cup X_n$, $X_3 \cup X_{n-1}$ and $X_{a+1} \cup E^{0,n-a-3}_{n-a+1}$ with $1 \leq a \leq n-3$. By (28) and (29), we only need to consider the last type. By (8), we have $m(X_{a+1} \cup E^{0,n-a-3}_{n-a+1}, 2) = a(n-a) + n - 2 - a = (a+1)(n-a) - 2 \triangleq f_{15}(a)$. If $a = 1$, then we have

$$m(X_2 \cup E^{0,n-4}_{n}, 2) = f_{15}(1) = 2n-4 < 2n-3. \tag{31}$$

If $2 \leq a \leq n-3$, then we get

$$m(X_{a+1} \cup E^{0,n-a-3}_{n-a+1}, 2) = f_{15}(a) \geq \min\{f_{15}(2), f_{15}(n-3)\} = 3n-8 \geq 2n-3. \tag{32}$$

Subcase (ii.iii). $\widehat{G}_2 = E_{n+1}^{1,n-4}$.

As $o(\widehat{G}) = 2$, $\widehat{G}$ may be one among the four types: $X_3 \cup X_{n-1}$, $X_4 \cup X_{n-2}$, $X_2 \cup E_n^{0,n-4}$, and $X_{a+1} \cup E_{n-a+1}^{1,n-a-4}$ with $1 \le a \le n-4$. By (29)–(31), we only need to consider the last type. We get $m(X_{a+1} \cup E_{n-a+1}^{1,n-a-4}, 2) = (a+2)(n-a) - 6 \triangleq f_{16}(a) \ge \min\{f_{16}(1), f_{16}(n-4)\} = 3n-9 \ge 2n-3$ for $1 \le a \le n-4$.

Subcase (ii.iv). $\widehat{G}_2 = F_{n+1}^{0,0,n-4}$.

As $o(\widehat{G}) = 2$, $\widehat{G}$ may be one among the four types: $X_3 \cup X_{n-1}$, $X_2 \cup E_n^{0,n-4}$, $X_{n-2} \cup E_4^{0,0}$, and $X_{a+1} \cup F_{n-a+1}^{0,0,n-a-4}$ with $1 \le a \le n-4$. By (29), (31) and (32), we only need to consider the last type. We get $m(X_{a+1} \cup F_{n-a+1}^{0,0,n-a-4}, 2) = (a+2)(n-a) - 5 \triangleq f_{17}(a) \ge \min\{f_{17}(1), f_{17}(n-4)\} = 3n-8 \ge 2n-3$ for $1 \le a \le n-4$.

Subcase (ii.v). $\widehat{G}_2 = F_{n+1}^{0,n-4,0}$.

If $o(\widehat{G}) = 2$, $\widehat{G}$ may be one among the four types: $X_2 \cup E_n^{0,n-4}$, $X_3 \cup E_{n-1}^{0,n-5}$, $X_{a+1} \cup F_{n-a+1}^{0,n-a-4,0}$ with $1 \le a \le n-4$, and $E_{a+3}^{0,a-1} \cup E_{n-a-1}^{0,n-a-5}$ with $1 \le a \le [\frac{n-4}{2}]$. By (31) and (32), we only need to consider the last two types. We get $m(X_{a+1} \cup F_{n-a+1}^{0,n-a-4,0}, 2) = (a+2)(n-a) - 5 \triangleq f_{18}(a) \ge \min\{f_{18}(1), f_{18}(n-4)\} = 3n-8 \ge 2n-3$ for $1 \le a \le n-4$ and $m(E_{a+3}^{0,a-1} \cup E_{n-a-1}^{0,n-a-5}, 2) = (a+2)(n-a-2) + n - 4 \triangleq f_{19}(a) \ge f_{19}(1) = 4n-13 \ge 2n-3$ for $1 \le a \le [\frac{n-4}{2}]$.

By the proofs of Subcases (ii.i)–(ii,v), if $\widehat{G} \ne X_2 \cup X_n, X_3 \cup X_{n-1}, X_2 \cup E_n^{0,n-4}$, then we have $m(\widehat{G}, 2) \ge 2n-3$. In conclusion, by the proofs of Cases (i) and (ii), we obtain Lemma 3.7. □

Let $\widehat{\mathcal{H}}$, $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{I}}$ be the sets of the special graphs in Lemmas 3.4, 3.5 and 3.7 respectively. Namely

$$\widehat{\mathcal{H}} = \{X_{n+1}, E_{n+1}^{0,n-3}, E_{n+1}^{1,n-4}, F_{n+1}^{0,0,n-4}, F_{n+1}^{0,n-4,0}\}, \tag{33}$$

$$\widehat{\mathcal{F}} = \{D_n^0, H_n, C_n^0, C_n^1\}, \tag{34}$$

$$\widehat{\mathcal{I}} = \{X_2 \cup C_{n-1}^0, X_2 \cup X_n, X_3 \cup X_{n-1}, X_2 \cup E_n^{0,n-4}\}. \tag{35}$$

Let $\widehat{\mathcal{A}} = \widehat{\mathcal{H}} \cup \widehat{\mathcal{F}} \cup \widehat{\mathcal{I}}$. By Lemmas 3.4–3.7, we have Lemma 3.8 as follows.

Lemma 3.8. *Let $G \in \mathcal{U}_{2n}$ with $n \ge 10$. If $\widehat{G} \notin \widehat{\mathcal{A}}$, then $m(\widehat{G}, 2) \ge 2n-3$.*

For each capped graph in $\widehat{\mathcal{F}}$, $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{I}}$, we can construct its original graph. Let $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{I}$ be the sets of the original graphs of the capped graphs in $\widehat{\mathcal{F}}$, $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{I}}$ respectively. By construction, we get

$$\mathcal{H} = \{A_{2n,1}, B_{2n,i}\,(1 \le i \le 4), C_{2n,i}\,(1 \le i \le 5), D_{2n,i}\,(1 \le i \le 7), E_{2n,i}\,(1 \le i \le 7)\},$$
$$\mathcal{F} = \{F_{2n,1}, G_{2n,1}, H_{2n,1}, I_{2n,1}\},$$
$$\mathcal{I} = \{J_{2n,i}\,(1 \le i \le 4), K_{2n,i}\,(1 \le i \le 3), L_{2n,i}\,(1 \le i \le 10), M_{2n,i}\,(1 \le i \le 19)\}.$$

In $\mathcal{H}$, we have $\widehat{A}_{2n,i} = X_{n+1}$, $\widehat{B}_{2n,i} = E_{n+1}^{0,n-3}$ $(1 \le i \le 4)$, $\widehat{C}_{2n,i} = E_{n+1}^{1,n-4}$ $(1 \le i \le 5)$, $\widehat{D}_{2n,i} = F_{n+1}^{0,0,n-4}$ $(1 \le i \le 7)$, and $\widehat{E}_{2n,i} = F_{n+1}^{0,n-4,0}$ $(1 \le i \le 7)$. In $\mathcal{F}$, we get $\widehat{F}_{2n,1} = D_n^0$, $\widehat{G}_{2n,1} = C_n^0$, $\widehat{H}_{2n,1} = H_n$, and $\widehat{I}_{2n,1} = C_n^1$. In $\mathcal{I}$, we obtain $\widehat{J}_{2n,i} = X_2 \cup X_n$ $(1 \le i \le 4)$, $\widehat{K}_{2n,i} = X_2 \cup C_{n-1}^0$ $(1 \le i \le 3)$, $\widehat{L}_{2n,i} = X_3 \cup X_{n-1}$ $(1 \le i \le 10)$, and $\widehat{M}_{2n,i} = X_2 \cup E_n^{0,n-4}$ $(1 \le i \le 19)$.

Let $\mathcal{A} = \mathcal{H} \cup \mathcal{F} \cup \mathcal{I}$. It is noted that there are 64 graphs in $\mathcal{A}$. All the graphs in $\mathcal{A}$ are shown in Appendix A. For each graph in Appendix A, the dashed line with a label represents the number of P_3 attached at the vertex of the graph. There exist a balanced graph and an unbalanced one corresponding to each unicyclic graph in $\mathcal{A}$. Let $\mathcal{A}^+$ and $\mathcal{A}^-$ be the sets of all the corresponding balanced and unbalanced graphs, respectively. For the unbalanced graphs in Appendix A, we denote the positive edge by a plain line and the negative edge by a dotted line.

Lemma 3.9. *Let $G \in \mathcal{U}_{2n}$ with $n \ge 10$ and S its signed graph. Let l be the girth of G and r, j be integers with $r, j \ge 1$.*
(i) For $l = 2r+1$, if $G \notin \mathcal{A}$, then $\mathcal{E}(F_{2n,1}^-) < \mathcal{E}(G) = \mathcal{E}(G^+) = \mathcal{E}(G^-)$;
(ii) For $l = 4j+2$, if S is balanced and $S \notin \mathcal{A}^+$, then $\mathcal{E}(F_{2n,1}^-) < \mathcal{E}(S)$;
(iii) For $l = 4j$, if S is unbalanced and $S \notin \mathcal{A}^-$, then $\mathcal{E}(F_{2n,1}^-) < \mathcal{E}(S)$.

