



Function Spaces Based on L -Sets

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Abstract. For a commutative, integral, and divisible quantale L , a concept of top L -convergence spaces based on L -sets other than crisp sets is proposed by using a kind of L -filters, namely limited L -filters defined in the paper. Our main result is the existence of function spaces in the the concrete category of top L -convergence spaces over the slice category $\mathbf{Set}\downarrow L$ rather than the category \mathbf{Set} of sets, such that the concrete category of top L -convergence spaces over the slice category $\mathbf{Set}\downarrow L$ is Cartesian closed. In order to support the existence of top L -convergence spaces, some nontrivial examples of limited L -filters and top L -convergence spaces are presented also.

1. Motivation and introduction.

Lowen [29] pointed out that the category of stratified L -topological spaces is not completely satisfactory from a structural point of view, that is, there is no natural function space structure for the sets of morphisms. In classical theory, this deficiency can be overcome by considering the category of convergence spaces [35], which is a super-category of the category of topological spaces [31]. In the lattice-valued case of the underlying lattice $L=[0,1]$, Lowen et al. [28, 29] consider fuzzy convergence spaces as a generalization of Choquent [2] convergence spaces and the resulting category has, among other things, function space, where prime prefilters play a crucial role as Jäger [22] pointed out. Beside prefilter, there are other kinds of lattice-valued filters in literature of fuzzy topology or nontrivial lattice-valued convergence theory, see Eklund and Gähler [3], Fang [5, 6, 8–10], Flores and others [4], Höhle and Šostak [16], Jäger [21, 22, 24, 25], Jäger and Burton [23], Li [26, 27], Pang [32–34], Yao [42] and others.

Besides Lowen's function space of fuzzy convergence spaces, there are some significant works of function spaces established by using different lattice-valued filters. For examples: (1) Based on the concept of stratified L -filters [16], Jäger [21, 22] developed a theory of L -generalized convergence spaces in case of L being a complete Heyting algebra, and the resulting category of stratified L -generalized convergence spaces has the function space as desired. And after Jäger's function space, Fang [5] proposed the category of L -ordered convergence spaces as a reflective full subcategory of the category of L -generalized convergence spaces, which has the function space [11]; (2) By using L -filters of ordinary sets, Yao [42] defined L -fuzzifying convergence spaces and showed the resulting category has the corresponding function space in case of L being a complete Heyting algebra. And after Yao's work, Wu with coauthor Fang [40] constructed their function spaces in the category of L -ordered fuzzifying convergence spaces, which is considered as a

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reflective full subcategory of the category of L -fuzzifying convergence spaces; (3) In 2013, Pang generalized the concept of L -fuzzy Q -convergence spaces [32] to that of (L, M) -fuzzy Q -convergence spaces [33] by using L -filters, and showed the existence of his function space, here the underlying lattices are completely distributive.

Following these works about function spaces, Yu and Fang [39] pointed out that the existence of function spaces in these work, seriously, depend on the idempotency of the meet operation \wedge of the underlying Heyting algebra or completely distributive lattice, and then constructed their function spaces in the category of \top -convergence spaces in case of the underlying lattice being a MV-algebra, here the semigroup operation of a MV-algebra is not idempotent but commutative. For more works about \top -convergence spaces, we refer to [12], [36], [30] and others.

From the constructions of function spaces described above, there exists a common phenomenon that the domain set of function spaces is a set of suitable maps from one crisp set to another, so one could say the base category of constructing function spaces is the category **Set** of sets and maps. We hesitate to mention all L -sets as objects form a slice category $\mathbf{Set}\downarrow L$ of **Set** over a appropriate lattice L . Then a interest question is how to construct function spaces so that the base category involved is the slice category $\mathbf{Set}\downarrow L$ rather than the category **Set** of sets.

By this paper, we want to answer the existence of function spaces in a concrete category over the slice category $\mathbf{Set}\downarrow L$. For this, we firstly recover the concept of \top -filters in [12, 15] on an L -set rather than a crisp set in case of the underlying lattice L being a commutative, integral, and divisible quantale, and obtain the concept of limited L -filters on an L -set, which is a new version of \top -filters defined on a crisp set. And then by means of limited L -filters, we introduce a concept of top L -convergence spaces and establish the concrete category of top L -convergence spaces over the slice category $\mathbf{Set}\downarrow L$. Our main result is the existence of function spaces in the the concrete category of top L -convergence spaces over the slice category $\mathbf{Set}\downarrow L$ such that the concrete category of top L -convergence spaces is Cartesian closed. In order to support the existence of our top L -convergence spaces, we present two kinds of examples to show how we can obtain firstly limited L -filters and then top L -convergence spaces, one of which is from stratified L -topological topologies and another stems from the classical convergence structures.

2. Preliminaries.

The main results in the paper depend on the quantaloid $\mathcal{D}(L)$ constructed from a commutative, integral, and divisible quantale L and the base category $\mathbf{Set}\downarrow L$. Hence we will divide the section into four subsections to introduce the definitions and notions needed.

