Filomat 34:11 (2020), 3767–3776
https://doi.org/10.2298/FIL2011767E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Gray's Decomposition on Doubly Warped Product Manifolds and Applications

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Abstract. A. Gray presented an interesting $O(n)$ invariant decomposition of the covariant derivative of the Ricci tensor. Manifolds whose Ricci tensor satisfies the defining property of each orthogonal class are called Einstein-like manifolds. In the present paper, we answered the following question: Under what condition(s), does a factor manifold $M_i, i = 1, 2$ of a doubly warped product manifold $M =_{f_2} M_1 \times_{f_1} M_2$ lie in the same Einstein-like class of M? By imposing sufficient and necessary conditions on the warping functions, an inheritance property of each class is proved. As an application, Einstein-like doubly warped product space-times of type $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{P}$ are considered.

1. An introduction

Alfred Gray in [22] presented $O(n)$ invariant orthogonal irreducible decomposition of the space W of all $(0, 3)$ tensors satisfying only the identities of the gradient of the Ricci tensor $\nabla_k R_{ij}$. The space W is decomposed into three orthogonal irreducible subspaces, that is, $W = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{I}$. This decomposition produces seven classes of Einstein-like manifolds, that is, manifolds whose Ricci tensor satisfies the defining identity of each subspace. They are the trivial class $\mathcal{P}$, the classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I}$ and three composite classes $\mathcal{I} \oplus \mathcal{A}$, $\mathcal{I} \oplus \mathcal{B}$ and $\mathcal{A} \oplus \mathcal{B}$.

In class $\mathcal{P}$, the Ricci tensor is parallel i.e. $\nabla_k R_{ij} = 0$ whereas class $\mathcal{A}$ contains manifolds whose Ricci tensor is Killing. The Ricci tensor of manifolds in class $\mathcal{B}$ is a Codazzi tensor i.e. $\nabla_k R_{ij} = \nabla_i R_{kj}$. The traceless part of the Ricci tensor vanishes in class $\mathcal{I}$ i.e. class $\mathcal{I}$ contains Sinyukov manifolds[26]. The tensor

$$\mathcal{L}_{ij} = R_{ij} - \frac{2R}{n+2} g_{ij}$$

is Killing in class $\mathcal{I} \oplus \mathcal{A}$ whereas the tensor

$$\mathcal{H}_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}$$

2010 *Mathematics Subject Classification*. Primary 53C21, 53C25; Secondary 53C50, 53C80
Keywords. Codazzi Ricci tensor, doubly warped manifolds, Killing Ricci tensor, doubly warped space-times, Einstein-like manifolds
Received: 30 November 2019; Revised: 17 January 2020; Accepted: 09 February 2020
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is a Codazzi tensor in class $\mathcal{I} \oplus \mathcal{B}$. The class $\mathcal{A} \oplus \mathcal{B}$ is identified by having constant scalar curvature. The same decomposition is discussed extensively in [3, Chapter 16] (see also [24, 26] and Section 3 for more details and equivalent conditions). Thereafter, Einstein-like manifolds have been studied by many authors such as G. Calvaruso in [7–10] Mantica et al in [24–26] and many others [2, 5, 6, 30, 36]. An interesting study in [26] shows Einstein-like generalized Robertson-Walker space-times are perfect fluid space-times except those in class $\mathcal{I}$ which are not restricted. Sufficient conditions on generalized Robertson-Walker space-times in this class to be a perfect fluid are derived in [13].

Doubly warped products is a generalization of singly warped products introduced in [4]. The geometric properties of doubly warped product manifolds have been investigated by many authors such as pseudo-convexity in [1], harmonic Weyl conformal curvature tensor in [18], conformal flatness in [20, 21], geodesic completeness in [35], doubly warped product submanifolds in [17, 28, 29, 31] and conformal vector fields in [15]. Doubly warped space-times are widely used as exact solutions of Einstein's field equations. Recently, the existence of compact Einstein doubly warped product manifolds is considered in [23].

Inspired by the above studies of Einstein-like metrics and doubly warped product manifolds, we studied doubly warped product manifolds equipped with Einstein-like metrics. The inheritance properties of the Einstein-like class type $\mathcal{P}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I} \oplus \mathcal{A}$, $\mathcal{I} \oplus \mathcal{B}$ or $\mathcal{A} \oplus \mathcal{B}$ are investigated. To assure that factor manifolds of a doubly warped product manifold inherits the Einstein-like class type, sufficient and necessary conditions are derived on the warping functions. Finally, we apply the results to doubly warped space-times.