Proof. Obviously, $\widehat{F}_{2n,1} = D_n^0$. It is noted that $m(D_n^0, 2) = 2n - 6$. Since each 2-matching of D_n^0 is adjacent to four edges of $M(F_{2n,1}^-)$ and $m(D_n^0, i) = 0$ for $3 \le i \le n$, we have

$$m(F_{2n,1}^-, k) = p + (2n-6)\binom{n-4}{k-2}. \tag{36}$$

From Lemma 3.1 and the fact that $F_{2n,1}^- - C_4^-$ is composed of $n-4$ independent edges and four isolated vertices, we obtain

$$\begin{aligned} b_{2k}(F_{2n,1}^-) &= m(F_{2n,1}^-, k) + 2m(F_{2n,1}^- - C_4^-, k-2) \\ &= m(F_{2n,1}^-, k) + 2\binom{n-4}{k-2} = p + (2n-4)\binom{n-4}{k-2}. \end{aligned} \tag{37}$$

(i) $l = 2r + 1$.

By Lemma 3.1, we have

$$\begin{aligned} b_{2k}(S) = m(S, k) &= p + m(\widehat{S}, 2)\binom{n-j}{k-2} + \sum_{i=3}^{n} m(\widehat{S}, i)\binom{n-j}{k-i} \\ &\ge p + m(\widehat{S}, 2)\binom{n-4}{k-2}. \end{aligned} \tag{38}$$

If $G \notin \mathcal{A}$, then by Lemma 3.8, we have $m(\widehat{S}, 2) \ge 2n-3$ for $n \ge 10$, where S is the signed graph of G. Therefore, it follows from (37) and (38) that $b_{2k}(F_{2n,1}^-) \le b_{2k}(S)$, where the equality does not hold for $0 \le k \le [n/2]$. For example, $b_4(F_{2n,1}^-) < b_4(S)$. Furthermore, from Lemma 3.1, we have $b_{2k+1}(F_{2n,1}^-) = 0 \le b_{2k+1}(S)$ for $0 \le k \le [n/2]$. Thus, by (4) and Theorem 3.2, we get Lemma 3.9(i).

By the methods similar to those for $l = 2r + 1$, we have Lemma 3.9(ii) and (iii). □

3.4. *Comparing the energies of two signed graphs among $\mathcal{A}^+ \cup \mathcal{A}^-$*

To obtain our final results, we need to compare their energies for the signed graphs in $\mathcal{A}^+ \cup \mathcal{A}^-$. The method of coefficient comparison in (4) will be used as the first choice. For two unicyclic signed graphs in $\mathcal{A}^+ \cup \mathcal{A}^-$, if their coefficients of the corresponding characteristic polynomials are incomparable, then their energies are directly performed on the basis of (2) by using the theorem of zero points, the integral formula in (14), and analytical techniques for the integral formula. In this section, the characteristic polynomial $\phi_S(x)$ of S is rewritten by $\phi(S, x)$.

Lemma 3.10 obtained by Li and Li [15] and Lemmas 3.11 and 3.12 derived by Wang [20] are simply quoted here.

Lemma 3.10. *[15] If $n \ge 5$, then $M_{2n,2}^+ \rightharpoonup A_{2n,1}$.*

Lemma 3.11. *[20] $A_{2n,1} \rightharpoonup B_{2n,1}^+$ for $n \ge 7$ while $B_{2n,1}^+ \rightharpoonup A_{2n,1}$ for $3 \le n \le 6$.*

Lemma 3.12. *[20] $B_{2n,1}^+ \rightharpoonup G_{2n,1} \rightharpoonup B_{2n,2}$ for $n \ge 4$.*

Lemma 3.13. *$G_{2n,1} \rightharpoonup B_{2n,4}$ for $n \ge 4$.*

Proof. By (7), we get

$$\phi(G_{2n,1}, x) = (x^2-1)^{n-4}[x^8 - (n+4)x^6 - 2x^5 + 3(n+1)x^4 + 2x^3 - (n+4)x^2 + 1], \tag{39}$$

$$\phi(B_{2n,4}, x) = (x^2-1)^{n-4}[x^8 - (n+4)x^6 - 2x^5 + (3n+4)x^4 + 6x^3 - (n+4)x^2 - 2x + 1]. \tag{40}$$

By (39), (40) and (4), we have $G_{2n,1} \rightharpoonup B_{2n,4}$ for $n \ge 4$. □

Lemma 3.14. *$B_{2n,4} \rightharpoonup B_{2n,2}$ for $n \geq 4$.*

Proof. By (7), we obtain

$$\phi(B_{2n,4}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6-2x^5+(3n+4)x^4+6x^3-(n+4)x^2-2x+1],$$
$$\phi(B_{2n,2}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6-2x^5+(3n+4)x^4+4x^3-(n+5)x^2-2x+1].$$

In (14), let $S_1 = B_{2n,4}$ and $S_2 = B_{2n,2}$. Therefore, by (14), we obtain

$$\mathcal{E}(B_{2n,4}) - \mathcal{E}(B_{2n,2}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \log \frac{p_1^2(x)+q_1^2(x)}{p_2^2(x)+q_2^2(x)} \mathrm{d}x, \tag{41}$$

where $p_1(x) = x^8+(n+4)x^6+(3n+4)x^4+(n+4)x^2+1$, $q_1(x) = 2x^5+6x^3+2x$, $p_2(x) = x^8+(n+4)x^6+(3n+4)x^4+(n+5)x^2+1$, and $q_2(x) = 2x^5+4x^3+2x$. For $n \geq 4$, we get $[p_1^2(x)+q_1^2(x)]-[p_2^2(x)+q_2^2(x)] = -x^2[2x^8+2nx^6+(6n-12)x^4+(2n+1)x^2+2] \leq 0$. Therefore, it follows from (41) that $B_{2n,4} \rightharpoonup B_{2n,2}$ for $n \geq 4$. □

Lemma 3.15. *$B_{2n,2} \rightharpoonup J_{2n,3}$ for $n \geq 4$.*

Proof. By (7), we deduce

$$\phi(B_{2n,2}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6-2x^5+(3n+4)x^4+4x^3-(n+5)x^2-2x+1], \tag{42}$$
$$\phi(J_{2n,3}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6-2x^5+(3n+5)x^4+6x^3-(n+5)x^2-2x+1]. \tag{43}$$

By (42), (43) and (4), we get $B_{2n,2} \rightharpoonup J_{2n,3}$ for $n \geq 4$. □

Lemma 3.16. *$J_{2n,3} \rightharpoonup B^-_{2n,1}$ for $n \geq 4$.*

Proof. It follows from (7) that

$$\phi(J_{2n,3}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6-2x^5+(3n+5)x^4+6x^3-(n+5)x^2-2x+1], \tag{44}$$
$$\phi(B^-_{2n,1}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6+(3n+6)x^4-(n+7)x^2+1]. \tag{45}$$

In (14), let $S_1 = J_{2n,3}$ and $S_2 = B^-_{2n,1}$. Therefore, by (14), we obtain

$$\mathcal{E}(J_{2n,3}) - \mathcal{E}(B^-_{2n,1}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \log \frac{p_1^2(x)+q_1^2(x)}{p_2^2(x)} \mathrm{d}x, \tag{46}$$

where $p_1(x) = x^8+(n+4)x^6+(3n+5)x^4+(n+5)x^2+1$, $q_1(x) = 2x^5+6x^3+2x$ and $p_2(x) = x^8+(n+4)x^6+(3n+6)x^4+(n+7)x^2+1$. Since $[p_1^2(x)+q_1^2(x)]-p_2^2(x) = -x^4[2x^8+(2n+8)x^6+(10n+3)x^4+(14n-10)x^2+4n+2] \leq 0$ for $n \geq 4$, it follows from (46) that $J_{2n,3} \rightharpoonup B^-_{2n,1}$ for $n \geq 4$. □

Lemmas 3.17 and 3.18 are simply quoted, which are derived by Wang [20] and Zhu [25] respectively.

Lemma 3.17. *[20] $B_{2n,3} \rightharpoonup F^+_{2n,1}$ for $n \geq 45$ while $F^+_{2n,1} \rightharpoonup B_{2n,3}$ for $4 \leq n \leq 44$.*

Lemma 3.18. *[25] If $n \geq 4$, then $B_{2n,2} \rightharpoonup J_{2n,1}$.*

Lemma 3.19. *$B^-_{2n,1} \rightharpoonup J^+_{2n,2}$ for $n \geq 43$ while $J^+_{2n,2} \rightharpoonup B^-_{2n,1}$ for $4 \leq n \leq 42$.*

Proof. By (7), we have

$$\begin{aligned}\phi(B^-_{2n,1}, x) &= (x^2-1)^{n-4}[x^8-(n+4)x^6+(3n+6)x^4-(n+7)x^2+1] \\ &\triangleq (x^2-1)^{n-4}g_1(x),\end{aligned} \tag{47}$$
$$\begin{aligned}\phi(J^+_{2n,2}, x) &= x^2(x^2-1)^{n-3}[x^4-(n+3)x^2+2n] \\ &\triangleq x^2(x^2-1)^{n-3}g_2(x).\end{aligned} \tag{48}$$

