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Best Simultaneous Approximation on Metric Spaces via Monotonous Norms

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Abstract. For a Banach space X, $L^{\Phi}(T,X)$ denotes the metric space of all X-valued Φ-integrable functions $f : T \to X$, where the measure space $(T,\sum,\mu)$ is a complete positive σ-finite and Φ is an increasing subadditive continuous function on $[0,\infty)$ with $\Phi(0) = 0$. In this paper we discuss the proximinality problem for the monotonous norm on best simultaneous approximation from the closed subspace $Y \subseteq X$ to a finite number of elements in X.

1. Introduction

Many authors studied the problem of best simultaneous approximation for functions and operators in Banach spaces, also in metric linear spaces, e.g. [1], [2], [6]-[12] .

A function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ is called a modulus function if it satisfies the following conditions:

1. $\Phi(x) = 0$ iff $x = 0$;
2. $\Phi(x + y) \le \Phi(x) + \Phi(y)$;
3. Φ is continuous and increasing.

The functions $\Phi(x) = x^p$, $0 < p < 1$, and $\Phi(x) = \ln(x + 1)$ are examples of modulus functions. Further the composition of two modulus functions is a modulus function.

Let $(T,\sum,\mu)$ be a complete positive σ-finite measure space, X be a Banach space and let$Y$ be a closed subspace of X. If Φ is a modulus function, then $L^{\Phi}(T,X)$ denotes the space of all $X-$valued Φ-integrable functions $f : T \to X$ on the measure space $(T,\sum,\mu)$ *i.e.*

$$L^{\Phi}(T,X) = \left\{ f : T \to X \ : \int_T \Phi\left(\left\| f(t)\right\|\right) d\mu < \infty \right\}.$$

Also, the sequence space $l^{\Phi}(T,X)$ is defined by:

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$$l^{\Phi}(T,X)=\left\{x=(x_k)_{k=1}^{\infty}\ :\sum_{k=1}^{\infty}\Phi(\|x_k\|)<\infty,\ x_k\in X\right\}.$$

For $x=(x_k)_{k=1}^{\infty}\in l^{\Phi}(T,X)$ and $f\in L^{\Phi}(T,X)$, set

$$\|x\|_{\Phi}=\sum_{k=1}^{\infty}\Phi(\|x_k\|)\ \text{and}\ \|f\|_{\Phi}=\int_T\Phi\left(\|f(t)\|\right)d\mu.$$

The spaces $(L^{\Phi}(T,X),\|\cdot\|_{\Phi})$ and $\left(l^{\Phi}(T,X),\|\cdot\|_{\Phi}\right)$ are complete metric linear spaces. It is well known that $l^{\Phi}(T,X)\subseteq l^{1}(T,X)$, $L^{\Phi}(T,X)\supseteq L^{1}(T,X)$. For more about these spaces see [4], [5].

We start with the following definitions:

We say that a norm ρ in $\mathbb{R}^n$ is monotonous if for every $x=(x_k)_{1\le k\le n}$, $y=(y_k)_{1\le k\le n}$ in $\mathbb{R}^n$ such that $|x_k|\le|y_k|$, for $k=1,\ldots,n$, we have

$$\rho(x)\le\rho(y).$$

Note that the usual norms in $\mathbb{R}^n$ are monotonous.

Let $x_1,x_2,\ldots,x_n$ be n elements in X, we set

$$dis(x_1,x_2,\ldots,x_n,Y)=\inf_{y\in Y}\rho\left(\Phi\|x_1-y\|,\Phi\|x_2-y\|,\ldots,\Phi\|x_n-y\|\right)$$

We say that Y is ρ-simultaneous proximinal in X, if for any n elements $x_1,x_2,\ldots,x_n$ in X, there exists $y_0\in Y$ such that

$$\rho\left(\Phi\|x_1-y_0\|,\Phi\|x_2-y_0\|,\ldots,\Phi\|x_n-y_0\|\right)=dis(x_1,x_2,\ldots,x_n,Y).$$

In this case we say that y_0 is a best ρ-simultaneous approximation from Y of the elements $x_1,x_2,\ldots,x_n$ in X.

Also, we say that $L^{\Phi}(T,Y)$ is ρ-simultaneous proximinal in $L^{\Phi}(T,X)$, if for any n elements $f_1,f_2,\ldots,f_n$ in $L^{\Phi}(T,X)$, there exists $g_0\in L^{\Phi}(T,Y)$ such that:

$$\begin{aligned}dis\left(f_1,f_2,\ldots,f_n,L^{\Phi}(T,Y)\right)&=\inf_{g\in L^{\Phi}(T,Y)}\rho\left(\|f_1-g\|_{\Phi},\|f_2-g\|_{\Phi},\ldots,\|f_n-g\|_{\Phi}\right)\\&=\rho\left(\|f_1-g_0\|_{\Phi},\|f_2-g_0\|_{\Phi},\ldots,\|f_n-g_0\|_{\Phi}\right).\end{aligned}$$

In this case we say that g_0 is a best ρ-simultaneous approximation from $L^{\Phi}(T,Y)$ of the elements $f_1,f_2,\ldots,f_n$ in $L^{\Phi}(T,X)$.