2. Preliminaries

A doubly warped product manifold is the (pseudo-)Riemannian product manifold $M = M_1 \times M_2$ of two (pseudo-)Riemannian manifolds (M_i, g_i, D_i), $i = 1, 2$, furnished with the metric tensor

$$g = (f_2 \circ \pi_2)^2 \pi_1^* (g_1) \oplus (f_1 \circ \pi_1)^2 \pi_2^* (g_2),$$

where the functions $f_i : M_i \to (0, \infty)$, $i = 1, 2$ are the warping functions of M. M is denoted by ${}_{f_2}M_1 \times_{f_1} M_2$. The maps $\pi_i : M_1 \times M_2 \to M_i$ are the natural projections M onto M_i whereas * denotes the pull-back operator on tensors. In particular, if for example $f_2 = 1$, then $M = M_1 \times_{f_1} M_2$ is called a singly warped product manifold (see [15, 35] for doubly warped products and [4, 14, 16, 27, 33, 34] for singly warped products).

Notation 2.1. *Throughout this work, we use the following notations*

1. *All tensor fields on M_i are identified with their lifts to M. For example, we use f_i for a function on M_i and for its lift $(f_i \circ \pi_i)$ on M.*
2. *The manifolds M_i has dimensions n_i where $n = n_1 + n_2$.*
3. Ric *is the Ricci curvature tensor on M and* Ric^i *is the Ricci tensor on M^i.*
4. *The gradient of f_i on M_i is denoted by $\nabla^i f_i$ and the Laplacian by $\triangle^i f_i$ whereas $f_i^\diamond = f_i \triangle^i f_i + \left(n_j - 1\right) g_i \left(\nabla^i f_i, \nabla^i f_i\right)$, $i \neq j$.*
5. *The indices i and j to denote the geometric objects of the factor manifolds M_i and M_j.*
6. *The $(0, 2)$ tensors $\mathcal{F}^i$ is defined as*

$$\mathcal{F}^i (X_i, Y_i) = \frac{n_j}{f_i} H^{f_i} (X_i, Y_i),$$

for $X_i, Y_i \in \mathfrak{X}(M_i)$ and $i, j = 1, 2, i \neq j$.

The Levi-Civita connection D on $M = {}_{f_2} M_1 \times_{f_1} M_2$ is given by

$$\begin{aligned} D_{X_i} X_j &= X_i (\ln f_i) X_j + X_j \left(\ln f_j\right) X_i, \\ D_{X_i} Y_i &= D^i_{X_i} Y_i - \frac{f_j^2}{f_i^2} g_i (X_i, Y_i) \nabla^j (\ln f_j), \end{aligned}$$

where $i \neq j$ and $X_i, Y_i \in \mathfrak{X}(M_i)$. Then the Ricci curvature tensor Ric on M is given by

$$\begin{aligned}\mathrm{Ric}(X_i, Y_i) &= \mathrm{Ric}^i(X_i, Y_i) - \frac{n_j}{f_i} H^{f_i}(X_i, Y_i) - \frac{f_j^\diamond}{f_i^2} g_i(X_i, Y_i),\\ \mathrm{Ric}\left(X_i, Y_j\right) &= (n-2) X_i(\ln f_i) Y_j\left(\ln f_j\right),\end{aligned}$$

where $i \neq j$ and $X_i, Y_i \in \mathfrak{X}(M_i)$. The reader is referred to [11, 12, 19] for some studies of curvature conditions on warped product manifolds.

3. Einstein-like doubly warped product manifolds

The Einstein-like doubly warped product manifolds $M =_{f_2} M_1 \times_{f_1} M_2$ are investigated in this section. Every subsection is devoted to the study of a class of Einstein-like doubly warped product manifolds. Sufficient and necessary conditions are derived on the warping functions f_i for factor manifolds M_i to acquire the same Einstein-like class type.