Obviously, we get

$$
\begin{aligned}
&g_1(\sqrt{\tfrac{1}{2n}}) = \frac{1-8n+22n^2-44n^3+8n^4}{16n^4} > 0 && (n \geq 5),\\
&g_1(\sqrt{\tfrac{1}{n}}) = \frac{1-4n+5n^2-4n^3}{n^4} < 0 && (n \geq 4),\\
&g_1(\sqrt{0.38}) = -0.992237 - 0.001672n < 0 && (n \geq 4),\\
&g_1(\sqrt{0.385}) = -1.01195 + 0.00260837n > 0 && (n \geq 388),\\
&g_1(\sqrt{2.61}) = -1.11102 + 0.046719n > 0 && (n \geq 24),\\
&g_1(\sqrt{2.62}) = -0.972525 - 0.011528n < 0 && (n \geq 4),\\
&g_1(\sqrt{n}) = 1 - 7n + 5n^2 - n^3 < 0 && (n \geq 4),\\
&g_1(\sqrt{n+1}) = -3 - 2n + 2n^2 > 0 && (n \geq 4).
\end{aligned}
$$

According to the theorem of zero points, we obtain

$$
\begin{aligned}
\mathcal{E}(B^-_{2n,1}) &= 2(n-4) + \sum_{i=1}^{8}|x_i| \\
&< 2(n-4) + 2(\sqrt{\frac{1}{n}} + \sqrt{0.385} + \sqrt{2.62} + \sqrt{n+1}), \qquad (n \geq 388). \qquad (49)
\end{aligned}
$$

Since $g_2(\sqrt{1.99}) = -2.0099 + 0.01n > 0$ for $n \geq 201$, $g_2(\sqrt{2}) = -2 < 0$, $g_2(\sqrt{n+1}) = -2 < 0$, and $g_2(\sqrt{n+2}) = -2 + n > 0$ for $n \geq 3$, we have

$$
2(n-3) + 2(\sqrt{1.99} + \sqrt{n+1}) < \mathcal{E}(J^+_{2n,2}), \qquad (n \geq 201). \qquad (50)
$$

It follows from $2(n-4) + 2(\sqrt{\frac{1}{n}} + \sqrt{0.385} + \sqrt{2.62} + \sqrt{n+1}) < 2(n-3) + 2(\sqrt{1.99} + \sqrt{n+1})$ that the right-handed side of (49) is less than the left-handed side of (50) for $n \geq 388$. Therefore, $B^-_{2n,1} \rightharpoonup J^+_{2n,2}$ for $n \geq 388$. The calculation yields $B^-_{2n,1} \rightharpoonup J^+_{2n,2}$ for $43 \leq n \leq 387$ while $J^+_{2n,2} \rightharpoonup B^-_{2n,1}$ for $4 \leq n \leq 42$. □

Lemma 3.20. *$J^+_{2n,2} \rightharpoonup B_{2n,3}$ for $n \geq 3$.*

Proof. By (7), we have

$$
\phi(J^+_{2n,2}, x) = x^2(x^2-1)^{n-3}[x^4 - (n+3)x^2 + 2n], \qquad (51)
$$
$$
\phi(B_{2n,3}, x) = (x^2-1)^{n-3}[x^6 - (n+3)x^4 - 2x^3 + (2n+1)x^2 + 2x - 1]. \qquad (52)
$$

It follows from (51), (52) and (4) that $J^+_{2n,2} \rightharpoonup B_{2n,3}$ for $n \geq 3$. □

Lemma 3.21. *$B_{2n,3} \rightharpoonup J_{2n,4}$ for $n \geq 3$.*

Proof. By (7), we get

$$
\phi(B_{2n,3}, x) = (x^2-1)^{n-3}[x^6 - (n+3)x^4 - 2x^3 + (2n+1)x^2 + 2x - 1], \qquad (53)
$$
$$
\phi(J_{2n,4}, x) = (x^2-1)^{n-3}[x^6 - (n+3)x^4 - 2x^3 + (2n+2)x^2 + 4x - 1]. \qquad (54)
$$

It follows from (53), (54) and (4) that $B_{2n,3} \rightharpoonup J_{2n,4}$ for $n \geq 3$. □

Lemma 3.22. *$J_{2n,4} \rightharpoonup J^-_{2n,2}$ for $n \geq 3$.*

Proof. By (7), we obtain

$$\phi(J_{2n,4},x) = (x^2-1)^{n-3}[x^6-(n+3)x^4-2x^3+(2n+2)x^2+4x-1], \tag{55}$$

$$\phi(J^-_{2n,2},x) = (x^2-1)^{n-3}[x^6-(n+3)x^4+(2n+4)x^2-4]. \tag{56}$$

In (14), let $S_1 = J_{2n,4}$ and $S_2 = J^-_{2n,2}$. Therefore, by (14), we obtain

$$\mathcal{E}(J_{2n,4}) - \mathcal{E}(J^-_{2n,2}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \log \frac{p_1^2(x)+q_1^2(x)}{p_2^2(x)} \mathrm{d}x, \tag{57}$$

where $p_1(x) = x^6+(n+3)x^4+(2n+2)x^2+1$, $q_1(x) = 2x^3+4x$ and $p_2(x) = x^6+(n+3)x^4+(2n+4)x^2+4$. For $n \geq 3$, we have $[p_1^2(x)+q_1^2(x)] - p_2^2(x) = -4x^8-(4n+14)x^6-(14n+14)x^4-(12n+12)x^2-15 < 0$. Therefore, by (57), we get $J_{2n,4} \rightharpoonup J^-_{2n,2}$ for $n \geq 3$. □

Lemma 3.23. *$J^-_{2n,2} \rightharpoonup C^+_{2n,1}$ for $n \geq 721$ while $C^+_{2n,1} \rightharpoonup J^-_{2n,2}$ for $4 \leq n \leq 720$.*

Proof. By (7), we deduce

$$\phi(J^-_{2n,2},x) = (x^2-1)^{n-3}[x^6-(n+3)x^4+(2n+4)x^2-4], \tag{58}$$

$$\phi(C^+_{2n,1},x) = (x^2-1)^{n-4}[x^8-(n+4)x^6+(4n-2)x^4-(n+3)x^2+1]. \tag{59}$$

By (14), we obtain

$$\mathcal{E}(J^-_{2n,2}) - \mathcal{E}(C^+_{2n,1}) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \log f_1(n,x)\mathrm{d}x = \frac{2}{\pi}\int_{0}^{+\infty} \log f_1(n,x)\mathrm{d}x, \tag{60}$$

where

$$f_1(n,x) = \frac{(x^2+1)[x^6+(n+3)x^4+(2n+4)x^2+4]}{x^8+(n+4)x^6+(4n-2)x^4+(n+3)x^2+1}. \tag{61}$$

Then

$$\frac{\partial f_1(n,x)}{\partial n} = \frac{-x^2(x^2+2)(x^2+1)\left(x^6+9x^4+6x^2+1\right)}{[x^8+(n+4)x^6+(4n-2)x^4+(n+3)x^2+1]^2} \leq 0. \tag{62}$$

Thus $f_1(n,x)$ is a decreasing function of n. When $n \geq 721$, it follows from (60) and (62) that

$$\begin{aligned}\mathcal{E}(J^-_{2n,2}) - \mathcal{E}(C^+_{2n,1}) &= \frac{2}{\pi}\int_{0}^{+\infty} \log f_1(n,x)\mathrm{d}x \leq \frac{2}{\pi}\int_{0}^{+\infty} \log f_1(721,x)\mathrm{d}x \\ &\doteq \frac{2}{\pi} \times (-0.0000107) < 0.\end{aligned} \tag{63}$$

It follows from (63) that $J^-_{2n,2} \rightharpoonup C^+_{2n,1}$ for $n \geq 721$ while the calculation yields $C^+_{2n,1} \rightharpoonup J^-_{2n,2}$ for $4 \leq n \leq 720$. □

Lemma 3.24. *$C^+_{2n,1} \rightharpoonup F^+_{2n,1} \rightharpoonup D^+_{2n,2} \rightharpoonup I_{2n,1}$ for $n \geq 4$.*

Proof. By (7), we get

$$\phi(C^+_{2n,1},x) = (x^2-1)^{n-4}[x^8-(n+4)x^6+(4n-2)x^4-(n+3)x^2+1], \tag{64}$$

$$\phi(F^+_{2n,1},x) = (x^2-1)^{n-4}[x^8-(n+4)x^6+(4n-2)x^4-(n+4)x^2+1], \tag{65}$$

$$\phi(D^+_{2n,2},x) = (x^2-1)^{n-4}[x^8-(n+4)x^6+(4n-1)x^4-(n+4)x^2+1], \tag{66}$$

$$\phi(I_{2n,1},x) = (x^2-1)^{n-4}[x^8-(n+4)x^6-2x^5+(4n-1)x^4+2x^3-(n+4)x^2+1]. \tag{67}$$

By (64)–(67) and (4), we get Lemma 3.24. □

Lemma 3.25. *$I_{2n,1} \rightharpoonup F^-_{2n,1}$ for $n \geq 4$.*

Proof. By (7), we have

$$\phi(I_{2n,1}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6-2x^5+(4n-1)x^4+2x^3-(n+4)x^2+1], \tag{68}$$

$$\phi(F^-_{2n,1}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6+(4n+2)x^4-(n+4)x^2+1]. \tag{69}$$