We shall denote the set of all such best ρ-simultaneous approximation to $x_1,x_2,\ldots,x_n$ by $P_Y(x_1,x_2,\ldots,x_n)$ *i.e.*

$$P_Y(x_1,x_2,\ldots,x_n)=\left\{\begin{array}{c}y\in Y:dist(x_1,x_2,\ldots,x_n,Y)=\\\rho\left(\Phi\|x_1-y\|,\Phi\|x_2-y\|,\ldots,\Phi\|x_n-y\|\right)\end{array}\right\}$$

Also

$$P_{L^{\Phi}(T,Y)}(f_1,f_2,\ldots,f_n)=\left\{\begin{array}{l}g\in L^{\Phi}(T,Y):dis\left(f_1,f_2,\ldots,f_n,L^{\Phi}(T,G)\right)\\=\rho\left(\|f_1-g\|_{\Phi},\|f_2-g\|_{\Phi},\ldots,\|f_n-g_0\|_{\Phi}\right)\end{array}\right\}.$$

It is clear that Y is ρ-simultaneous proximinal in X if and only if $P_Y(x_1, x_2, \ldots, x_n)$ is nonempty for every n elements $x_1, x_2, \ldots, x_n$ in X.

Let $x_1, x_2, \ldots, x_n$ be n elements in X, we say that the sequence $(y_k)_{k=1}^{\infty} \subseteq Y$ is ρ-simultaneously approximating for $x_1, x_2, \ldots, x_n$ in Y, if

$$\underset{k\longrightarrow\infty}{Lim}\, \rho\left(\Phi\left\|x_1 - y_k\right\|, \Phi\left\|x_2 - y_k\right\|, \ldots, \Phi\left\|x_n - y_k\right\|\right) = dis\left(x_1, x_2, \ldots, x_n, Y\right).$$

The set $Y \subseteq X$ is said to be approximatively compact, if for every n elements $x_1, x_2, \ldots, x_n$ in X, and each ρ-simultaneously approximating sequence $(y_k)_{k=1}^{\infty} \subseteq Y$, there exists a subsequence of $(y_k)_{k=1}^{\infty} \subseteq Y$ that converges to an element in Y.

2. Main results

Theorem 2.1. *Y is ρ-simultaneous proximinal in X, if Y is a compact subspace of X.*

Proof. For any $x_1, x_2, \ldots, x_n \in X$, define the function $g : Y \longrightarrow \mathbb{R}$ by

$$g(y) = \rho\left(\Phi\left\|x_1 - y\right\|, \Phi\left\|x_2 - y\right\|, \ldots, \Phi\left\|x_n - y\right\|\right).$$

It is clear that the function g is continuous, since Φ, ρ, $\|\cdot\|$ are continuous functions of Y and thus, the infimum is attained. i.e., there exists $y_0 \in Y$ such that

$$g(y_0) = \inf_{y\in Y} g(y) = \inf_{y\in Y} \rho\left(\Phi\left\|x_1 - y\right\|, \Phi\left\|x_2 - y\right\|, \ldots, \Phi\left\|x_n - y\right\|\right).$$

Thus Y is ρ-simultaneous proximinal in X. □

The following Lemma deals with the boundedness and the closeness of the set $P_Y(x_1, x_2, \ldots, x_n)$.

Lemma 2.2. *The set $P_Y(x_1, x_2, \ldots, x_n)$ is bounded and closed if Y is a closed subspace.*

Proof. Let $x_1, x_2, \ldots, x_n \in X$ and suppose that $y_1, y_2, \ldots, y_n \in P_Y(x_1, x_2, \ldots, x_n)$. Using the fact that Φ is an increasing function and $0 \in Y$, then for each $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\rho(1, 1, \ldots, 1)\ \Phi\left\|y_i\right\| &= \rho\left(\Phi\left\|y_i\right\|, \ldots, \Phi\left\|y_i\right\|\right) \\
&\leq \rho\left(\Phi\left\|x_1 - y_i\right\| + \Phi\left\|x_1\right\|, \ldots, \Phi\left\|x_n - y_i\right\| + \Phi\left\|x_n\right\|\right) \\
&\leq \rho\left(\Phi\left\|x_1 - y_i\right\|, \ldots, \Phi\left\|x_n - y_i\right\|\right) + \rho\left(\Phi\left\|x_1\right\|, \ldots, \Phi\left\|x_n\right\|\right) \\
&= dis\left(x_1, x_2, \ldots, x_n, Y\right) + \rho\left(\Phi\left\|x_1\right\|, \ldots, \Phi\left\|x_n\right\|\right).
\end{aligned}$$

Thus, for each $i = 1, 2, \ldots, n$,

$$\rho(1, 1, \ldots, 1)\ \Phi\left\|y_i\right\| \leq dis\left(x_1, x_2, \ldots, x_n, Y\right) + \rho\left(\Phi\left\|x_1\right\|, \ldots, \Phi\left\|x_n\right\|\right).$$

Hence, $P_Y(x_1, x_2, \ldots, x_n)$ is bounded. Suppose Y is a closed subspace and let $(y_k)_{k=1}^{\infty}$ be a sequence in $P_Y(x_1, x_2, \ldots, x_n)$ such that $\underset{k\longrightarrow\infty}{Lim}\, y_k = y_0$. Since $(y_k)_{k=1}^{\infty} \subseteq P_Y(x_1, x_2, \ldots, x_n)$, we have

$$\rho\left(\Phi\left\|x_1 - y_k\right\|, \ldots, \Phi\left\|x_n - y_k\right\|\right) = \inf_{y\in Y} \rho\left(\Phi\left\|x_1 - y\right\|, , \ldots, \Phi\left\|x_n - y\right\|\right),$$

for all $k \geq 1$.