3.1. Class $\mathcal{A}$

A doubly warped product manifold (M, g) whose Ricci tensor is Killing, that is,

$$(D_X \mathrm{Ric})(Y, Z) + (D_Y \mathrm{Ric})(Z, X) + (D_Z \mathrm{Ric})(X, Y) = 0,$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$ is called Einstein-like doubly warped product manifold of class $\mathcal{A}$. This condition equivalent to

$$(D_X \mathrm{Ric})(X, X) = 0,$$

for any vector field $X \in \mathfrak{X}(M)$ and the Ricci tensor is also called cyclic parallel. The legacy of factor manifolds of M in class $\mathcal{A}$ is as follows.

Theorem 3.1. *In a doubly warped product manifold $M =_{f_2} M_1 \times_{f_1} M_2$ where M is of class type $\mathcal{A}$, a factor manifold (M_i, g_i) is an Einstein-like manifold of class $\mathcal{A}$ if and only if*

$$\left(D^i_{X_i} \mathcal{F}^i\right)(X_i, X_i) = \frac{2}{f_i^3} X_i(f_i) g_i(X_i, X_i)\left[f_j^\diamond + (n-2)\left(\nabla^j f_j\right)\left(f_j\right)\right],$$

where $i, j = 1, 2, i \neq j$ and $X_i \in \mathfrak{X}(M_i)$.

Proof. In a doubly warped product manifold $M =_{f_2} M_1 \times_{f_1} M_2$ of class $\mathcal{A}$, it is

$$\begin{aligned}0 &= (D_X \mathrm{Ric})(X, X)\\ &= X(\mathrm{Ric}(X, X)) - 2\mathrm{Ric}(D_X X, X).\end{aligned}$$

Thus, for a the special case where $X = X_i$ lands on one factor, one may get

$$\begin{aligned}0 &= (D_{X_i} \mathrm{Ric})(X_i, X_i)\\ &= X_i\left(\mathrm{Ric}^i(X_i, X_i) - \mathcal{F}^i(X_i, X_i) - \frac{f_j^\diamond}{f_i^2} g_i(X_i, X_i)\right)\\ &\quad -2\mathrm{Ric}^i\left(D^i_{X_i} X_i, X_i\right) + 2\mathcal{F}^i\left(D^i_{X_i} X_i, X_i\right) + 2\frac{f_j^\diamond}{f_i^2} g_i\left(D^i_{X_i} X_i, X_i\right)\\ &\quad +2(n-2)\frac{1}{f_i^3}\left(\nabla^j f_j\right)\left(f_j\right) X_i(f_i) g_i(X_i, X_i).\end{aligned}$$

Thus, after lengthy computations, it is

$$\begin{aligned} 0 &= \left(D^i_{X_i}\mathrm{Ric}^i\right)(X_i, X_i) - \left(D^i_{X_i}\mathcal{F}^i\right)(X_i, X_i) \\ &\quad + \frac{2}{f_i^3} X_i(f_i)\, g_i(X_i, X_i)\left[f_j^\diamond + (n-2)\left(\nabla^j f_j\right)\left(f_j\right)\right]. \end{aligned}$$

These equations complete the proof. □

It is now easy to recover a similar result on singly warped product manifolds.

Corollary 3.2. *In a singly warped product manifold $M = M_1 \times_{f_1} M_2$ where M is of class type $\mathcal{A}$, (M_1, g_1) is an Einstein-like manifold of class $\mathcal{A}$ if and only if $\mathcal{F}^i$ is Killing. In addition, (M_2, g_2) is of class type $\mathcal{A}$.*

3.2. Class $\mathcal{B}$

Let M be as Einstein-like doubly warped product manifold of class $\mathcal{B}$. Then, the Ricci tensor is a Codazzi tensor,that is,

$$(D_X\mathrm{Ric})(Y, Z) = (D_Y\mathrm{Ric})(X, Z).$$

The above condition is equivalent to:

1. M has a harmonic Riemann tensor, that is, $\nabla_\varepsilon \mathcal{R}_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} = 0$, or
2. M admits a harmonic Weyl conformal tensor and the scalar curvature is constant, that is, $\nabla_\varepsilon C_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} = 0$ and $\nabla_\varepsilon R = 0$.

The base manifold and the fiber manifold gain the Einstein-like class type $\mathcal{B}$ according to.