2.1. GL -quantales

A commutative quantale is a pair $(L, *)$, where L is a complete lattice with respect to a partial order \leq on it, with the top element $\top (= \bigwedge \emptyset)$ and the bottom element $\perp (= \bigvee \emptyset)$, and $*$ is a commutative semigroup operation on L such that

$$\alpha * \left(\bigvee_{j \in J} \beta_j \right) = \bigvee_{j \in J} \alpha * \beta_j,$$

for all $\alpha \in L$ and $\{\beta_j \mid j \in J\} \subseteq L$. $(L, *)$ is unital if there exists an element I with $I * \alpha = \alpha$ for all α , and the I is unique if exists, called the unit.

For a given commutative quantale $(L, *)$ there exists a binary operation $\rightarrow: L \times L \rightarrow L$, defined by

$$\alpha \rightarrow \beta = \bigvee \{x \in L \mid \alpha * x \leq \beta\},$$

called the implication (operation). Further, $*$ and \rightarrow form an adjoint pair in the sense that the following adjointness holds:

$$\alpha * \gamma \leq \beta \iff \gamma \leq (\alpha \rightarrow \beta) \text{ for all } \alpha, \beta, \gamma \in L. \tag{Ad}$$

A commutative unital quantale $(L, *, I)$ is said to be divisible if it satisfies the following condition

$$\forall \alpha, \beta \in L, \alpha \wedge \beta = \alpha * (\alpha \rightarrow \beta). \tag{Div}$$

From [20], we know (Div) is equivalent one of (Div1)-(Div2) below:

$$\forall \alpha, \beta \in L, \alpha \leq \beta \Rightarrow \alpha = \beta * (\beta \rightarrow \alpha) \tag{Div1}$$

$$\forall \alpha, \beta, \gamma \in L, \alpha, \gamma \leq \beta \Rightarrow \gamma * (\beta \rightarrow \alpha) = \alpha * (\beta \rightarrow \gamma) \tag{Div2}$$

and divisible $(L, *, I)$ or L simply, must be integral, i.e., $I = \top$. Hence L will be called a GL-quantale.

Lemma 2.1 (Fang [7], Höhle[20]). *In a GL-quantale L , (1) $\alpha \wedge (\bigvee_{j \in J} \beta_j) = \bigvee_{j \in J} \alpha \wedge \beta_j$, (2) $\alpha * (\beta \wedge \gamma) = (\alpha * \beta) \wedge (\alpha * \gamma)$ for $\alpha, \beta, \gamma, \beta_j \in L, \forall j \in J$.*

Standing assumption. In this paper, if not otherwise specified, L always stands for a GL-quantale.

A quantaloid $\mathcal{D}(L)$ [37] (called the quantaloid of diagonals of L) is constructed in [18], which means that it is a category by the following data:

- (2) Objects: elements in L .
- (2) Morphism sets: $\mathcal{D}(L)(\delta, \eta) = \{\delta \in L \mid \delta \leq \alpha \wedge \beta\}$ for all $\alpha, \beta \in L$.
- (3) Composition: the composition \circ_q of morphisms is defined by

$$\epsilon \circ_q \delta := \delta * (\beta \rightarrow \epsilon) \stackrel{(Div2)}{=} \epsilon * (\beta \rightarrow \delta)$$

for each $\delta \in \mathcal{D}(L)(\alpha, \beta)$ and $\epsilon \in \mathcal{D}(L)(\beta, \gamma)$.

- (4) Identities: the unit of $\mathcal{D}(L)(\alpha, \alpha)$ is α for every $\alpha \in L$.
- (5) Local orders: the partial order of $\mathcal{D}(L)(\delta, \eta)$ is inherited from L such that
 - (i) $\mathcal{D}(L)(\delta, \eta)$ is a complete lattice;
 - (ii) for all $\delta, \delta_j \in \mathcal{D}(L)(\alpha, \beta)$ ($j \in J$), all $\beta, \epsilon_j \in \mathcal{D}(L)(\beta, \gamma)$ ($j \in J$),

$$\left(\bigvee_{j \in J} \epsilon_j\right) \circ_q \delta = \bigvee_{j \in J} (\epsilon_j \circ_q \delta), \quad \epsilon \circ_q \bigvee_{j \in J} \delta_j = \bigvee_{j \in J} (\epsilon \circ_q \delta_j).$$

Remark 2.2. Let $\mathcal{D}(L)$ be the quantaloid of diagonals of a GL-quantale L .

- (i) Given arrows $\delta \in \mathcal{D}(L)(\alpha, \beta)$ and $\epsilon \in \mathcal{D}(L)(\beta, \gamma)$, supremum-preserving maps

$$(-) \circ_q \delta : \mathcal{D}(L)(\beta, \gamma) \rightarrow \mathcal{D}(L)(\alpha, \gamma) \quad \text{and} \quad \epsilon \circ_q (-) : \mathcal{D}(L)(\alpha, \beta) \rightarrow \mathcal{D}(L)(\alpha, \gamma)$$

have right adjoints

$$(-) \swarrow \delta : \mathcal{D}(L)(\alpha, \gamma) \rightarrow \mathcal{D}(L)(\beta, \gamma) \quad \text{and} \quad \epsilon \searrow (-) : \mathcal{D}(L)(\alpha, \gamma) \rightarrow \mathcal{D}(L)(\alpha, \beta),$$

called the left implication and the right implication, respectively, Hence we have

$$(\beta \circ_q \alpha) \leq \gamma \Leftrightarrow \beta \leq (\gamma \swarrow \alpha) \Leftrightarrow \alpha \leq (\beta \searrow \gamma).$$