Therefore,

$$\begin{aligned}
&\inf_{y\in Y}\rho\left(\Phi\left\|x_1-y\right\|,\ldots,\Phi\left\|x_n-y\right\|\right)\\
&=\underset{k\longrightarrow\infty}{Lim}\,\rho\left(\Phi\left\|x_1-y_k\right\|,\ldots,\Phi\left\|x_n-y_k\right\|\right)\\
&=\ \rho\left(\Phi\left\|x_1-\underset{k\longrightarrow\infty}{Lim}\,y_k\right\|,\ldots,\Phi\left\|x_n-\underset{k\longrightarrow\infty}{Lim}\,y_k\right\|\right)\\
&=\rho\left(\Phi\left\|x_1-y_0\right\|,\ldots,\Phi\left\|x_n-y_0\right\|\right).
\end{aligned}$$

Hence, $y_0\in P_Y(x_1,x_2,\ldots,x_n)$, which gives the required result.

The following lemmas and theorems introduce some classes of ρ-simultaneous proximinal subspaces: □

Lemma 2.3. *Let $x_1,x_2,\ldots,x_n$ be n elements in X and let the sequence $(y_k)_{k=1}^{\infty}\subseteq Y$ be an ρ-simultaneously approximating sequence to $x_1,x_2,\ldots,x_n$ in Y. If $(y_k)_{k=1}^{\infty}$is weakly convergent to $y_0\in Y$, then y_0 is a best ρ-simultaneous approximation from Y of the elements $x_1,x_2,\ldots,x_n$.*

Proof. Since $\|\cdot\|$ is weakly lower semicontinuous (see [3]), then for each $i=1,2,\ldots,n$, we have

$$\left\|x_i-y_0\right\|\le\underset{k}{Lim}\inf\left\|x_i-y_k\right\|.$$

Since Φ is continuous and increasing function, then for each $i=1,2,\ldots,n$, we have

$$\Phi\left\|x_i-y_0\right\|\le\underset{k}{Lim}\inf\Phi\left\|x_i-y_k\right\|.$$

Using the monotonicity and the continuity of the norm ρ, we get

$$\begin{aligned}
&\rho\left(\Phi\left\|x_1-y_0\right\|,\ldots,\Phi\left\|x_n-y_0\right\|\right)\\
&\le\rho(Lim_{k\longrightarrow\infty}\inf\Phi\left\|x_1-y_k\right\|,\ldots,Lim_{k\longrightarrow\infty}\inf\Phi\left\|x_n-y_k\right\|\\
&=Lim_{k\longrightarrow\infty}\inf\rho\left(\Phi\left\|x_1-y_k\right\|,\ldots,\Phi\left\|x_n-y_k\right\|\right)\\
&=Lim_{k\longrightarrow\infty}\rho\left(\Phi\left\|x_1-y_k\right\|,\ldots,\Phi\left\|x_n-y_k\right\|\right)\\
&=dis\,(x_1,x_2,\ldots,x_n,Y).
\end{aligned}$$

Which means that y_0 is a best ρ-simultaneous approximation from Y of the elements $x_1,x_2,\ldots,x_n$ in X. □

Theorem 2.4. *Y is ρ-simultaneous proximinal in X, if Y is approximatively compact subspace of X.*

Proof. Let $x_1,x_2,\ldots,x_n$ be elements in X. Then by the definition of

$$dis\,(x_1,x_2,\ldots,x_n,Y)=\inf_{y\in Y}\rho\left(\Phi\left\|x_1-y\right\|,\ldots,\Phi\left\|x_n-y\right\|\right),$$

we can find $(y_k)_{k=1}^{\infty}\subseteq Y$ such that

$$\underset{k\longrightarrow\infty}{Lim}\,\rho\left(\Phi\left\|x_1-y_k\right\|,\ldots,\Phi\left\|x_n-y_k\right\|\right)=dis\,(x_1,x_2,\ldots,x_n,Y).$$

Then $(y_k)_{k=1}^{\infty}$is a ρ-simultaneously approximating sequence to $x_1,x_2,\ldots,x_n$ in Y. Since Y is approximately compact, then there exists a subsequence (y_{k_n}) of $(y_k)_{k=1}^{\infty}$that converges to $y_0\in Y\left(i.e\ \underset{k_n\longrightarrow\infty}{Lim}\,y_{k_n}=y_0\right)$. Thus

$$\begin{aligned}
\rho\left(\Phi\left\|x_1-y_0\right\|,\ldots,\Phi\left\|x_n-y_0\right\|\right)&=\rho\left(\Phi\left\|x_1-\underset{k_n\longrightarrow\infty}{Lim}\,y_{k_n}\right\|,\ldots,\Phi\left\|x_n-\underset{k_n\longrightarrow\infty}{Lim}\,y_{k_n}\right\|\right)\\
&=\underset{k_n\longrightarrow\infty}{Lim}\,\rho\left(\Phi\left\|x_1-y_{k_n}\right\|,\ldots,\Phi\left\|x_n-y_{k_n}\right\|\right)\\
&=dis\,(x_1,x_2,\ldots,x_n,Y).
\end{aligned}$$