Theorem 3.3. *In a doubly warped product manifold $M =_{f_2} M_1 \times_{f_1} M_2$ where M is of class type $\mathcal{B}$, the factor manifold (M_i, g_i) is an Einstein-like manifold of class $\mathcal{B}$ if and only if*

$$\begin{aligned} \left(D^i_{X_i}\mathcal{F}^i\right)(Y_i, Z_i) &= \left(D^i_{Y_i}\mathcal{F}^i\right)(X_i, Z_i) \\ &\quad + \frac{1}{f_i^3} X_i(f_i)\, g_i(Y_i, Z_i)\left(2f_j^\diamond - (n-2)\left(\nabla^j f_j\right) f_j\right) \\ &\quad - \frac{1}{f_i^3} Y_i(f_i)\, g_i(X_i, Z_i)\left(2f_j^\diamond - (n-2)\left(\nabla^j f_j\right) f_j\right), \end{aligned}$$

where $i, j = 1, 2, i \neq j$ and $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$.

Proof. Let us define the deviation tensor $B(X, Y, Z)$ as follows

$$B(X, Y)Z = (D_X\mathrm{Ric})(Y, Z) - (D_Y\mathrm{Ric})(X, Z).$$

There are three different cases. Let us consider the first case, that is,

$$B(X_i, Y_i, Z_i) = (D_{X_i}\mathrm{Ric})(Y_i, Z_i) - (D_{Y_i}\mathrm{Ric})(X_i, Z_i). \tag{1}$$

It is enough to find $(D_{X_i}\text{Ric})(Y_i, Z_i)$ as

$$\begin{aligned}(D_{X_i}\text{Ric})(Y_i, Z_i) &= X_i\left(\text{Ric}^i(Y_i, Z_i)\right) - X_i\left(\mathcal{F}^i(Y_i, Z_i)\right) - f_j^\diamond X_i\left(\frac{1}{f_i^2} g_i(Y_i, Z_i)\right) \\ &-\text{Ric}^i\left(D^i_{X_i}Y_i, Z_i\right) + \mathcal{F}^i\left(D^i_{X_i}Y_i, Z_i\right) + \frac{f_j^\diamond}{f_i^2} g_i\left(D^i_{X_i}Y_i, Z_i\right) \\ &-\text{Ric}^i\left(Y_i, D^i_{X_i}Z_i\right) + \mathcal{F}^i\left(Y_i, D^i_{X_i}Z_i\right) + \frac{f_j^\diamond}{f_i^2} g_i\left(Y_i, D^i_{X_i}Z_i\right) \\ &+ (n-2)\frac{1}{f_i^3} g_i(X_i, Y_i)\nabla^j f_j\left(f_j\right) Z_i(f_i) \\ &+ (n-2)\frac{1}{f_i^3} g_i(X_i, Z_i)\nabla^j f_j\left(f_j\right) Y_i(f_i).\end{aligned}$$

Simplifying this expression, it is

$$\begin{aligned}(D_{X_i}\text{Ric})(Y_i, Z_i) &= \left(D^i_{X_i}\text{Ric}^i\right)(Y_i, Z_i) - \left(D^i_{X_i}\mathcal{F}^i\right)(Y_i, Z_i) + 2\frac{f_j^\diamond}{f_i^3} X_i(f_i)\, g_i(Y_i, Z_i) \\ &+ (n-2)\frac{1}{f_i^3} g_i(X_i, Y_i)\nabla^j f_j\left(f_j\right) Z_i(f_i) \\ &+ (n-2)\frac{1}{f_i^3} g_i(X_i, Z_i)\nabla^j f_j\left(f_j\right) Y_i(f_i). \end{aligned} \tag{2}$$

By exchanging X_i and Y_i in the last equation and substitution in Equation (1), one gets the deviation tensor. For Einstein-like manifolds of class $\mathcal{B}$, the deviation tensor vanishes from which the result hold. □

It is easy to retrieve a similar result on a singly warped product manifold.

Corollary 3.4. *In a singly warped product manifold $M = M_1 \times_{f_1} M_2$ where M is of class type $\mathcal{B}$, (M_1, g_1) is an Einstein-like manifold of class $\mathcal{B}$ if and only if*

$$\left(D^1_{X_1}\mathcal{F}^1\right)(Y_1, Z_1) = \left(D^1_{Y_1}\mathcal{F}^1\right)(X_1, Z_1),$$

where $X_1, Y_1, Z_1 \in \mathfrak{X}(M_1)$. In addition, (M_2, g_2) is Einstein-like of class type $\mathcal{B}$.