In (14), let $S_1 = I_{2n,1}$ and $S_2 = F^-_{2n,1}$. By (14), we get

$$\mathcal{E}(I_{2n,1}) - \mathcal{E}(F^-_{2n,1}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \log \frac{p_1^2(x)+q_1^2(x)}{p_2^2(x)}\mathrm{d}x, \tag{70}$$

where $p_1(x) = x^8+(n+4)x^6+(4n-1)x^4+(n+4)x^2+1$, $q_1(x) = 2x^5+2x^3$ and $p_2(x) = x^8+(n+4)x^6+(4n+2)x^4+(n+4)x^2+1$. For $n \geq 4$, since $[p_1^2(x)+q_1^2(x)]-p_2^2(x) = -x^4[6x^8+(6n+20)x^6+(24n-5)x^4+(6n+20)x^2+6] \leq 0$, it follows from (70) that $I_{2n,1} \rightharpoonup F^-_{2n,1}$ for $n \geq 4$. □

Lemma 3.26. *$F^-_{2n,1} \rightleftharpoons L^-_{2n,1}$ for $n \geq 4$.*

Proof. By (7), we obtain $\phi(F^-_{2n,1}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6+(4n+2)x^4-(n+4)x^2+1] = \phi(L^-_{2n,1}, x)$. Obviously, we get Lemma 3.26. □

Lemma 3.27. *$F^-_{2n,1} \rightharpoonup E^+_{2n,3}$ for $n \geq 35$ while $E^+_{2n,3} \rightharpoonup F^-_{2n,1}$ for $6 \leq n \leq 34$.*

Proof. By (7), we get

$$\phi(F^-_{2n,1}, x) = (x^2-1)^{n-4}[x^8-(n+4)x^6+(4n+2)x^4-(n+4)x^2+1], \tag{71}$$

$$\begin{aligned}\phi(E^+_{2n,3}, x) = (x^2-1)^{n-6}[&x^{12}-(n+6)x^{10}+(6n+8)x^8-(11n-1)x^6 \\ &+(6n+5)x^4-(n+5)x^2+1].\end{aligned} \tag{72}$$

By (14), we obtain

$$\mathcal{E}(F^-_{2n,1}) - \mathcal{E}(E^+_{2n,3}) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \log f_2(n,x)\mathrm{d}x = \frac{2}{\pi}\int_0^{+\infty} \log f_2(n,x)\mathrm{d}x, \tag{73}$$

where

$$f_2(n,x) = \frac{(x^2+1)^2[x^8+(n+4)x^6+(4n+2)x^4+(n+4)x^2+1]}{\phi_2}$$

with $\phi_2 = x^{12}+(n+6)x^{10}+(6n+8)x^8+(11n-1)x^6+(6n+5)x^4+(n+5)x^2+1$. Then

$$\frac{\partial f_2(n,x)}{\partial n} = -\frac{x^4(4x^2+1)(x^4+3x^2+1)(x^4+4x^2+1)(x^2+1)^2}{(\phi_2)^2} \leq 0. \tag{74}$$

Thus $f_2(n,x)$ is a decreasing function of n. When $n \geq 35$, we get

$$\begin{aligned}\mathcal{E}(F^-_{2n,1}) - \mathcal{E}(E^+_{2n,3}) &= \frac{2}{\pi}\int_0^{+\infty} \log f_2(n,x)\mathrm{d}x \leq \frac{2}{\pi}\int_0^{+\infty} \log f_2(35,x)\mathrm{d}x \\ &\doteq \frac{2}{\pi} \times (-0.0011112) < 0.\end{aligned} \tag{75}$$

Therefore, by (75), we obtain $F^-_{2n,1} \rightharpoonup E^+_{2n,3}$ for $n \geq 35$ while the calculation yields $E^+_{2n,3} \rightharpoonup F^-_{2n,1}$ for $6 \leq n \leq 34$. □

Lemma 3.28. $E^+_{2n,3} \rightharpoonup E^+_{2n,2} \rightharpoonup D^+_{2n,3}$ *for* $n \geq 6$.

Proof. By (7), we have

$$\begin{aligned}\phi(E^+_{2n,3}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} + (6n+8)x^8 - (11n-1)x^6 \\ &\quad + (6n+5)x^4 - (n+5)x^2 + 1], \quad (76)\\ \phi(E^+_{2n,2}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} + (6n+8)x^8 - (11n+2)x^6 \\ &\quad + (7n+4)x^4 - (n+6)x^2 + 1], \quad (77)\\ \phi(D^+_{2n,3}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} + (6n+8)x^8 - (11n+2)x^6 \\ &\quad + 8nx^4 - (2n+2)x^2 + 1]. \quad (78)\end{aligned}$$

It follows from (76)–(78) and (4) that Lemma 3.28 holds. □

Lemma 3.29. $E^+_{2n,3} \rightharpoonup H_{2n,1}$ *for* $n \geq 6$.

Proof. By (7), we get

$$\begin{aligned}\phi(E^+_{2n,3}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} + (6n+8)x^8 - (11n-1)x^6 \\ &\quad + (6n+5)x^4 - (n+5)x^2 + 1], \quad (79)\\ \phi(H_{2n,1}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} - 2x^9 + (6n+9)x^8 + 8x^7 - (11n+3)x^6 \\ &\quad - 8x^5 + (6n+9)x^4 + 2x^3 - (n+6)x^2 + 1]. \quad (80)\end{aligned}$$

It follows from (79), (80) and (4) that $E^+_{2n,3} \rightharpoonup H_{2n,1}$ for $n \geq 6$. □

Lemma 3.30. $E^+_{2n,3} \rightharpoonup M^+_{2n,7}$ *for* $n \geq 6$.

Proof. By (7), we obtain

$$\begin{aligned}\phi(E^+_{2n,3}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} + (6n+8)x^8 - (11n-1)x^6 \\ &\quad + (6n+5)x^4 - (n+5)x^2 + 1], \quad (81)\\ \phi(M^+_{2n,7}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} + (6n+9)x^8 - (11n+3)x^6 \\ &\quad + (6n+9)x^4 - (n+6)x^2 + 1]. \quad (82)\end{aligned}$$

Thus, $E^+_{2n,3} \rightharpoonup M^+_{2n,7}$ for $n \geq 6$ follows from (81), (82) and (4). □

Lemma 3.31. $M^+_{2n,7} \rightharpoonup M^+_{2n,3} \rightharpoonup M^+_{2n,4}$ *for* $n \geq 6$.

Proof. By (7), we get

$$\begin{aligned}\phi(M^+_{2n,7}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} + (6n+9)x^8 - (11n+3)x^6 \\ &\quad + (6n+9)x^4 - (n+6)x^2 + 1], \quad (83)\\ \phi(M^+_{2n,3}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} + (6n+9)x^8 - (11n+4)x^6 \\ &\quad + (7n+5)x^4 - (n+6)x^2 + 1], \quad (84)\\ \phi(M^+_{2n,4}, x) &= (x^2-1)^{n-6}[x^{12} - (n+6)x^{10} + (6n+9)x^8 - 12nx^6 \\ &\quad + (9n-3)x^4 - (2n+2)x^2 + 1]. \quad (85)\end{aligned}$$

Thus, Lemma 3.31 directly follows from (83)–(85) and (4). □

Lemma 3.32. $F^-_{2n,1} \rightharpoonup M^+_{2n,5}$ *for* $n \geq 31$ *while* $M^+_{2n,5} \rightharpoonup F^-_{2n,1}$ *for* $5 \leq n \leq 30$.

Proof. By (7), we have

$$\begin{aligned}\phi(F^-_{2n,1}, x) &= (x^2-1)^{n-4}[x^8-(n+4)x^6+(4n+2)x^4-(n+4)x^2+1]\\ &\triangleq (x^2-1)^{n-4}g_3(x), \qquad (86)\\ \phi(M^+_{2n,5}, x) &= x^2(x^2-1)^{n-5}[x^8-(n+5)x^6+(5n+4)x^4-(7n-6)x^2+2n-2]\\ &\triangleq x^2(x^2-1)^{n-5}g_4(x). \qquad (87)\end{aligned}$$

By calculation, we have $g_3(\sqrt{\frac{1}{n}}) > 0$ for $n \geq 4$, $g_3(\sqrt{\frac{2}{n}}) < 0$ for $n \geq 8$, $g_3(\sqrt{0.26}) < 0$ for $n \geq 5$, $g_3(\sqrt{0.27}) > 0$ for $n \geq 4$, $g_3(\sqrt{3.73}) > 0$ for $n \geq 4$, $g_3(\sqrt{3.74}) < 0$ for $n \geq 5$, $g_3(\sqrt{n-1}) < 0$ for $n \geq 5$, and $g_3(\sqrt{n}) > 0$ for $n \geq 4$. According to the theorem of zero points, we have

$$\begin{aligned}\mathcal{E}(F^-_{2n,1}) &= 2(n-4) + \sum_{i=1}^{8}|x_i|\\ &< 2(n-4) + 2(\sqrt{\frac{2}{n}} + \sqrt{0.27} + \sqrt{3.74} + \sqrt{n}), \qquad (n \geq 8). \qquad (88)\end{aligned}$$