Which gives the required result. □

For $x \in X$ and $r > 0$, let $B(x, r)$ denotes the closed ball with center x and radius r. Recall that $Y \subseteq X$ is locally weakly compact(resp. boundedly weakly compact) if for each $y \in Y$ (resp. for each $r > 0$), there exists $\delta > 0$ such that $B(y, \delta) \cap Y$ (resp.$B(0, r) \cap Y$) is locally weakly compact.

Now, we introduce the following Lemma which gives the relation between locally weakly compact and boundedly weakly compact, [9].

Lemma 2.5. *For a closed subspace Y of X, the following statements are equivalent:*
(i) *Y is locally weakly compact.*
(ii) *Y is boundedly weakly compact.*
(iii) *There exists a point $y \in Y$ and $\delta > 0$ such that $B(y, \delta) \cap Y$ is locally weakly compact.*

Theorem 2.6. *Y is ρ-simultaneously proximinal in X if Y is a locally weakly compact closed subspace of X.*

Proof. Let $x_1, x_2, \ldots, x_n$ be elements in X. By the definition of

$$dis(x_1, x_2, \ldots, x_n, Y) = \inf_{y \in Y} \rho\left(\Phi \left\|x_1 - y\right\|, \Phi \left\|x_2 - y\right\|, \ldots, \Phi \left\|x_n - y\right\|\right),$$

we can find $(y_k)_{k=1}^{\infty} \subseteq Y$, such that

$$\underset{k \longrightarrow \infty}{Lim} \rho\left(\Phi \left\|x_1 - y_k\right\|, \ldots, \Phi \left\|x_n - y_k\right\|\right) = dis(x_1, x_2, \ldots, x_n, Y).$$

Then $(y_k)_{k=1}^{\infty}$ is a ρ-simultaneously approximating sequence to $x_1, x_2, \ldots, x_n$ in Y. Thus, there exists a positive number α, such that

$$\rho\left(\Phi \left\|x_1 - y_k\right\|, \ldots, \Phi \left\|x_n - y_k\right\|\right) \leq \alpha,$$

for all k. Using the fact that Φ is a modulus function and the norm ρ is monotonous, we have for each $k \geq 1$:

$$\begin{aligned}
\rho(1, 1, \ldots, 1)\ \Phi \left\|y_k\right\| &= \rho\left(\Phi \left\|y_k\right\|, \ldots, \Phi \left\|y_k\right\|\right) \\
&\leq \rho\left(\Phi\left(\left\|x_1 - y_k\right\| + \|x_1\|\right), \ldots, \Phi\left(\left\|x_n - y_k\right\| + \|x_n\|\right)\right) \\
&\leq \rho\left(\Phi \left\|x_1 - y_k\right\| + \Phi \|x_1\|, \ldots, \Phi \left\|x_n - y_k\right\| + \Phi \|x_n\|\right) \\
&\leq \rho\left(\Phi \left\|x_1 - y_k\right\|, \ldots, \Phi \left\|x_n - y_k\right\|\right) + \rho\left(\Phi \|x_1\|, \ldots, \Phi \|x_n\|\right) \\
&\leq \alpha + \rho\left(\Phi \|x_1\|, \ldots, \Phi \|x_n\|\right).
\end{aligned}$$

This shows that $(y_k)_{k=1}^{\infty} \subseteq Y$ is a bounded sequence. Since Y is locally weakly compact, it follows from Lemma (2.5) that $(y_k)_{k=1}^{\infty}$ has a weakly convergent subsequence with weak limit y_0. Since Y is a closed subspace of X, then $y_0 \in Y$, it follows from Lemma (2.3) that y_0 is a best ρ-simultaneous approximation from Y of $x_1, x_2, \ldots, x_n$. □

Lemma 2.7. *Let $f_1, \ldots, f_n \in L^{\Phi}(T, X)$ and define $H : T \to \mathbb{R}$ by $H(t) = dis(f_1(t), \ldots, f_n(t), Y)$. Then H is a measurable function.*

Proof. Let $f_1, \ldots, f_n \in L^{\Phi}(T, X)$, then there exist sequences of simple functions $\left(f_m^i\right)$, $(i = 1, 2 \ldots, n)$ in $L^{\Phi}(T, X)$ which converges to f_i ,$(i = 1, 2 \ldots, n)$ for almost all $t \in T$ *i.e*:

$$\underset{m \longrightarrow \infty}{Lim} \left\|f_m^1(t) - f_i(t)\right\| = 0, \ \ i = 1, 2 \ldots, n,$$

for almost all $t \in T$.