3.3. Class $\mathcal{P}$

Let M be an Einstein-like doubly warped product manifold of class $\mathcal{P}$. Thus, M has a parallel Ricci tensor, that is,

$$(D_X\text{Ric})(Y, Z) = 0.$$

Manifolds in this class are usually called Ricci symmetric.

Theorem 3.5. *In a doubly warped product manifold $M =_{f_2} M_1 \times_{f_1} M_2$ where M is of class type $\mathcal{P}$, (M_i, g_i) is an Einstein-like manifold of class $\mathcal{P}$ if and only if*

$$\begin{aligned}\left(D^i_{X_i}\mathcal{F}^i\right)(Y_i, Z_i) &= \frac{n-2}{f_i^3}\left[g_i(X_i, Y_i) Z_i(f_i) + g_i(X_i, Z_i) Y_i(f_i)\right]\left(\nabla^j f_j\right) f_j \\ &+2\frac{f_j^\diamond}{f_i^3} X_i(f_i)\, g_i(Y_i, Z_i),\end{aligned}$$

where $i, j = 1, 2, i \neq j$ and $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$.

Proof. Let $M = {}_{f_2} M_1 \times_{f_1} M_2$ be a Ricci symmetric doubly warped product manifold, that is,

$$0 = (D_X \mathrm{Ric})(Y, Z)$$

Equation (2) infers

$$\begin{aligned}(D_{X_i}\mathrm{Ric})(Y_i, Z_i) &= \left(D^i_{X_i}\mathrm{Ric}^i\right)(Y_i, Z_i) - \left(D^i_{X_i}\mathcal{F}^i\right)(Y_i, Z_i) + 2\frac{f_j^\diamond}{f_i^3} X_i(f_i)\, g_i(Y_i, Z_i) \\ &+ (n-2)\frac{1}{f_i^3} g_i(X_i, Y_i)\nabla^j f_j\left(f_j\right) Z_i(f_i) \\ &+ (n-2)\frac{1}{f_i^3} g_i(X_i, Z_i)\nabla^j f_j\left(f_j\right) Y_i(f_i).\end{aligned}$$

Thus, having a parallel Ricci tensor implies

$$\begin{aligned}\left(D^i_{X_i}\mathrm{Ric}^i\right)(Y_i, Z_i) &= \left(D^i_{X_i}\mathcal{F}^i\right)(Y_i, Z_i) - 2\frac{f_j^\diamond}{f_i^3} X_i(f_i)\, g_i(Y_i, Z_i) \\ &-\frac{n-2}{f_i^3}\left[g_i(X_i, Y_i) Z_i(f_i) + g_i(X_i, Z_i) Y_i(f_i)\right]\nabla^j f_j\left(f_j\right).\end{aligned}$$

This equation completes the proof. □

The corresponding result on singly warped product manifolds is as follows.

Corollary 3.6. *In a singly warped product manifold $M = M_1 \times_{f_1} M_2$ where M is of class type $\mathcal{P}$. Then (M_1, g_1) is an Einstein-like manifold of class $\mathcal{P}$ if and only if*

$$\left(D^1_{X_1}\mathcal{F}^1\right)(Y_1, Z_1) = 0,$$

where $X_1, Y_1, Z_1 \in \mathfrak{X}(M_1)$. Also, (M_2, g_2) is Einstein-like of class type $\mathcal{P}$.

3.4. *Class $\mathcal{I} \oplus \mathcal{B}$*

A doubly warped product manifold M is of class type $\mathcal{I} \oplus \mathcal{B}$ if its Ricci tensor satisfies

$$\nabla_\gamma\left[R_{\alpha\beta} - \frac{R}{2(n-1)} g_{\alpha\beta}\right] = \nabla_\alpha\left[R_{\gamma\beta} - \frac{R}{2(n-1)} g_{\gamma\beta}\right],$$

that is, the tensor $\mathcal{H}_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{2(n-1)} g_{\alpha\beta}$ is a Codazzi tensor. This condition is equivalent to

$$\nabla_\varepsilon C_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} = 0,$$