Since $g_4(\sqrt{0.38}) > 0$ for $n \geq 5$, $g_4(\sqrt{0.39}) < 0$ for $n \geq 24$, $g_4(\sqrt{1.99}) < 0$ for $n \geq 204$, $g_4(\sqrt{2}) > 0$ for $n \geq 5$, $g_4(\sqrt{2.6}) > 0$ for $n \geq 65$, $g_4(\sqrt{2.62}) < 0$ for $n \geq 5$, $g_4(\sqrt{n}) < 0$ for $n \geq 5$, and $g_4(\sqrt{n+1}) > 0$ for $n \geq 5$, we have

$$2(n-5) + 2(\sqrt{0.38} + \sqrt{1.99} + \sqrt{2.6} + \sqrt{n}) < \mathcal{E}(M^+_{2n,5}), \qquad (n \geq 204). \qquad (89)$$

It follows from $2(n-4) + 2(\sqrt{\frac{2}{n}} + \sqrt{0.27} + \sqrt{3.74} + \sqrt{n}) < 2(n-5) + 2(\sqrt{0.38} + \sqrt{1.99} + \sqrt{2.6} + \sqrt{n})$ that the right-handed side of (88) is less than the left-handed side of (89) for $n \geq 204$. Therefore, $F^-_{2n,1} \rightharpoonup M^+_{2n,5}$ for $n \geq 204$. The calculation yields $F^-_{2n,1} \rightharpoonup M^+_{2n,5}$ for $31 \leq n \leq 204$ while $M^+_{2n,5} \rightharpoonup F^-_{2n,1}$ for $5 \leq n \leq 30$. □

Lemma 3.33. *$M^+_{2n,5} \rightharpoonup M^+_{2n,6} \rightleftharpoons L^+_{2n,2}$ for $n \geq 5$.*

Proof. By (7), we obtain

$$\begin{aligned}\phi(M^+_{2n,5}, x) &= x^2(x^2-1)^{n-5}[x^8-(n+5)x^6+(5n+4)x^4-(7n-6)x^2+2n-2], \qquad (90)\\ \phi(M^+_{2n,6}, x) &= x^2(x^2-1)^{n-5}[x^8-(n+5)x^6+(5n+4)x^4-(7n-3)x^2+3n-3]\\ &= \phi(L^+_{2n,2}, x). \qquad (91)\end{aligned}$$

Therefore, Lemma 3.33 directly follows from (90), (91) and (4). □

Lemma 3.34. *$F^-_{2n,1} \rightharpoonup C_{2n,5}$ for $n \geq 4$.*

Proof. It follows from (7) that

$$\begin{aligned}\phi(C_{2n,5}, x) &= (x^2-1)^{n-4}[x^8-(n+4)x^6-2x^5+4nx^4+(2n-2)x^3-(n+4)x^2-2x+1]\\ &\triangleq (x^2-1)^{n-4}g_5(x). \qquad (92)\end{aligned}$$

By calculation, we have $g_5(-\sqrt{n+1}) > 0$ for $n \geq 4$, $g_5(-\sqrt{n}) < 0$ for $n \geq 4$, $g_5(-1.49) < 0$ for $n \geq 4$, $g_5(-1.48) > 0$ for $n \geq 382$, $g_5(-1) = 0$, $g_5(-\sqrt{\frac{2}{n}}) < 0$ for $n \geq 5$, $g_5(-\sqrt{\frac{1}{n}}) > 0$ for $n \geq 4$, $g_5(\sqrt{\frac{1}{2n}}) > 0$ for $n \geq 7$, $g_5(\sqrt{\frac{1}{n}}) < 0$ for $n \geq 4$, $g_5(0.31) < 0$ for $n \geq 4$, $g_5(0.32) > 0$ for $n \geq 32$, $g_5(2.17) > 0$ for $n \geq 7399$, $g_5(2.18) < 0$ for $n \geq 4$, $g_5(\sqrt{n}) < 0$ for $n \geq 4$, and $g_5(\sqrt{n+1}) > 0$ for $n \geq 7$, we have

$$2(n-4) + \sqrt{n} + 1.48 + 1 + \sqrt{\frac{1}{n}} + \sqrt{\frac{1}{2n}} + 0.31 + 2.17 + \sqrt{n} < \mathcal{E}(C_{2n,5}), \quad (n \geq 7399). \qquad (93)$$

It follows from $2(n-4)+2(\sqrt{\frac{2}{n}}+\sqrt{0.27}+\sqrt{3.74}+\sqrt{n}) < 2(n-4)+\sqrt{n}+1.48+1+\sqrt{\frac{1}{n}}+\sqrt{\frac{1}{2n}}+0.31+2.17+\sqrt{n}$ that the right-handed side of (88) is less than the left-handed side of (93) for $n \geq 7399$. The calculation yields $F^-_{2n,1} \rightharpoonup C_{2n,5}$ for $4 \leq n \leq 7398$. Therefore, $F^-_{2n,1} \rightharpoonup C_{2n,5}$ for $n \geq 4$. □

Lemma 3.35. *$C_{2n,5} \rightharpoonup D_{2n,6}$ for $n \geq 4$.*

Proof. By (7), we get

$$\begin{aligned}\phi(C_{2n,5}, x) &= (x^2-1)^{n-4}[x^8 - (n+4)x^6 - 2x^5 + 4nx^4 + (2n-2)x^3 \\ &\quad - (n+4)x^2 - 2x + 1], \qquad (94)\\ \phi(D_{2n,6}, x) &= (x^2-1)^{n-4}[x^8 - (n+4)x^6 - 2x^5 + (4n+1)x^4 + (2n-2)x^3 \\ &\quad - (n+5)x^2 - 2x + 1]. \qquad (95)\end{aligned}$$

Thus, $C_{2n,5} \rightharpoonup D_{2n,6}$ for $n \geq 4$ directly follows from (94), (95) and (4). □

3.5. The preceding 18 signed graphs in the increasing order by their minimal energies in $\mathcal{U}^\sigma_{2n}$ with $n \geq 721$

By Lemmas 3.10–3.25, we get the first 18 signed graphs in the increasing order according to their minimal energies among $\mathcal{U}^\sigma_{2n}$ with $n \geq 721$, which are showed in Theorem 3.36. In the proof of Theorem 3.36, we need to compare $\mathcal{E}(F^-_{2n,1})$ with the energies of some signed graphs among $\mathcal{A}^+ \cup \mathcal{A}^-$ by using the method of coefficient comparison. Due to the tedious calculations for the coefficients of their characteristic polynomials, we directly list their coefficients in (A1)–(A37) in Appendix B, which are derived by using Lemma 3.1.

Theorem 3.36. *Let $S \in \mathcal{U}^\sigma_{2n}$ and $G \neq J_{2n,1}, C_{2n,2}, D_{2n,1}, D_{2n,4}, D_{2n,7}, C_{2n,3}$, where G is the underlying graph of S. Let l be the girth of S and r, j be integers with $r, j \geq 1$. If one of the following three conditions holds: (i) $l = 2r+1$; (ii) $l = 4j+2$ and S is balanced; (iii) $l = 4j$ and S is unbalanced, then for $n \geq 721$, we have*

$$\begin{aligned}M^+_{2n,2} &\rightharpoonup A_{2n,1} \rightharpoonup B^+_{2n,1} \rightharpoonup G_{2n,1} \rightharpoonup B_{2n,4} \rightharpoonup B_{2n,2} \rightharpoonup J_{2n,3} \rightharpoonup B^-_{2n,1} \rightharpoonup J^+_{2n,2} \rightharpoonup B_{2n,3} \\ &\rightharpoonup J_{2n,4} \rightharpoonup J^-_{2n,2} \rightharpoonup C^+_{2n,1} \rightharpoonup F^+_{2n,1} \rightharpoonup D^+_{2n,2} \rightharpoonup I_{2n,1} \rightharpoonup F^-_{2n,1} \rightleftharpoons L^-_{2n,1} \rightharpoonup S. \qquad (96)\end{aligned}$$

Proof. Let $n \geq 721$. By Lemmas 3.10–3.16 and 3.19–3.25, we have the first to the sixteenth inequalities in (96). By Lemma 3.26, we have $F^-_{2n,1} \rightleftharpoons L^-_{2n,1}$ in (96). Next, we prove the last inequality $F^-_{2n,1} \rightharpoonup S$ in (96), where S does not contain the preceding terms in (96).

Let $\mathcal{B}$ be the set of $J_{2n,1}$, $C_{2n,2}$, $C_{2n,3}$, $D_{2n,1}, D_{2n,4}, D_{2n,7}$, and all the 18 graphs listed in (96) before S. Let $C = (\mathcal{A}^+ \cup \mathcal{A}^-) \setminus \mathcal{B}$. We divide C into three subsets according to the girth of the graph. Namely $C = C_1 \cup C_2 \cup C_3$, where

$$\begin{aligned}C_1 &= \{C_{2n,4}, C_{2n,5}, D_{2n,5}, D_{2n,6}, E_{2n,1}, E_{2n,i}(4 \leq i \leq 7)\} \cup \{H_{2n,1}\} \\ &\quad \cup \{K_{2n,i}(1 \leq i \leq 3), L_{2n,i}(3 \leq i \leq 10), M_{2n,i}(8 \leq i \leq 19)\},\\ C_2 &= \{C^-_{2n,1}, D^-_{2n,2}, D^+_{2n,3}, D^-_{2n,3}, E^+_{2n,2}, E^-_{2n,2}, E^+_{2n,3}, E^-_{2n,3}\} \\ &\quad \cup \{L^+_{2n,2}, L^-_{2n,2}, M^+_{2n,i}(3 \leq i \leq 7), M^-_{2n,i}(2 \leq i \leq 7)\},\\ C_3 &= \{L^+_{2n,1}, M^+_{2n,1}, M^-_{2n,1}\}.\end{aligned}$$

It is noted that all the graphs in C_1 have odd girth 3 or 5. All the graphs in C_2 and in C_3 have even girths 4 and 6, respectively. Three cases are considered according to the girth l of the signed graph.