The continuity of the distance function $dis(x_1, x_2, \ldots, x_n, Y)$ implies that:

$$\underset{m \longrightarrow \infty}{Lim} \left| dis\left(f_m^1(t), \ldots, f_m^n(t), Y\right) - dis(f_1(t), \ldots, f_n(t), Y) \right| = 0,$$

for almost all $t \in T$.

Furthermore, for each $m \in N$ the function:$t \longrightarrow dis\left(f_m^1(t), \ldots, f_m^n(t), Y\right)$ is a simple function, therefore H is a measurable function. □

Lemma 2.8. *Let $f_1, \ldots, f_n \in L^\Phi(T, X)$ be n elements of simple functions. Then*

$$dist\left(f_1, \ldots, f_n, L^\Phi(T, Y)\right) \le \int_T dist(f_1(t), \ldots, f_n(t), Y)\, d\mu(t).$$

Proof. Assume that $f_i = \sum_{k=1}^{m} x_k^i \chi_{A_k}, (i = 1, 2 \ldots, n)$, where the A_k's are pairwise disjoint measurable sets of T with $\bigcup_{k=1}^{m} A_k = T$, and the set$\left\{x_k^i\right\}_{k=1}^m \subseteq X,\ (i = 1, 2 \ldots, n)$, $\mu(A_k) < \infty$ whenever $x_k^i \neq 0$ because

$$\left\|f_i\right\|_\Phi = \sum_{k=1}^{m} \Phi \left\|x_k^i\right\| \mu(A_k) < \infty.$$

Thus, we may assume $0 < \mu(A_k) < \infty$, for each $k = 1, 2, ..., m$. Since

$$dis(x_1, x_2, \ldots, x_n, Y) = \inf_{y \in Y} \rho\left(\Phi\left\|x_1 - y\right\|, \Phi\left\|x_2 - y\right\|, \ldots, \Phi\left\|x_n - y\right\|\right),$$

then, we can select $y_k \in Y$ such that:

$$\rho\left(\Phi\left\|x_k^1 - y_k\right\|, \ldots, \Phi\left\|x_k^n - y_k\right\|\right) < dis\left(x_k^1, \ldots, x_k^n, Y\right) + \frac{\varepsilon}{m\,\mu(A_k)},$$

for each $k = 1, 2, \ldots, m$.

Set $g_0 = \sum_{k=1}^{m} y_k\, \chi_{A_k}$, clearly $g_0 \in L^\Phi(T, Y)$. Then

$$\begin{aligned}
& dis\left(f_1, \ldots, f_n, L^\Phi(T, Y)\right) \\
& \le \rho\left(\left\|f_1 - g_0\right\|_\Phi, \ldots, \left\|f_n - g_0\right\|_\Phi\right) \\
& = \rho\left(\int_T \Phi\left\|f_1(t) - g_0(t)\right\| d\mu(t), \ldots, \int_T \Phi\left\|f_n(t) - g_0(t)\right\| d\mu(t)\right) \\
& = \rho\left(\sum_{k=1}^{m} \mu(A_k)\, \Phi\left\|x_k^1 - y_k\right\|, \ldots, \sum_{k=1}^{m} \mu(A_k)\, \Phi\left\|x_k^n - y_k\right\|\right) \\
& \le \sum_{k=1}^{m} \mu(A_k)\, \rho\left(\Phi\left\|x_k^1 - y_k\right\|, \ldots, \Phi\left\|x_k^n - y_k\right\|\right) \\
& < \sum_{k=1}^{m} \mu(A_k)\left(dis\left(x_k^1, \ldots, x_k^n, Y\right) + \frac{\varepsilon}{m\,\mu(A_k)}\right) \\
& = \sum_{k=1}^{m} \mu(A_k)\, dis\left(x_k^1, \ldots, x_k^n, Y\right) + \varepsilon \\
& = \int_T dis(f_1(t), \ldots, f_n(t), Y)\, d\mu(t) + \varepsilon.
\end{aligned}$$

Therefore,

$$dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right) < \int_T dist\left(f_1(t),\ldots,f_n(t),Y\right)\ d\mu(t)+\varepsilon$$

Since ε is arbitrary, let $\varepsilon \to 0$, then

$$dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right) \le \int_T dist\left(f_1(t),\ldots,f_n(t),Y\right)\ d\mu(t).$$

□

Theorem 2.9. *Let $(T,\sum,\mu)$ be a complete positive finite measure space and $f_1,\ldots,f_n$ any n elements in $L^{\Phi}(T,X)$, then*

$$dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right) \le \int_T dist\left(f_1(t),\ldots,f_n(t),Y\right)\ d\mu(t).$$

Proof. Since $\mu(T)$ is finite, let $\mu(T)=\alpha$. Using the fact that simple functions are dense in $L^{\Phi}(T,X)$, then for any $\varepsilon>0$ there are n simple functions $f_1^*,\ldots,f_n^*$ in $L^{\Phi}(T,X)$ such that for each $i=1,2\ldots,n$, and for almost all $t\in T$, we have

$$\Phi\left\|f_i^*(t)-f_i(t)\right\| < \frac{\varepsilon}{\alpha}. \tag{1}$$

Threrefore,

$$\begin{aligned}\left\|f_i^*-f_i\right\|_{\Phi} &= \int_T \Phi\left\|f_i^*(t)-f_i(t)\right\| d\mu(t) \\ &< \int_T \frac{\varepsilon}{\alpha} d\mu(t)=\varepsilon.\end{aligned} \tag{2}$$