where C is the Weyl conformal curvature tensor and $n \geq 3$, i.e., M has a harmonic Weyl tensor. Let $g_{\beta\gamma} = \varphi^2 \bar{g}_{\beta\gamma}$ be a conformal change of on a manifold M. It is well known that the Weyl tensor $C_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon}$ remains invariant, that is, $\bar{C}_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} = C_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon}$ however $C_{\alpha\beta\gamma\varepsilon} = \varphi^2 \bar{C}_{\alpha\beta\gamma\varepsilon}$. The divergence of the Weyl tensor is given by[3]

$$\nabla_\varepsilon C_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} = \bar{\nabla}_\varepsilon \bar{C}_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} - \frac{n-3}{\varphi}(\nabla_\varepsilon \varphi)\bar{C}_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon}. \tag{3}$$

The doubly warped product metric may be rewritten as follows

$$\begin{aligned} g &= f_1^2 f_2^2\left(f_1^{-2} g_1 + f_2^{-2} g_2\right) \\ &= f_1^2 f_2^2\left(\bar{g}_1 + \bar{g}_2\right) \\ &= f_1^2 f_2^2 \bar{g}\end{aligned}$$

where $g_i = f_i^2 \bar{g}_i$ and $\bar{g} = \bar{g}_1 + \bar{g}_2$. The doubly warped product manifold (M, g) has harmonic Weyl tensor if and only

$$\bar{\nabla}_\varepsilon \bar{C}_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} = \frac{n-3}{\varphi} \left(\nabla_\varepsilon \varphi\right) C_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} \tag{4}$$

where $\varphi = f_1 f_2$. Assume that $\nabla_\varepsilon \left(f_1 f_2\right) C_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} = 0$, then

$$\bar{\nabla}_\varepsilon \bar{C}_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} = 0. \tag{5}$$

having a harmonic Weyl tensor is equivalent to the condition

$$\begin{aligned} 0 &= \bar{\mathcal{T}}_{\alpha\beta\gamma} \\ &= \bar{\nabla}_\gamma \bar{R}_{\alpha\beta} - \bar{\nabla}_\gamma \bar{R}_{\alpha\beta} - \frac{1}{2(n-1)} \left[\left(\bar{\nabla}_\gamma \bar{R}\right) \bar{g}_{\alpha\beta} - \left(\bar{\nabla}_\gamma \bar{R}\right) \bar{g}_{\alpha\beta}\right], \end{aligned}$$

where $\bar{\mathcal{T}}$ is the Cotton tensor. The metric $\bar{g}$ splits as $\bar{g} = \bar{g}_1 + \bar{g}_2$ and consequently the divergence of the Cotton tensor $\bar{\mathcal{T}}$ splits on the factor manifolds $(M_i, \bar{g}_i)$ as

$$0 = \bar{\mathcal{T}}^i_{\alpha\beta\gamma} + \frac{n_2}{2(n-1)(n_1-1)} \left[\left(\bar{\nabla}^i_\gamma \bar{R}^i\right) (\bar{g}_i)_{\alpha\beta} - \left(\bar{\nabla}^i_\gamma \bar{R}^i\right) (\bar{g}_i)_{\alpha\beta}\right]. \tag{6}$$

In this case, $\left(\bar{\nabla}^i_\gamma \bar{R}^i\right) (\bar{g}_i)_{\alpha\beta} - \left(\bar{\nabla}^i_\gamma \bar{R}^i\right) (\bar{g}_i)_{\alpha\beta} = 0$, that is, $\bar{R}^i$ is constant if and only if the cotton tensor $\bar{\mathcal{T}}^i$ on the doubly warped factor manifolds $\left(M^i, \bar{g}_i\right)$ vanishes i.e.

$$\bar{\nabla}_\varepsilon \bar{C}^{i\ \ \ \ \varepsilon}_{\alpha\beta\gamma} = 0.$$

The Weyl tensors C^i on doubly warped product factor manifolds (M_i, g_i) satisfy

$$\begin{aligned} 0 &= \bar{\nabla}_\varepsilon \bar{C}^{i\ \ \ \ \varepsilon}_{\alpha\beta\gamma} \\ &= \nabla_\varepsilon C^{i\ \ \ \ \varepsilon}_{\alpha\beta\gamma} + \frac{n_i - 3}{f_i} \left(\nabla^i_\varepsilon f_i\right) C^{i\ \ \ \ \varepsilon}_{\alpha\beta\gamma}. \end{aligned} \tag{7}$$

It is time now to write the following result.