Case (i). $l = 2r+1$.

Since $l = 2r+1$, by Theorem 3.2, for a fixed graph $G \in \mathcal{A}$, we have $\mathcal{E}(G^-) = \mathcal{E}(G^+) = \mathcal{E}(G)$. Therefore, we only need to prove $F^-_{2n,1} \rightharpoonup G$ in (96), where G has girth $2r+1$ and G does not contain the preceding terms in (96).

If $G \notin \mathcal{A}$, then by Lemma 3.9(i), we have $F^-_{2n,1} \rightharpoonup G$. If $G \in \mathcal{A} \setminus \mathcal{B}$, then $G \in C_1$ since G has girth $l = 2r+1$. Next, we prove $F^-_{2n,1} \rightharpoonup G$ for $G \in C_1$.

By Lemmas 3.34 and 3.35, we have $F^-_{2n,1} \rightharpoonup C_{2n,5} \rightharpoonup D_{2n,6}$. By Lemmas 3.27 and 3.29, we have $F^-_{2n,1} \rightharpoonup E^+_{2n,3} \rightharpoonup H_{2n,1}$. Furthermore, by using the method of coefficient comparison and comparing (A2) with (A9), (A10) and (A11), we obtain $F^-_{2n,1} \rightharpoonup E^+_{2n,3} \rightharpoonup G$, where $G \in \{E_{2n,1}, E_{2n,4}, E_{2n,5}\}$. By comparing (A3) with (A4), (A5), (A12), (A13), (A6), (A7), and (A8), we get $F^-_{2n,1} \rightharpoonup H_{2n,1} \rightharpoonup G$, where $G \in \{C_{2n,4}, D_{2n,5}, E_{2n,6}, E_{2n,7}, K_{2n,i}\ (1 \leq i \leq 3)\}$. By comparing (A1) with (A14)–(A33), we obtain $F^-_{2n,1} \rightharpoonup G$, where $G \in \{L_{2n,i}\ (3 \leq i \leq 10), M_{2n,i}\ (8 \leq i \leq 19)\}$.

Case (ii). $l = 4j$ and S is unbalanced.

If $S \notin \mathcal{A}^-$, then by Lemma 3.9(iii), we have $E(F^-_{2n,1}) < E(S)$. If $S \in \mathcal{A}^- \setminus \mathcal{B}$, since S has girth $l = 4j$ and S is unbalanced, then we only need to prove $F^-_{2n,1} \rightharpoonup S$, where $S \in \{C^-_{2n,1}, D^-_{2n,2}, D^-_{2n,3}, E^-_{2n,2}, E^-_{2n,3}\} \cup \{L^-_{2n,2}, M^-_{2n,i}\ (2 \leq i \leq 7)\}$.

By comparing (A1) with (A36), (A37) and (A35), we get $F^-_{2n,1} \rightharpoonup C^-_{2n,1}, D^-_{2n,2}, M^-_{2n,2}$ for $n \geq 6$. We have the following inequalities: $F^-_{2n,1} \rightharpoonup D^+_{2n,3} \rightharpoonup D^-_{2n,3}$ (by Lemmas 3.27, 3.28 and Theorem 3.2(ii)), $F^-_{2n,1} \rightharpoonup E^-_{2n,2}$ (by Lemmas 3.27, 3.28 and Theorem 3.2(ii)), $F^-_{2n,1} \rightharpoonup E^-_{2n,3}$ (by Lemma 3.27 and Theorem 3.2(ii)), $F^-_{2n,1} \rightharpoonup L^-_{2n,2}$ (by Lemmas 3.32, 3.33 and Theorem 3.2(ii)), $F^-_{2n,1} \rightharpoonup M^-_{2n,3}$ and $F^-_{2n,1} \rightharpoonup M^-_{2n,4}$ (by Lemmas 3.27, 3.30, 3.31, and Theorem 3.2(ii)), $F^-_{2n,1} \rightharpoonup M^-_{2n,5}$ (by Lemma 3.32 and Theorem 3.2(ii)), $F^-_{2n,1} \rightharpoonup M^-_{2n,6}$ (by Lemmas 3.32, 3.33, and Theorem 3.2(ii)), and $F^-_{2n,1} \rightharpoonup M^-_{2n,7}$ (by Lemmas 3.27, 3.30 and Theorem 3.2(ii)).

Case (iii). $l = 4j + 2$ and S is balanced.

If $S \notin \mathcal{A}^+$, then by Lemma 3.9(ii), we have $F^-_{2n,1} \rightharpoonup S$. If $S \in \mathcal{A}^+ \setminus \mathcal{B}$, since S has girth $l = 4j + 2$ and S is balanced, then $S \in \{L^+_{2n,1}, M^+_{2n,1}\}$. By Lemma 3.26 and Theorem 3.2(iii), we get $F^-_{2n,1} \rightleftharpoons L^-_{2n,1} \rightharpoonup L^+_{2n,1}$. By (A1) and (A34) and by Theorem 3.2(iii), we obtain $F^-_{2n,1} \rightharpoonup M^-_{2n,1} \rightharpoonup M^+_{2n,1}$.

By the proofs of Cases (i)–(iii), we get Theorem 3.36. □

It should be pointed out that when $6 \leq n \leq 721$, for the graphs among $\mathcal{U}^\sigma_{2n}$, the increasing order according to their minimal energies are complicated and irregular. Bearing $\mathcal{E}(F^+_{2n,1}) = \mathcal{E}(F_{2n,1})$ in mind, we can, by Theorems 3.2 and 3.36, directly obtain Theorem 3.37 as follows. Namely, among $\mathcal{U}_{2n}$ with $n \geq 721$, the first 12 graphs in the increasing order according to their minimal energies are obtained. In 2011, Wang [20] obtained the first 7 graphs, which are shown in Theorem 1 in [20]. Therefore, Theorem 3.37 extends the results of Theorem 1 in [20] and updates the proof of Theorem 1 in [20].

Theorem 3.37. *Let $G \in \mathcal{U}_{2n}$ with $n \geq 721$ and $G \neq J_{2n,1}, C_{2n,2}, D_{2n,1}, D_{2n,4}, D_{2n,7}, C_{2n,3}$. Let l be the girth of G and r, j be integers with $r, j \geq 1$. If $l = 2r + 1$ or $l = 4j + 2$, then we have*

$$\begin{aligned} M_{2n,2} &\rightharpoonup A_{2n,1} \rightharpoonup B_{2n,1} \rightharpoonup G_{2n,1} \rightharpoonup B_{2n,4} \rightharpoonup B_{2n,2} \rightharpoonup J_{2n,3} \rightharpoonup J_{2n,2} \rightharpoonup B_{2n,3} \\ &\rightharpoonup J_{2n,4} \rightharpoonup C_{2n,1} \rightharpoonup F_{2n,1} \rightharpoonup G. \end{aligned} \tag{97}$$

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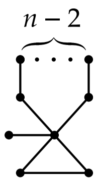
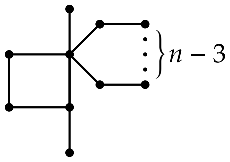
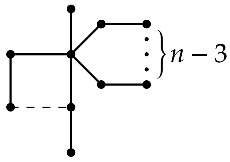
Appendix A : Figures for signed graphs among $\mathscr{A}^+ \cup \mathscr{A}^-$