- (ii) If α, δ, β are in L with $\delta \leq \alpha \wedge \beta$, then δ has two roles in $\mathcal{D}(L)$: one is a morphism from α to β , i.e., $\delta \in \mathcal{D}(L)(\alpha, \beta)$, and another is a morphism from β to α , i.e., $\delta \in \mathcal{D}(L)(\beta, \alpha)$. Thus for $\delta \leq (\alpha \wedge \gamma)$ and $\epsilon \leq (\beta \wedge \gamma)$, the left implication $(\epsilon \swarrow \delta) \in \mathcal{D}(L)(\alpha, \beta)$ for $\delta \in \mathcal{D}(L)(\gamma, \alpha), \epsilon \in \mathcal{D}(L)(\gamma, \beta)$ and the right implication $(\delta \searrow \epsilon) \in \mathcal{D}(L)(\beta, \alpha)$ for $\delta \in \mathcal{D}(L)(\alpha, \gamma), \epsilon \in \mathcal{D}(L)(\beta, \gamma)$, are well-defined. Further, $(\epsilon \swarrow \delta)$ is equal to $(\delta \searrow \epsilon)$, and both are smaller than $\alpha \wedge \beta$. In fact, both $(\epsilon \swarrow \delta)$ and $(\delta \searrow \epsilon)$ are computed by

$$(\epsilon \swarrow \delta) = (\delta \searrow \epsilon) = \alpha \wedge \beta \wedge ((\beta \rightarrow \delta) \rightarrow \epsilon), \tag{Im}$$

Some basic properties of operations, such as \searrow, \swarrow and \circ_q are needed, which are collected in the lemma below. They can be found in many works, for instance, [37].

Lemma 2.3. *In the quantaloid $\mathcal{D}(L)$, the following formulas hold for all arrows $\epsilon \in \mathcal{Q}(\beta, \gamma)$, $\delta, \delta_1, \delta_j \in \mathcal{Q}(\alpha, \beta)$, $\eta, \eta_j \in \mathcal{Q}(\alpha, \gamma)$ for all $j \in J$.*

- (1) $\beta \circ_q \delta = \delta = \delta \circ_q \alpha, (\delta \swarrow \alpha) = \delta = (\beta \searrow \delta)$.
- (2) $\alpha \leq (\delta \searrow \delta_1) \Leftrightarrow \delta \leq \delta_1 \Leftrightarrow \beta \leq (\delta_1 \swarrow \delta)$.
- (3) $(\bigwedge_j \eta_j) \swarrow (\bigwedge_j \delta_j) \geq \bigwedge_j (\eta_j \swarrow \delta_j), (\bigvee_j \eta_j) \swarrow (\bigvee_j \delta_j) \geq \bigwedge_j (\eta_j \swarrow \delta_j)$.
- (4) $(\eta \swarrow \delta) \circ_q \delta \leq \eta$ and $\epsilon \circ_q (\epsilon \searrow \eta) \leq \eta$.

2.2. The base category $\mathbf{Set}\downarrow L$.

An L -set A is a map from a domain set A_{dom} to the underlying GL-quantale L , which is precisely an L -fuzzy set in terminology of Goguen [14]. And intuitively, the valued $A(x) \in L$ is interpreted as the degree to which the element $x \in A_{\text{dom}}$ belong to A .

For two L -sets A and B , if a map $\varphi : A_{\text{dom}} \rightarrow B_{\text{dom}}$ with the property of $A(x) = B(\varphi(x))$ for all $x \in A_{\text{dom}}$, we say that it is degree-preserving from L -set A to B , denoted by $\varphi : A \rightarrow B$ as usual. All L -sets and degree-preserving maps between them form the the base category $\mathbf{Set}\downarrow L$ in our paper. In details, $\mathbf{Set}\downarrow L$ is given by the following data:

- objects: all L -sets, denoted by A, B, C, \dots , in the paper.
- morphisms: the set of morphisms from an L -set A to B is denoted and determined by

$$[A, B]_{\mathbf{Set}\downarrow L} := \{\varphi \mid \varphi : A_{\text{dom}} \rightarrow B_{\text{dom}} \text{ is a map with } A = B \circ \varphi\} \\ = \{\varphi \mid \varphi : A \rightarrow B \text{ is a degree-preserving map}\}.$$

- composition: $(\psi \circ \varphi)(x) = \psi(\varphi(x))$ for all $x \in A_{\text{dom}}$ when $\varphi \in [A, B]_{\mathbf{Set}\downarrow L}$ and $\psi \in [B, C]_{\mathbf{Set}\downarrow L}$.
- identity: for $A \in \mathbf{Set}\downarrow L$, the identity id_A in $[A, A]_{\mathbf{Set}\downarrow L}$ is the identity map $\text{id}_{A_{\text{dom}}}$ on the domain set A_{dom} .

Remark 2.4. In $\mathbf{Set}\downarrow L$, the terminal object, is the L -set id_L , the identity map on L , with its domain set L .

If an L -set B fulfills $B_{\text{dom}} \subseteq A_{\text{dom}}$ and $A(x) = B(x)$ for all $x \in B_{\text{dom}}$, then we say B is a saturated L -subset of A . For an L -set A and $\delta \in L$, we write A^δ , for the saturated L -subset of A such that $A^\delta(s) = \delta$ whenever

$$s \in (A^\delta)_{\text{dom}} := \{x \in A_{\text{dom}} \mid A(x) = \delta\},$$

called the δ -saturated L -subset of A concretely. No loss of generality, we assume the domain set of A is nonempty. Under these notions, we have the following proposition.