Theorem 3.7. *In a doubly warped product manifold $M =_{f_2} M_1 \times_{f_1} M_2$ where M is of class type $\mathcal{I} \oplus \mathcal{B}$. Assume that $\nabla_\varepsilon (f_1 f_2) C_{\alpha\beta\gamma}^{\ \ \ \ \varepsilon} = 0$ and the conformal change $\left(M_i, f_i^{-2} g_i\right)$ has a constant scalar curvature. Then (M_i, g_i) is an Einstein-like manifold of class $\mathcal{I} \oplus \mathcal{B}$ if and only if $\left(\nabla^i_\varepsilon f_i\right) C^{i\ \ \ \ \varepsilon}_{\alpha\beta\gamma} = 0$ for each $i = 1, 2$.*

A. Gebarowski proved an inheritance property of this class in [18, Theorem 2].

3.5. Class $\mathcal{I} \oplus \mathcal{A}$

Doubly warped product manifolds where the tensor

$$\mathcal{L} = \mathrm{Ric} - \frac{2R}{n+2} g$$

is Killing lies the class $\mathcal{I} \oplus \mathcal{A}$. The above condition is equivalent to

$$0 = \left(D_X \mathcal{L}\right)(X, X).$$

The following theorem draw the inheritance property of this class.

Theorem 3.8. *In a doubly warped product manifold* $M = {}_{f_2} M_1 \times_{f_1} M_2$ *where* M *is of class type* $\mathcal{I} \oplus \mathcal{A}$, *the factor manifold* (M_i, g_i) *is of class type* $\mathcal{I} \oplus \mathcal{A}$ *if and only if*

$$\begin{aligned} \left(D^i_{X_i}\mathcal{F}^i\right)(X_i, X_i) &= \frac{2}{f_i^3} X_i(f_i)\, g_i(X_i, X_i)\left[f_j^\diamond + (n-2)\left(\nabla^j f_j\right)\left(f_j\right)\right] \\ &\quad -\frac{2}{n+2}\left(D_{X_i}R - \frac{n+2}{n_i+2} D^i_{X_i}R^i\right) g_i(X_i, X_i)\,. \end{aligned}$$

Proof. Assume that $M = {}_{f_2} M_1 \times_{f_1} M_2$ be a doubly warped product manifold of class type $\mathcal{I} \oplus \mathcal{A}$. Then

$$\begin{aligned} 0 &= (D_X)\left(\mathrm{Ric} - \frac{2R}{n+2} g\right)(X, X) \\ &= (D_X \mathrm{Ric})(X, X) - \frac{2}{n+2} g(X, X)\, D_X R. \end{aligned}$$

Using equation (2), it is

$$\begin{aligned} 0 &= \left(D^i_{X_i}\mathrm{Ric}^i\right)(X_i, X_i) - \left(D^i_{X_i}\mathcal{F}^i\right)(X_i, X_i) \\ &\quad + \frac{2}{f_i^3} X_i(f_i)\, g_i(X_i, X_i)\left[f_j^\diamond + (n-2)\left(\nabla^j f_j\right)\left(f_j\right)\right] \\ &\quad - \frac{2}{n+2}\left(D_{Xi}R\right) g_i(X_i, X_i) \end{aligned}$$

and consequently, one has

$$\begin{aligned} 0 &= \left(D^i_{X_i}\mathrm{Ric}^i\right)(X_i, X_i) - \frac{2}{n_i+2} g_i(X_i, X_i)\, D^i_{X_i}R^i \\ &\quad - \left(D^i_{X_i}\mathcal{F}^i\right)(X_i, X_i) \\ &\quad + \frac{2}{f_i^3} X_i(f_i)\, g_i(X_i, X_i)\left[f_j^\diamond + (n-2)\left(\nabla^j f_j\right)\left(f_j\right)\right] \\ &\quad - \frac{2}{n+2}\left(D_{Xi}R - \frac{n+2}{n_i+2} D^i_{X_i}R^i\right) g_i(X_i, X_i) \end{aligned}$$