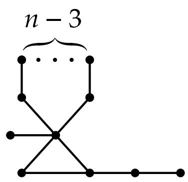
$A_{2n,1}$



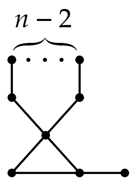
$B^+_{2n,1}$



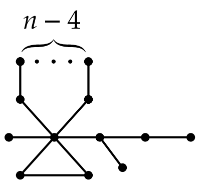
$B^-_{2n,1}$



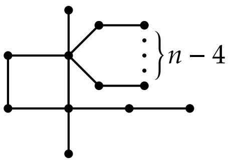
$B_{2n,2}$



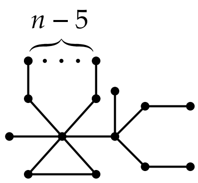
$B_{2n,3}$



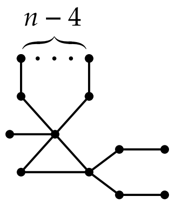
$B_{2n,4}$



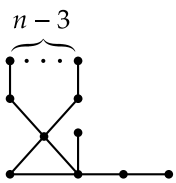
$C^+_{2n,1}$



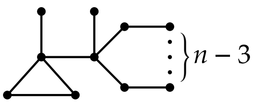
$C_{2n,2}$



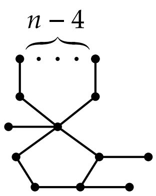
$C_{2n,3}$



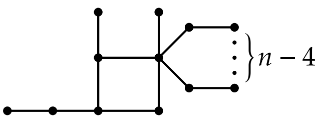
$C_{2n,4}$



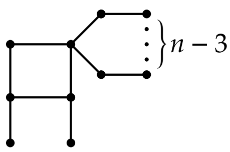
$C_{2n,5}$



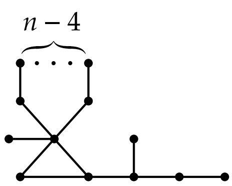
$D_{2n,1}$



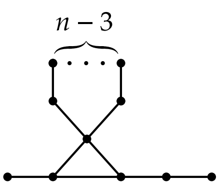
$D^+_{2n,2}$



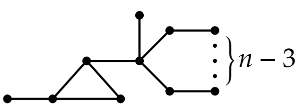
$D^+_{2n,3}$



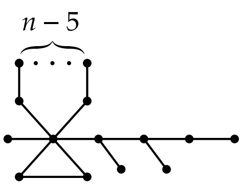
$D_{2n,4}$



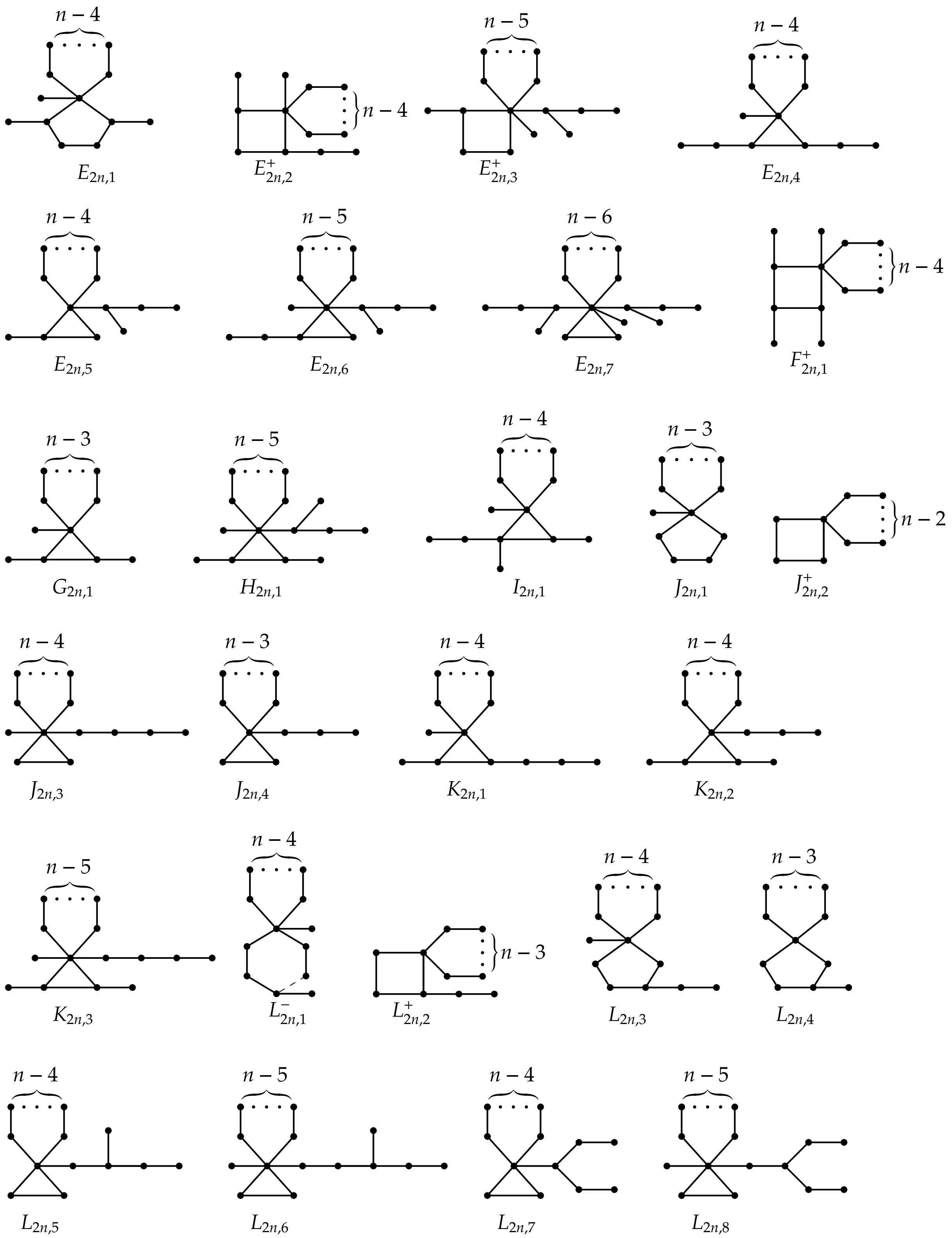
$D_{2n,5}$



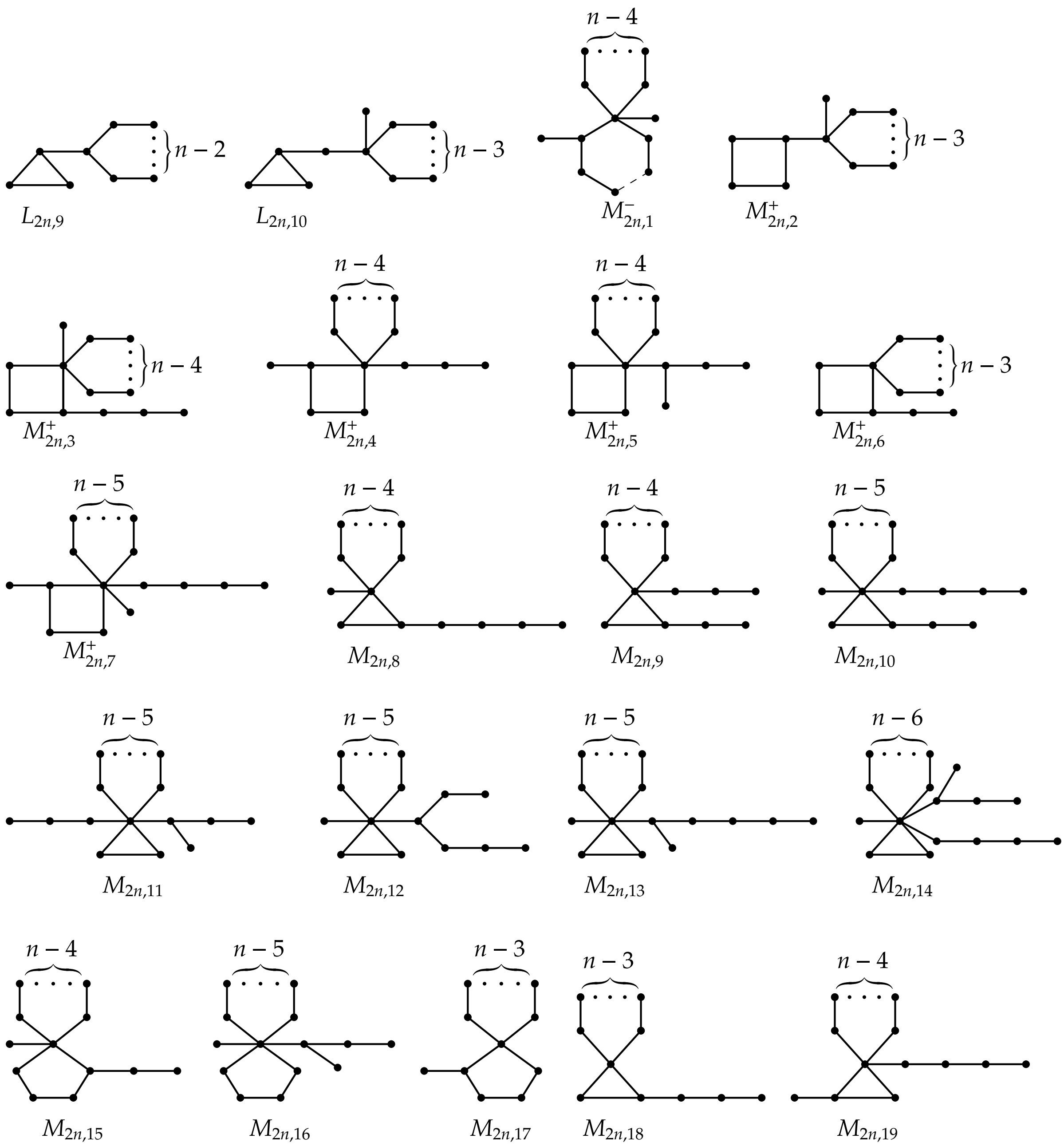
$D_{2n,6}$



$D_{2n,7}$



$n-4$
$n-4$
$n-5$
$n-4$
$E_{2n,1}$
$E^+_{2n,2}$
$E^+_{2n,3}$
$E_{2n,4}$
$n-4$
$n-5$
$n-6$
$n-4$
$E_{2n,5}$
$E_{2n,6}$
$E_{2n,7}$
$F^+_{2n,1}$
$n-3$
$n-5$
$n-4$
$n-3$
$n-2$
$G_{2n,1}$
$H_{2n,1}$
$I_{2n,1}$
$J_{2n,1}$
$J^+_{2n,2}$
$n-4$
$n-3$
$n-4$
$n-4$
$J_{2n,3}$
$J_{2n,4}$
$K_{2n,1}$
$K_{2n,2}$
$n-5$
$n-4$
$n-3$
$n-4$
$n-3$
$K_{2n,3}$
$L^-_{2n,1}$
$L^+_{2n,2}$
$L_{2n,3}$
$L_{2n,4}$
$n-4$
$n-5$
$n-4$
$n-5$
$L_{2n,5}$
$L_{2n,6}$
$L_{2n,7}$
$L_{2n,8}$