Proposition 2.5. Let $\{A^j\}_{j \in J}$ be a set-indexed family of L -sets. The product of $\{A^j\}_{j \in J}$ in $\mathbf{Set}\downarrow L$ is an L -set P , given by

(i) $P = \text{id}_L$ if $J = \emptyset$.

(ii) When $J \neq \emptyset$, $P_{\text{dom}} = \bigcup_{\alpha \in L} \{(x_j)_{j \in J} \mid A^j(x_j) = \alpha \text{ for all } j \in J\}$ such that $P((x_j)_{j \in J}) = \alpha$ whenever

$$(x_j)_{j \in J} \in \{(x_j)_{j \in J} \mid A^j(x_j) = \alpha \text{ for all } j \in J\}$$

with some $\alpha \in L$. In the case, the projection $p_j : P \rightarrow A_j$ from the product P to A^j in $\mathbf{Set}\downarrow L$, here $j \in J$, is determined by $p_j((x_j)_{j \in J}) = x_j$ whenever $(x_j)_{j \in J} \in P_{\text{dom}}$. In particular, The product of two L -sets A and B is denoted by $A \times B$ simply, and of course

$$(A \times B)_{\text{dom}} = \{(x, y) \in A_{\text{dom}} \times B_{\text{dom}} \mid A(x) = B(y)\}.$$

Notice that in Proposition 2.5, for the product of two L -sets A and B , it is possible that the domain set $(A \times B)_{\text{dom}}$ is empty as a referee pointed out by the example: let an L -set A with $A_{\text{dom}} = \{x\}$ and $A(x) = \alpha$, and B with $B_{\text{dom}} = \{y\}$ and $B(y) = \beta$, where $\alpha \neq \beta$. Then $(A \times B)_{\text{dom}} = \emptyset$ according to the definition of $A \times B$. In this case, we confirm here that $A \times B$ as an object in $\mathbf{Set}\downarrow L$, is the unique empty map $\emptyset \rightarrow L$ since the empty set \emptyset is the initial object in \mathbf{Set} , and both projections $p_A : A \times B \rightarrow A$ and $p_B : A \times B \rightarrow B$ are empty. By the way, $\emptyset \rightarrow L$ is the initial object in $\mathbf{Set}\downarrow L$, precisely.

2.3. The power L -set

For an L -set A , an L -set $\mathcal{P}A$ [19] could be constructed by

$$(\mathcal{P}A)_{\text{dom}} := \bigcup_{\delta \in L} \{(\delta, f) \in L \times L^{A_{\text{dom}}} \mid f \leq \delta \wedge A\}$$

with $\mathcal{P}A(\delta, f) = \delta$ for $(\delta, f) \in (\mathcal{P}A)_{\text{dom}}$. And $\mathcal{P}A$ will be called the power L -set of A , and a pair $(\delta, f) \in (\mathcal{P}A)_{\text{dom}}$ is called a δ -limited L -subset (or a limited L -subset) of A sometimes. Note that for a limited L -subset (δ, f) of A and an $x \in A_{\text{dom}}$, $f(x)$ is a morphism in $\mathcal{D}(L)(A(x), \delta)$, and we write $(\delta, f) = (\epsilon, g)$ for $\delta = \epsilon$ and $f = g$.

Remark 2.6. There exists a partial order \leq on $(\mathcal{P}A)_{\text{dom}}$ defined by

$$\forall (\delta, f), (\epsilon, g) \in (\mathcal{P}A)_{\text{dom}}, \quad (\delta, f) \leq (\epsilon, g) \Leftrightarrow \delta = \epsilon \text{ and } f \leq g,$$

which is called the underlying order on $\mathcal{P}A$. Further, if we still write \leq for the restriction of the underlying order \leq to the set $(\mathcal{P}A)_{\text{dom}}^{\delta}$ with $\delta \in L$, $((\mathcal{P}A)_{\text{dom}}^{\delta}, \leq)$ is a complete lattice with the bottom element $(\delta, \perp_{A_{\text{dom}}})$ and the top element $(\delta, \delta \wedge A)$, where $\perp_{A_{\text{dom}}}$ and $\delta \wedge A$ are determined by $\perp_{A_{\text{dom}}}(x) = \perp$ and $(\delta \wedge A)(x) = \delta \wedge A(x)$ for all $x \in A_{\text{dom}}$, respectively. Thus for a family of $\{(\delta, f_j) \in (\mathcal{P}A)_{\text{dom}}^{\delta} \mid j \in J\}$, the supremum and the infimum of it are given by

$$\bigvee_{j \in J} (\delta, f_j) = \left(\delta, \bigvee_{j \in J} f_j\right) \text{ and } \bigwedge_{j \in J} (\delta, f_j) = \left(\delta, \bigwedge_{j \in J} f_j\right),$$

here $\left(\bigvee_{j \in J} f_j\right)(x) = \bigvee_{j \in J} f_j(x)$, $\left(\bigwedge_{j \in J} f_j\right)(x) = \bigwedge_{j \in J} f_j(x)$ for $x \in A_{\text{dom}}$.