Assume that

$$f_i^* = \sum_{k=1}^{m} x_k^i\ \chi_{A_k},\ \ (i=1,2\ldots,n)$$

where the A_k's are pairwise disjoint measurable sets of T with $\bigcup\limits_{k=1}^{m} A_k = T$, and $\left(x_k^i\right)_{k=1}^{m} \subseteq X$, $(i=1,2\ldots,n)$.
To complete the proof we need the following steps:

Step 1: We show that

$$\int_T dis\left(f_1^*(t),\ldots,f_n^*(t),Y\right)\ d\mu(t) \le \int_T dis\left(f_1(t),\ldots,f_n(t),Y\right)\ d\mu(t).$$

To show this, let $t\in T$, then for any $y\in Y$, we have

$$dis\left(f_1^*(t),\ldots,f_n^*(t),Y\right) \le \rho\left(\Phi\left\|f_1^*(t)-y\right\|,\ldots,\Phi\left\|f_n^*(t)-y\right\|\right).$$

Since Φ is a modulus function, we have

$$\Phi\left\|f_i^*(t)-y\right\| \le \Phi\left\|f_i^*(t)-f_1(t)\right\| + \Phi\left\|f_i(t)-y\right\|,$$

for $t\in T$ and for each $i=1,2\ldots,n$.

Using the fact that the norm ρ is monotonous, we get

$$\begin{aligned}
&\rho\left(\Phi\left\|f_1^*(t)-y\right\|,\ldots,\Phi\left\|f_n^*(t)-y\right\|\right)\\
&\leq \rho\left(\Phi\left\|f_1^*(t)-f_1(t)\right\|+\Phi\left\|f_1(t)-y\right\|,\ldots,\Phi\left\|f_n^*(t)-f_n(t)\right\|+\Phi\left\|f_n(t)-y\right\|\right)\\
&\leq \rho\left(\Phi\left\|f_1^*(t)-f_1(t)\right\|,\ldots,\Phi\left\|f_n^*(t)-f_n(t)\right\|\right)\\
&+\rho\left(\Phi\left\|f_1(t)-y\right\|,\ldots,\Phi\left\|f_n(t)-y\right\|\right)\\
&<\rho\left(\frac{\varepsilon}{\alpha},\ldots,\frac{\varepsilon}{\alpha}\right)+\rho\left(\Phi\left\|f_1(t)-y\right\|,\ldots,\Phi\left\|f_n(t)-y\right\|\right)\\
&<\frac{\varepsilon}{\alpha}\rho(1,1,\ldots,1)+\rho\left(\Phi\left\|f_1(t)-y\right\|,\ldots,\Phi\left\|f_n(t)-y\right\|\right),\text{for each } t\in T.
\end{aligned}$$

Therefore,

$$dis\left(f_1^*(t),\ldots,f_n^*(t),Y\right)<\frac{\varepsilon}{\alpha}\rho(1,1,\ldots,1)+\rho\left(\Phi\left\|f_1(t)-y\right\|,\ldots,\Phi\left\|f_n(t)-y\right\|\right),\ t\in T$$

Taking the infimum over all such $y\in Y$, we have

$$dis\left(f_1^*(t),\ldots,f_n^*(t),Y\right)\leq\frac{\varepsilon}{\alpha}\rho(1,1,\ldots,1)+dis\left(f_1(t),\ldots,f_2(t),Y\right),\ t\in T$$

Using Lemma (2.8), we can take the integral

$$\begin{aligned}
\int_T dis\left(f_1^*(t),\ldots,f_n^*(t),Y\right)\,d\mu(t)&\leq\int_T\left(\frac{\varepsilon}{\alpha}\rho(1,1,\ldots,1)+dis\left(f_1(t),\ldots,f_2(t),Y\right)\right)d\mu(t)\\
&=\varepsilon\,\rho(1,1,\ldots,1)+\int_T dis\left(f_1(t),\ldots,f_2(t),Y\right)\,d\mu(t).
\end{aligned}$$

Since ε arbitrary let $\varepsilon\to 0$, then

$$\int_T dis\left(f_1^*(t),\ldots,f_n^*(t),Y\right)\,d\mu(t)\leq\int_T dis\left(f_1(t),\ldots,f_2(t),Y\right)\,d\mu(t). \tag{3}$$

Step 2: We show that

$$dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right)\leq\int_T dis\left(f_1(t),\ldots,f_2(t),Y\right)\,d\mu(t).$$