which completes the proof. □

3.6. *Class* $\mathcal{A} \oplus \mathcal{B}$

This class is identified by having a constant scalar curvature. Let $M = {}_{f_2} M_1 \times_{f_1} M_2$ be a doubly warped product manifold of class type $\mathcal{A} \oplus \mathcal{B}$, that is, the scalar curvature R of M is constant, say c. The use of Equation 7 in [18] implies that M_i is of class $\mathcal{A} \oplus \mathcal{B}$ if there are two constants c_i and c_j such that

$$\frac{c_i}{f_j^2} + \frac{c_j}{f_i^2} - \frac{n_i(n_i-1)}{f_j^2}\Delta_j f_j - \frac{n_j\left(n_j-1\right)}{f_i^2}\Delta_i f_i - \frac{2n_i}{f_j}F_j - \frac{2n_j}{f_i}F_i = c,$$

where $F_i = g_i^{\alpha\beta}\nabla^i_\alpha \nabla^i_\beta f_i$.

4. Einstein-like doubly warped Relativistic space-times

Let (M, g) be a Riemannian manifold, $f : M \to (0, \infty)$ and $\sigma : I \to (0, \infty)$ are smooth functions. The manifold $\bar{M} =_f I \times_\sigma M$ furnished with the metric tensor $\bar{g} = -f^2 dt^2 \oplus \sigma^2 g$ is called a doubly warped space-time. For $U, V \in \mathfrak{X}(M)$, the covariant derivative $\bar{D}$ on $\bar{M}$ is given by

$$\begin{aligned}
\bar{D}_{\partial_t}\partial_t &= \frac{f}{\sigma^2}\nabla f, \\
\bar{D}_{\partial_t}U &= D_U\partial_t = \frac{\dot{\sigma}}{\sigma}U + \frac{1}{f}U(f)\,\partial_t, \\
\bar{D}_U V &= D_U V - \frac{\sigma\dot{\sigma}}{f^2}g(U, V)\,\partial_t,
\end{aligned}$$

whereas the Ricci tensor $\bar{\mathrm{Ric}}$ on $\bar{M}$ is given by

$$\begin{aligned}
\bar{\mathrm{Ric}}(\partial_t, \partial_t) &= \frac{n}{\sigma}\ddot{\sigma} + \frac{f^\diamond}{\sigma^2}, \\
\bar{\mathrm{Ric}}(U, V) &= \mathrm{Ric}(U, V) - \frac{1}{f}H^f(U, V) - \frac{\sigma^\diamond}{f^2}g(U, V), \\
\bar{\mathrm{Ric}}(\partial_t, U) &= (n-1)\frac{\dot{\sigma}}{\sigma}U(\ln f).
\end{aligned}$$

For the definition and relativistic significance of doubly warped space-times, the reader is referred to [15, 32] and references therein.

Theorem 4.1. *In a doubly warped space-time $\bar{M} =_f I \times_\sigma M$ of class type $\mathcal{A}$, M is an Einstein-like manifold of class type $\mathcal{A}$ if and only if*

$$(D_V\mathcal{F})(V, V) = \left((n-1)\dot{\sigma}^2 + \sigma^\diamond\right)\frac{2}{f^3}V(f)\,g(V, V).$$

Theorem 4.2. *In a doubly warped space-time $\bar{M} =_f I \times_\sigma M$ of class type $\mathcal{B}$, M is an Einstein-like manifold of class type $\mathcal{B}$ if and only if*

$$\begin{aligned}
(D_W\mathcal{F})(U, V) &= (D_U\mathcal{F})(W, V) + \left(2\sigma^\diamond - (n-1)\dot{\sigma}^2\right)\frac{1}{f^3}W(f)\,g(U, V) \\
&\quad - \left(2\sigma^\diamond + (n-1)\dot{\sigma}^2\right)\frac{1}{f^3}U(f)\,g(W, V).
\end{aligned}$$

Theorem 4.3. *In a doubly warped space-time $\bar{M} =_f I \times_\sigma M$ of class type $\mathcal{P}$, M is an Einstein-like manifold of class type $\mathcal{P}$ if and only if*

$$(D_W\mathcal{F})(U, V) = 2\frac{\sigma^\diamond}{f^3}W(f)\,g(U, V) + \frac{\dot{\sigma}^2}{f^3}(n-1)\left(g(W, V)\,U(f) + g(W, U)\,V(f)\right).$$

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