There exists an L -relation on $\mathcal{P}A$, denoted by $\mathcal{P}A(-, -)$ or $\mathcal{P}A$ simply, in the sense that $\mathcal{P}A : (\mathcal{P}A)_{\text{dom}} \times (\mathcal{P}A)_{\text{dom}} \rightarrow L$ is a map such that

$$\mathcal{P}A((\delta, f), (\epsilon, g)) \in \mathcal{D}(L)(\delta, \epsilon),$$

which is defined by

$$\begin{aligned} \mathcal{P}A((\delta, f), (\epsilon, g)) &= \bigwedge_{x \in A_{\text{dom}}} (g(x) \swarrow f(x)) \\ &\stackrel{(\text{lm})}{=} \bigwedge_{x \in A_{\text{dom}}} (\delta \wedge \epsilon \wedge ((\delta \rightarrow f(x)) \rightarrow g(x))) \end{aligned}$$

In particular, $\mathcal{P}A((\delta, f), (\delta, g)) = \bigwedge_{x \in A_{\text{dom}}} \delta \wedge ((\delta \rightarrow f(x)) \rightarrow g(x))$, and further, by using the condition (Div), $\mathcal{P}A((\delta, f), (\delta, g)) = \bigwedge_{x \in A_{\text{dom}}} \delta * (f(x) \rightarrow g(x))$. Note that $(\delta, f) \leq (\epsilon, g)$ if and only if $\mathcal{P}A((\delta, f), (\epsilon, g)) = \delta = \epsilon$.

To keep notations simple, **we will write $x \in A$ instead of $x \in A_{\text{dom}}$ for an element x in the domain set A_{dom} of any L -set A if no confusion exists.**

Each degree-preserving map $\varphi : C \rightarrow D$ between L -sets can induce two degree-preserving maps between the power L -sets. One is denoted by $\varphi^{\rightarrow} : \mathcal{P}C \rightarrow \mathcal{P}D$, such that $\varphi^{\rightarrow}(\delta, g) = (\delta, \varphi^{\rightarrow}(g))$ for $(\delta, g) \in \mathcal{P}C$, here $\varphi^{\rightarrow}(g)$ is determined by for each $y \in D$, $\varphi^{\rightarrow}(g)(y) = \bigvee_{\varphi(x)=y} g(x)$, or equivalently,

$$\varphi^{\rightarrow}(g)(y) = \begin{cases} \bigvee \{g(x) \mid \varphi(x) = y, x \in C^{D(y)}\}, & y \in \varphi(C^{D(y)}), \\ \perp, & \text{others,} \end{cases}$$

since $\varphi : C \rightarrow D$ is a degree-preserving map. And another is denoted by $\varphi^{\leftarrow} : \mathcal{P}D \rightarrow \mathcal{P}C$ such that $\varphi^{\leftarrow}(\epsilon, f) = (\epsilon, \varphi^{\leftarrow}(f))$ for $(\epsilon, f) \in \mathcal{P}D$, here $\varphi^{\leftarrow}(f)$ is defined by $\varphi^{\leftarrow}(f)(x) = f(\varphi(x))$ for an $x \in C$. The pair of degree-preserving maps $\varphi^{\rightarrow} : \mathcal{P}C \rightarrow \mathcal{P}D$ and $\varphi^{\leftarrow} : \mathcal{P}D \rightarrow \mathcal{P}C$ form an adjunction in the sense that for $(\delta, g) \in \mathcal{P}C$, $(\epsilon, f) \in \mathcal{P}D$, $\mathcal{P}D(\varphi^{\rightarrow}(\delta, g), (\epsilon, f)) = \mathcal{P}C((\delta, g), \varphi^{\leftarrow}(\epsilon, f))$ holds.

At the end of the section, we offer a proposition to collect the properties of L -relation $\mathcal{P}A$; and they can be found in many works, for instance [19, 38], which is useful to the following sections.

Proposition 2.7. For all $(\delta, f), (\epsilon, g), (\eta, h) \in \mathcal{PA}$, the following are valid.

- (1) $\delta = \mathcal{PA}((\delta, f), (\delta, f))$.
- (2) $\mathcal{PA}((\epsilon, g), (\eta, h)) \circ_q \mathcal{PA}((\delta, f), (\epsilon, g)) \leq \mathcal{PA}((\delta, f), (\eta, h))$.
- (3) For each δ -limited L -subset (δ, f) , $\mathcal{PA}((\delta, f), -)$ is order-preserving and $\mathcal{PA}(-, (\delta, f))$ is order-inverse, and further for a family $\{(\epsilon, g_j)\}_{j \in J}$,
 - (a) $\bigwedge_{j \in J} \mathcal{PA}((\delta, f), (\epsilon, g_j)) = \mathcal{PA}((\delta, f), \bigwedge_{j \in J} (\epsilon, g_j))$,
 - (b) $\bigwedge_{j \in J} \mathcal{PA}((\epsilon, g_j), (\delta, f)) = \mathcal{PA}(\bigvee_{j \in J} (\epsilon, g_j), (\delta, f))$.