Using inequality (2), we have

$$\begin{aligned}
&dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right)\\
&\leq\rho\left(\left\|f_1-g\right\|_{\Phi},\ldots,\left\|f_n-g\right\|_{\Phi}\right)\\
&\leq\rho\left(\left\|f_1-f_1^*\right\|_{\Phi}+\left\|f_1^*-g\right\|_{\Phi},\ldots,\left\|f_n-f_n^*\right\|_{\Phi}+\left\|f_n^*-g\right\|_{\Phi}\right)\\
&\leq\rho\left(\left\|f_1-f_1^*\right\|_{\Phi},\left\|f_2-f_2^*\right\|_{\Phi}\right)+\ldots+\rho\left(\left\|f_n^*-g\right\|_{\Phi},\left\|f_n^*-g\right\|_{\Phi}\right)\\
&\leq\rho(\varepsilon,\ldots,\varepsilon)+\rho\left(\left\|f_1^*-g\right\|_{\Phi},\ldots,\left\|f_n^*-g\right\|_{\Phi}\right)\\
&\leq\varepsilon\,\rho(1,1,\ldots,1)+\rho\left(\left\|f_1^*-g\right\|_{\Phi},\ldots,\left\|f_n^*-g\right\|_{\Phi}\right),
\end{aligned}$$

for any $g\in L^{\Phi}(T,Y)$.Thus

$$dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right)\leq\varepsilon\,\rho(1,1,\ldots,1)+\rho\left(\left\|f_1^*-g\right\|_{\Phi},\ldots,\left\|f_n^*-g\right\|_{\Phi}\right).$$

Taking the infimum over all such g in $L^{\Phi}(T,Y)$, we get

$$dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right)\leq\varepsilon\,\rho(1,1,\ldots,1)+dist\left(f_1^*,\ldots,f_n,L^{\Phi}(T,Y)\right)\tag{4}$$

Lemma (2.8) and inequality (4) imply that

$$\begin{aligned}dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right)&\leq\varepsilon\,\rho(1,1,\ldots,1)+dist\left(f_1^*,\ldots,f_n,L^{\Phi}(T,Y)\right)\\&\leq\varepsilon\,\rho(1,1,\ldots,1)+\int_T dis\left(f_1^*(t),\ldots,f_n^*(t),Y\right)\,d\mu(t).\end{aligned}$$

Since ε arbitrary, let $\varepsilon\to0$, we have

$$dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right)\leq\int_T dis\left(f_1^*(t),\ldots,f_n^*(t),Y\right)\,d\mu(t).$$

Using inequality(3), we get

$$dis\left(f_1,\ldots,f_n,L^{\Phi}(T,Y)\right)\leq\int_T dis\left(f_1(t),\ldots,f_2(t),Y\right)\,d\mu(t).$$

Thus we get the result. □

Theorem 2.10. *Let $g\in L^{\Phi}(T,Y)$ be the best ρ-simultaneous approximation from $L^{\Phi}(T,Y)$ of the elements $f_1,\ldots,f_n\in L^{\Phi}(T,X)$, then for any measurable subset A of T , and for every $h\in L^{\Phi}(T,Y)$, we have*

$$\int_A\Phi\left\|f_i(t)-g(t)\right\|d\mu(t)\leq\int_A\Phi\left\|f_i(t)-h(t)\right\|d\mu(t),\tag{5}$$

for some $i\in\{1,2\ldots,n\}$.

Proof. Assume that $\mu(A)>0$, for some $A\subseteq T$. Suppose that there is $h_0\in L^{\Phi}(T,Y)$ that doesn't satisfy inequality (5), then we can define $g_0\in L^{\Phi}(T,Y)$ such that

$$g_0(t)=\begin{Bmatrix}h_0(t), & t\in A\\ g(t), & t\in T-A\end{Bmatrix}$$

Thus, for $i=1,2\ldots,n$, we have

$$\begin{aligned}\int_T\Phi\left\|f_i(t)-g_0(t)\right\|d\mu(t)&=\int_A\Phi\left\|f_i(t)-h_0(t)\right\|d\mu(t)+\int_{T-A}\Phi\left\|f_i(t)-g(t)\right\|d\mu(t)\\&<\int_T\Phi\left\|f_i(t)-g(t)\right\|d\mu(t).\end{aligned}$$

Which implies that

$$\left\|f_i-g_0\right\|_{\Phi}<\left\|f_i-g\right\|_{\Phi},$$

for $i=1,2,\ldots,n$. Using the fact that the norm ρ is monotonous, we have

$$\rho\left(\left\|f_1-g_0\right\|_{\Phi},\ldots,\left\|f_n-g_0\right\|_{\Phi}\right)<\rho\left(\left\|f_n-g\right\|_{\Phi},\ldots,\left\|f_n-g\right\|_{\Phi}\right).$$

This contradicts the fact that g is the best ρ-simultaneous approximation from $L^{\Phi}(T,Y)$ of the elements $f_1,\ldots,f_n\in L^{\Phi}(T,X)$. □

The following result concerns the ρ-simultaneous approximation of $l^{\Phi}(T,Y)$ in $l^{\Phi}(T,X)$.