Appendix B: Expressions for coefficients of characteristic polynomials of some signed graphs among $\mathcal{A}^{+} \cup \mathcal{A}^{-}$

$$\begin{cases} b_{2i}(F^{-}_{2n,1}) = p + (2n-4)\binom{n-4}{i-2}, \\ b_{2i+1}(F^{-}_{2n,1}) = 0. \end{cases} \tag{A1}$$

$$\begin{cases} b_{2i}(E^{+}_{2n,3}) = p + (2n-7)\binom{n-4}{i-2} + (n-5)\binom{n-6}{i-3} - \binom{n-4}{i-3} - \binom{n-5}{i-3}, \\ b_{2i+1}(E^{+}_{2n,3}) = 0. \end{cases} \tag{A2}$$

$$\begin{cases} b_{2i}(H_{2n,1}) = p + (2n-6)\binom{n-4}{i-2} + (n-5)\binom{n-6}{i-3}, \\ b_{2i+1}(H_{2n,1}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-3}{i-1} + 2\binom{n-5}{i-2}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A3}$$

$$\begin{cases} b_{2i}(C_{2n,4}) = p + (2n-6)\binom{n-4}{i-2} + (n-3)\binom{n-4}{i-3}, \\ b_{2i+1}(C_{2n,4}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-3}{i-1} + 2\binom{n-3}{i-2}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A4}$$

$$\begin{cases} b_{2i}(D_{2n,5}) = p + (2n-5)\binom{n-4}{i-2} + (2n-5)\binom{n-4}{i-3}, \\ b_{2i+1}(D_{2n,5}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-3}{i-1} + 2\binom{n-3}{i-2}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A5}$$

$$\begin{cases} b_{2i}(K_{2n,1}) = p + (2n-5)\binom{n-4}{i-2} + 2\binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}, \\ b_{2i+1}(K_{2n,1}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-3}{i-1} + 2\binom{n-4}{i-2}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A6}$$

$$\begin{cases} b_{2i}(K_{2n,2}) = p + (2n-5)\binom{n-4}{i-2} + (n-2)\binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}, \\ b_{2i+1}(K_{2n,2}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-3}{i-1} + 2\binom{n-4}{i-2}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A7}$$

$$\begin{cases} b_{2i}(K_{2n,3}) = p + (2n-5)\binom{n-4}{i-2} + \binom{n-4}{i-3} + \binom{n-5}{i-3} + (n-5)\binom{n-6}{i-3}, \\ b_{2i+1}(K_{2n,3}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-3}{i-1} + 2\binom{n-5}{i-2}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A8}$$

$$\begin{cases} b_{2i}(E_{2n,1}) = p + (2n-5)\binom{n-4}{i-2} + \binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}, \\ b_{2i+1}(E_{2n,1}) = \begin{cases} 0, & 2i+1 < 5 \\ 2\binom{n-4}{i-2}, & 2i+1 \geq 5. \end{cases} \end{cases} \tag{A9}$$

$$\begin{cases} b_{2i}(E_{2n,4}) = p + (2n-5)\binom{n-4}{i-2} + 3\binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}, \\ b_{2i+1}(E_{2n,4}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-2}{i-1}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A10}$$

$$\begin{cases} b_{2i}(E_{2n,5}) = p + (2n-5)\binom{n-4}{i-2} + (n-3)\binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}, \\ b_{2i+1}(E_{2n,5}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-2}{i-1} + 2\binom{n-4}{i-2}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A11}$$

$$\begin{cases} b_{2i}(E_{2n,6}) = p + (2n-5)\binom{n-4}{i-2} + \binom{n-4}{i-3} + \binom{n-5}{i-3} + (n-5)\binom{n-6}{i-3}, \\ b_{2i+1}(E_{2n,6}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-2}{i-1} + 2\binom{n-4}{i-2}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A12}$$

$$\begin{cases} b_{2i}(E_{2n,7}) = p + (2n-5)\binom{n-4}{i-2} + (n-4)\binom{n-6}{i-3}, \\ b_{2i+1}(E_{2n,7}) = \begin{cases} 0, & 2i+1 < 3 \\ 2\binom{n-2}{i-1} + 4\binom{n-4}{i-2} + 2\binom{n-6}{i-3}, & 2i+1 \geq 3. \end{cases} \end{cases} \tag{A13}$$

$$b_{2i}(L_{2n,3}) = p + (2n-4)\binom{n-4}{i-2} + 3\binom{n-4}{i-3}. \tag{A14}$$

$$b_{2i}(L_{2n,4}) = p + (2n-4)\binom{n-4}{i-2} + (n-1)\binom{n-4}{i-3}. \tag{A15}$$

$$b_{2i}(L_{2n,5}) = p + (2n-4)\binom{n-4}{i-2} + (n-2)\binom{n-4}{i-3}. \tag{A16}$$

$$b_{2i}(L_{2n,6}) = p + (2n-4)\binom{n-4}{i-2} + \binom{n-4}{i-3}. \tag{A17}$$

$$b_{2i}(L_{2n,7}) = p + (2n-4)\binom{n-4}{i-2} + (2n-4)\binom{n-4}{i-3}. \tag{A18}$$

$$b_{2i}(L_{2n,8}) = p + (2n-4)\binom{n-4}{i-2} + 2\binom{n-4}{i-3}. \tag{A19}$$

$$b_{2i}(L_{2n,9}) = p + (2n-4)\binom{n-4}{i-2} + (2n-4)\binom{n-4}{i-3}. \tag{A20}$$

$$b_{2i}(L_{2n,10}) = p + (2n-4)\binom{n-4}{i-2} + 2\binom{n-4}{i-3}. \tag{A21}$$

$$b_{2i}(M_{2n,8}) = p + (2n-4)\binom{n-4}{i-2} + 3\binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}. \tag{A22}$$

$$b_{2i}(M_{2n,9}) = p + (2n-4)\binom{n-4}{i-2} + n\binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}. \tag{A23}$$

$$b_{2i}(M_{2n,10}) = p + (2n-4)\binom{n-4}{i-2} + 2\binom{n-4}{i-3} + 2\binom{n-5}{i-3} + (n-5)\binom{n-6}{i-3}. \tag{A24}$$

$$b_{2i}(M_{2n,11}) = p + (2n-4)\binom{n-4}{i-2} + (n-2)\binom{n-4}{i-3} + (n-3)\binom{n-5}{i-3}. \tag{A25}$$

$$b_{2i}(M_{2n,12}) = p + (2n-4)\binom{n-4}{i-2} + 2\binom{n-4}{i-3} + (n-3)\binom{n-5}{i-3}. \tag{A26}$$

$$b_{2i}(M_{2n,13}) = p + (2n-4)\binom{n-4}{i-2} + \binom{n-4}{i-3} + (n-3)\binom{n-5}{i-3}. \tag{A27}$$

$$b_{2i}(M_{2n,14}) = p + (2n-4)\binom{n-4}{i-2} + \binom{n-4}{i-3} + \binom{n-5}{i-3} + (n-4)\binom{n-6}{i-3}. \tag{A28}$$

$$b_{2i}(M_{2n,15}) = p + (2n-4)\binom{n-4}{i-2} + 4\binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}. \tag{A29}$$

$$b_{2i}(M_{2n,16}) = p + (2n-4)\binom{n-4}{i-2} + 2\binom{n-4}{i-3} + 2\binom{n-5}{i-3} + (n-5)\binom{n-6}{i-3}. \tag{A30}$$

$$b_{2i}(M_{2n,17}) = p + (2n-4)\binom{n-4}{i-2} + (2n-4)\binom{n-4}{i-3}. \tag{A31}$$

$$b_{2i}(M_{2n,18}) = p + (2n-4)\binom{n-4}{i-2} + (2n-4)\binom{n-4}{i-3}. \tag{A32}$$

$$b_{2i}(M_{2n,19}) = p + (2n-4)\binom{n-4}{i-2} + (n-1)\binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}. \tag{A33}$$

$$b_{2i}(M^{-}_{2n,1}) = p + (2n-4)\binom{n-4}{i-2} + \binom{n-4}{i-3} + (n-4)\binom{n-5}{i-3}. \tag{A34}$$

$$b_{2i}(M^{-}_{2n,2}) = p + (2n-2)\binom{n-4}{i-2} + 3\binom{n-3}{i-3} + (n+3)\binom{n-4}{i-3}. \tag{A35}$$

$$b_{2i}(C^{-}_{2n,1}) = p + (2n-4)\binom{n-4}{i-2} + 3\binom{n-4}{i-3}. \tag{A36}$$

$$b_{2i}(D^{-}_{2n,2}) = p + (2n-3)\binom{n-4}{i-2} + 4\binom{n-4}{i-3}. \tag{A37}$$