2.4. Categorical concepts.

The reader is referred to [1] for notions and results in category theory. In the subsection, some categorical concepts needed in the paper, are introduced. By a category we mean a construct \mathcal{C} whose objects are structured L -sets, i.e. pairs (A, ξ) where A is an L -set and ξ a \mathcal{C} -structure on A , whose morphisms $\varphi : (A, \xi) \rightarrow (B, \eta)$ are suitable degree-preserving maps from A to B and whose composition is the composition of degree-preserving maps. Hence a category in the paper is a concrete category over $\mathbf{Set} \downarrow L$ and the forgetful functor is obvious. We simply write \mathbf{A} for a categorical object (A, ξ) sometimes.

Definition 2.8. A category \mathcal{C} is said to be topological over $\mathbf{Set} \downarrow L$ if for any L -set A , any family $\{(A_j, \xi_j)\}_{j \in J}$ of \mathcal{C} -objects and any family $\{\varphi_j : A \rightarrow A_j\}_{j \in J}$ of degree-preserving maps, indexed by a class J , there exists a unique \mathcal{C} -structure ξ on A which is initial with respect to $\{\varphi_j : A \rightarrow (A_j, \xi_j)\}_{j \in J}$, i.e., for a \mathcal{C} -object (C, η) , a $\psi : (C, \eta) \rightarrow (A, \xi)$ is a \mathcal{C} -morphism if and only if for every $j \in J$ the composite $\varphi_j \circ \psi : (C, \eta) \rightarrow (A_j, \xi_j)$ is a \mathcal{C} -morphism.

Definition 2.9. A category \mathcal{C} is said to be Cartesian-closed provided

- (1) \mathcal{C} has the terminal object \mathbf{T} in the sense that there is exactly one \mathcal{C} -morphism from every \mathcal{C} -object \mathbf{C} to \mathbf{T} .
- (2) For each pair (\mathbf{A}, \mathbf{B}) of \mathcal{C} -objects there exists a product $\mathbf{A} \times \mathbf{B}$ in \mathcal{C} .
- (3) For each pair of \mathcal{C} -objects \mathbf{A} and \mathbf{B} , there exists a \mathcal{C} -object $\mathbf{B}^{\mathbf{A}}$, called function space, such that there exists a \mathcal{C} -morphism $E_{\mathbf{A}, \mathbf{B}} : \mathbf{B}^{\mathbf{A}} \times \mathbf{A} \rightarrow \mathbf{B}$ (called evaluation morphism) satisfying the universal property that for each \mathcal{C} -object \mathbf{C} and each \mathcal{C} -morphism $\psi : \mathbf{C} \times \mathbf{A} \rightarrow \mathbf{B}$, there is a unique \mathcal{C} -morphism $\bar{\psi} : \mathbf{C} \rightarrow \mathbf{B}^{\mathbf{A}}$ such that $E_{\mathbf{A}, \mathbf{B}} \circ (\bar{\psi} \times id_{\mathbf{A}}) = \psi$.

3. Limited L -filters on an L -set and their products.

The contents of the section are twofold: one is to generalize the concept of \top -filters from a crisp set to any L -set such that the concept of limited L -filters is obtained; Another is to determine the products of limited L -filters on L -sets.

3.1. Limited L -filters on an L -set.

In the subsection, a concept of limited L -filters is proposed and some examples are offered.

Definition 3.1 (Fang and Yue [13] for quantaloid-enriched categories). Let A be an L -set with the nonempty domain set and \mathbb{F} be a nonempty subset of $(\mathcal{PA})^\delta$ for an element $\delta \in L$. The pair (δ, \mathbb{F}) , or \mathbb{F} briefly, is said to be a δ -limited L -filter, or justly limited L -filter sometimes, on A provided the subset \mathbb{F} satisfies the following axioms:

- (T1) For all $(\delta, g) \in \mathbb{F}$, $\delta = \bigvee_{x \in A^\delta} g(x)$.
- (T2) $(\delta, f) \in (\mathcal{PA})^\delta$ with $\delta = \bigvee_{(\delta, g) \in \mathbb{F}} \mathcal{PA}((\delta, g), (\delta, f))$ means $(\delta, f) \in \mathbb{F}$.
- (T3) Both $(\delta, g) \in \mathbb{F}$ and $(\delta, f) \in \mathbb{F}$ mean $(\delta, g \wedge f) \in \mathbb{F}$.

Write $F_\top(A)$ for the L -set with the domain set of all limited L -filters on A , and for a limited L -filter (δ, \mathbb{F}) , $F_\top(A)(\delta, \mathbb{F}) = \delta$. For a fixed $\delta \in L$, the δ -saturated L -subset of $F_\top(A)$ is denoted by $F_\top^\delta(A)$.

According to Definition 3.1, there exists an order \leq_F on the domain set of $F_\top(A)$ defined by

$$(\delta, \mathbb{F}) \leq_F (\epsilon, \mathbb{G}) \text{ if and only if } \delta = \epsilon \text{ and } \mathbb{F} \subseteq \mathbb{G},$$

and then $(F_\top(A), \leq_F)$ becomes a partially ordered set. Here notice that the condition (T2) assures that the subset \mathbb{F} of a limited L -filter (δ, \mathbb{F}) must be an upper set with the underlying order on $\mathcal{P}A$.