Theorem 2.11. *$l^{\Phi}(T,Y)$ is ρ-simultaneous proximinal in $l^{\Phi}(T,X)$ if Y is ρ-simultaneous proximinal in X.*

Proof. Let $f_1,\ldots,f_n \in l^{\Phi}(T,X)$, where $f_i = (f_i(n))_{n=1}^{\infty}$. Since Y is ρ-simultaneous proximinal in X, then for each $k \in \mathbb{N}$ there exists $g(k) \in Y$ such that

$$\rho\left(\Phi\left\|f_1(k)-g(k)\right\|,\ldots,\Phi\left\|f_n(k)-g(k)\right\|\right) \le \rho\left(\Phi\left\|f_1(k)-y\right\|,\ldots,\Phi\left\|f_n(k)-y\right\|\right),$$

for every $y \in Y$. Since $0 \in Y$, we have

$$\rho\left(\Phi\left\|f_1(k)-g(k)\right\|,\ldots,\Phi\left\|f_n(k)-g(k)\right\|\right) \le \rho\left(\Phi\left\|f_1(k)\right\|,\ldots,\Phi\left\|f_n(k)\right\|\right). \tag{6}$$

Using inequality (6) and the fact that Φ is subadditive and increasing, we have

$$\begin{aligned}
&\rho(1,1,\ldots,1)\ \Phi\left\|g(k)\right\| \\
&= \rho\left(\Phi\left\|g(k)\right\|,\ldots,\Phi\left\|g(k)\right\|\right) \\
&\le \rho\left(\Phi\left\|f_1(k)-g(k)\right\| + \Phi\left\|f_1(k)\right\|,\ldots,\Phi\left\|f_n(k)-g(k)\right\| + \Phi\left\|f_n(k)\right\|\right) \\
&\le \rho\left(\Phi\left\|f_1(k)-g(k)\right\|,\ldots,\Phi\left\|f_n(k)-g(k)\right\|\right) \\
&+ \rho\left(\Phi\left\|f_1(k)\right\|,\ldots,\Phi\left\|f_n(k)\right\|\right) \\
&\le n\ \rho\left(\Phi\left\|f_1(k)\right\|,\ldots,\Phi\left\|f_n(k)\right\|\right).
\end{aligned}$$

Therefore, $g_0 = (g(k))_{k=1}^{\infty} \in l^{\Phi}(T,Y)$ .To show that g_0 is the best ρ-simultaneous approximation from $l^{\Phi}(T,Y)$ of $f_1,\ldots,f_n$ in $l^{\Phi}(T,X)$, let $h = (h(k))_{n=1}^{\infty} \in l^{\Phi}(T,Y)$. Then for each $i = 1,2,\ldots,n$, we have

$$\begin{aligned}
\left\|f_i - h\right\|_{\Phi} &= \sum_{k=1}^{\infty} \Phi\left\|f_i(k)-h(k)\right\| \\
&\ge \sum_{k=1}^{\infty} \Phi\left\|f_i(k)-g(k)\right\| \\
&\ge \left\|f_i - g_0\right\|_{\Phi}.
\end{aligned}$$

Using the monotonicity of the norm ρ, we have

$$\rho\left(\left\|f_1-h\right\|_{\Phi},\ldots,\left\|f_n-h\right\|_{\Phi}\right) \ge \rho\left(\left\|f_1-g_0\right\|_{\Phi},\ldots,\left\|f_n-g_0\right\|_{\Phi}\right).$$

Hence, we get the result. □

Conclusion 2.12. *We have established the ρ-simultaneous proximinality of the closed subspace Y in the Banach space X and give some results in the distance formula of the space $L^{\Phi}(T,X)$. It is not hard to extend our results to the case where ρ is any monotone norm of $\mathbb{R}^n$with n a finite positive integer.*

References

[1] Abu-Sirhan, *Best Simultaneous Approximation in function and operator space*, Turk J Math 36 (2012), 101 – 112.

[2] Al-Sharif, Sh., *Best Simultaneous Approximation in Metric Spaces*, Jordan Journal of Mathematics and Statistics(JJMS), **1**(1), (2008) 69-80.

[3] Day, M.M., Normed Linear Spaces, third ed., Springer, NewYork, 1973.

[4] Deeb, W. and Khalil, R., On the Tensor Product of Non Locally Topological Vector Spaces,Illinois J. Math. Proc. 30,594-601(1986).
[5] Deeb, W. and Younis Y., Extreme Points in a Class of Non Locally Convex Topological vector space,Math. Reb. Toyama Univ. 6, 95-103(1983, New York (1972).
[6] Khandaqji, M. and Al-Sharif, Sh., Best Simultaneous Approximation in Orlicz Spaces, International Journal of Mathematics and Mathematical Sciences, Article ID 68017 (2007).
[7] Khandaqji M., Awawdeh F. ,Jawdat J., Simultaneous proximinality of vector valued function spaces, Turk J Math, 36,(2012) 437 – 444 .
[8] Khandaqji M., Shatanawi W., Best p-Simultaneous approximation in some metric space, Turk J Math, 32,(2008) 1 – 9 .
[9] Luo X F, Li C, Xu H K, Yao J C., Existence of best simultaneous approximations in Lp(S;X); J. Approx. Theory, 163, 1300-1316 (2011).
[10] Mendoza J. Proximinality in Lp(μ;X), J. Approx. Theory, 93, 331-343 (1998).
[11] Mendoza J. ,Pakhrou T., Best simultaneous approximation in L1(μ, X), J. Approx. Theory 145 (2007) 212–220.
[12] Saidi F B, Hussein D, Khalil R., Best Simultaneous Approximation in Lp(I;E); J.Approx. Theory, 116, 369-379 (2002).