Definition 3.2. Let A be an L -set, and \mathbb{S} be a nonempty subset of $(\mathcal{P}A)^\delta$ for $\delta \in L$. The pair (δ, \mathbb{S}) , briefly \mathbb{S} sometimes, is said to be a δ -limited filter-base or limited filter-base on A if the subset \mathbb{S} satisfies the following conditions:

(TB1) For all $(\delta, g) \in \mathbb{S}, \delta = \bigvee_{x \in A^\delta} g(x),$

(TB2) For each $(\delta, f_1), (\delta, f_2) \in \mathbb{S}, \delta = \bigvee_{(\delta, g) \in \mathbb{S}} \mathcal{P}A((\delta, g), (\delta, f_1 \wedge f_2)).$

Every δ -limited filter-base (δ, \mathbb{S}) generates a δ -limited L -filter (denoted by $(\delta, \mathbb{F}_\mathbb{S})$ or $\mathbb{F}_\mathbb{S}$ simply), defined by

$$\mathbb{F}_\mathbb{S} := \{(\delta, g) \in (\mathcal{P}A)^\delta \mid \delta = \bigvee_{(\delta, f) \in \mathbb{S}} \mathcal{P}A((\delta, f), (\delta, g))\}.$$

In the case, (δ, \mathbb{S}) is called a δ -limited filter-base of $(\delta, \mathbb{F}_\mathbb{S})$. Certainly, every limited L -filter is a limited filter-base of itself. There are examples of limited L -filters useful to next sections.

Example 3.3. (1) For an L -set A with $x \in A$, a pair $(A(x), [x]_{A(x)})$, given by

$$[x]_{A(x)} := \{(A(x), f) \in (\mathcal{P}A)^{A(x)} \mid f(x) = A(x)\},$$

is a limited L -filter on A in the sense of Definition 3.1, called the degree-limited L -filter of x . Furthermore, we observe that the degree-limited L -filter $(A(x), [x]_{A(x)})$ has a limited filter-base of one element determined by $\{(A(x), f_x)\}$, here and in the following, f_x denote the map from the domain set of A to L such that for each y in the domain set of A , $f_x(y) = A(x)$ if $y = x$, and $= \perp$ otherwise. Notice that $(A(x), [x]_{A(x)})$ is ultrafilter with respect to \leq_F . In fact, if $(A(x), [x]_{A(x)}) \leq_F (A(x), \mathbb{F})$, i.e., $[x]_{A(x)} \subseteq \mathbb{F}$, for some limited \top -filter on A , then for each $(A(x), f) \in \mathbb{F}$, $(A(x), f_x \wedge f) \in \mathbb{F}$ follows from the $(A(x), f_x) \in [x]_{A(x)} \subseteq \mathbb{F}$. So,

$$f(x) = \bigvee_{y \in A} (f_x \wedge f)(y) = \bigvee_{y \in A} (f_x(y) \wedge f(y)) = f_x(x) = A(x),$$

which means that $(A(x), f) \in [x]_{A(x)}$ already.

(2) Let $\varphi : C \rightarrow D$ be a degree-preserving map between L -sets.

(i) For $(\delta, \mathbb{F}) \in F_\top(C)$, the pair $(\delta, \mathbb{B}_\mathbb{F})$ is a δ -limited filter-base on D , here $\mathbb{B}_\mathbb{F} := \{(\delta, \varphi^\rightarrow(f)) \mid (\delta, f) \in \mathbb{F}\}$. The limited L -filter $(\delta, \varphi^\rightarrow(\mathbb{F}))$ generated by $(\delta, \mathbb{B}_\mathbb{F})$ is called the image of (δ, \mathbb{F}) under φ , denoted by $\varphi^\rightarrow(\delta, \mathbb{F})$. Especially, when $x \in C$, the image of the degree-limited L -filter $(C(x), [x]_{C(x)})$ of x under φ is the degree-limited L -filter $(C(x), [\varphi(x)]_{C(x)})$, where $C(x) = D(\varphi(x))$.

(ii) The degree-preserving map $\varphi^\rightarrow : (F_\top(C), \leq_F) \rightarrow (F_\top(D), \leq_F)$ between partially ordered sets is order-preserving in the sense of $(\delta, \varphi^\rightarrow(\mathbb{F})) \leq_F (\delta, \varphi^\rightarrow(\mathbb{G}))$ when $(\delta, \mathbb{F}) \leq_F (\delta, \mathbb{G})$ in $(F_\top(C), \leq_F)$.

(iii) For $(\sigma, \mathbb{G}) \in F_\top(D)$, the pair $(\sigma, \mathbb{S}_\mathbb{G})$ determined by

$$\mathbb{S}_\mathbb{G} = \{(\sigma, \varphi^\leftarrow(g)) \in (\mathcal{P}C)^\sigma \mid (\sigma, g) \in \mathbb{G}\},$$

is a σ -limited filter-base on C provided $\bigvee_{x \in C^\sigma} g(\varphi(x)) = \sigma$ for any $(\sigma, g) \in \mathbb{G}$, and in this case the limited L -filter $(\sigma, \varphi^\leftarrow(\mathbb{G}))$ on C generated by $(\sigma, \mathbb{S}_\mathbb{G})$ is determined by

$$\varphi^\leftarrow(\mathbb{G}) = \{(\sigma, f) \in (\mathcal{P}C)^\sigma \mid \bigvee_{(\sigma, g) \in \mathbb{G}} \mathcal{P}C((\sigma, \varphi^\leftarrow(g)), (\sigma, f)) = \sigma\},$$

which is called the inverse image of (σ, \mathbb{G}) under φ and denoted by $\varphi^\leftarrow(\sigma, \mathbb{G})$.

Example 3.4. If A is an L -set with $A^\delta \neq \emptyset$ for a $\delta \in L$, and $i_\delta : A^\delta \rightarrow A$ denotes the inclusion degree-preserving map, then $(\delta, i_\delta^{\leftarrow}(\mathbf{G}))$ is a limited L -filter on A^δ for a limited L -filter (δ, \mathbf{G}) on A , and we have $(\delta, \mathbf{G}) \leq_F (\delta, i_\delta^{\rightarrow}(i_\delta^{\leftarrow}(\mathbf{G})))$ in general. Further, if (δ, \mathbf{F}) is a limited L -filter on A^δ , then $(\delta, i_\delta^{\leftarrow}(i_\delta^{\rightarrow}(\mathbf{F}))) = (\delta, \mathbf{F})$ holds.

Example 3.5. Let id_L be the terminal object (Cf. Remark 2.4) in $\mathbf{Set}\downarrow L$. Then the power L -set $\mathcal{P}id_L$ of id_L is determined by the following data:

- (i) the domain set of it is $\{(\alpha, \alpha \wedge S) \mid S \in L^L \text{ such that } S \leq id_L \text{ and } \alpha \in L\}$.
- (ii) $\mathcal{P}L(\alpha, \alpha \wedge S) = \alpha$ for each $(\alpha, \alpha \wedge S)$ in the domain set.

By using (T1) and (T2) of Definition 3.1, the degree-limited L -filter $(\alpha, [\alpha]_\alpha)$ of $\alpha \in L$ must have the form of

$$[\alpha]_\alpha = \{(\alpha, \alpha \wedge S) \mid S \in L^L \text{ such that } S \leq id_L \text{ and } S(\alpha) = \alpha\}.$$

3.2. The products of limited L -filters

We will focus on the products of limited L -filters on L -sets. For this, we need the lemma below.

Lemma 3.6. Let (δ, \mathbf{S}_i) be a δ -limited filter-base of $(\delta, \mathbb{F}_i) \in F_\top(C_i)$, here $i = 1, 2$ and $\delta \in L$. Then the pair $(\delta, \mathbf{S}_1 \times \mathbf{S}_2)$, determined by

$$\mathbf{S}_1 \times \mathbf{S}_2 = \{(\delta, f_1 \times f_2) \in (\mathcal{P}(C_1 \times C_2))^\delta \mid (\delta, f_1) \in \mathbf{S}_1 \text{ and } (\delta, f_2) \in \mathbf{S}_2\},$$

is a δ -limited filter-base, here for any $(\delta, f_1) \in (\mathcal{P}C_1)^\delta$ and $(\delta, f_2) \in (\mathcal{P}C_2)^\delta$,

$$f_1 \times f_2((x_1, x_2)) = f_1(x_1) \wedge f_2(x_2), \quad \forall (x_1, x_2) \in (C_1 \times C_2).$$

Concretely, the δ -limited filter-base $(\delta, \mathbf{S}_1 \times \mathbf{S}_2)$ generates a δ -limited L -filter, called the product of (δ, \mathbb{F}_1) and (δ, \mathbb{F}_2) and denoted by $(\delta, \mathbb{F}_1 \times \mathbb{F}_2)$ or $\mathbb{F}_1 \times \mathbb{F}_2$ simply, determined by

$$\mathbb{F}_1 \times \mathbb{F}_2 = \{(\delta, f) \in (\mathcal{P}(C_1 \times C_2))^\delta \mid \delta = \bigvee_{(\delta, f_i) \in \mathbf{S}_i, i=1,2} \mathcal{P}(C_1 \times C_2)((\delta, f_1 \times f_2), (\delta, f))\}.$$

In particular, the binary operation $(-) \times (-)$ of limited L -filters preserves the partial order \leq_F in each argument, i.e., $(\delta, \mathbb{H}_1 \times \mathbb{F}_2) \leq_F (\delta, \mathbb{H}_1 \times \mathbb{G}_2)$ for $(\delta, \mathbb{H}_1) \in F_\top(C_1)$ is $(\delta, \mathbb{F}_2 \leq_F (\delta, \mathbb{G}_2)$ and $(\delta, \mathbb{F}_1 \times \mathbb{H}_2) \leq_F (\delta, \mathbb{G}_1 \times \mathbb{H}_2)$ for $(\delta, \mathbb{H}_2) \in F_\top(C_2)$ if $(\delta, \mathbb{F}_1) \leq (\delta, \mathbb{G}_1)$.

Proof. For any pair $(\delta, f), (\delta, g) \in (\mathbf{S}_1 \times \mathbf{S}_2)$, there exist $(\delta, f_1), (\delta, g_1) \in \mathbf{S}_1$ and $(\delta, f_2), (\delta, g_2) \in \mathbf{S}_2$ such that $(\delta, f) = (\delta, f_1 \times f_2)$ and $(\delta, g) = (\delta, g_1 \times g_2)$. Firstly,

$$\delta = \mathcal{P}C_i((\delta, f_i), (\delta, f_i)) \leq \bigvee_{(\delta, h_i) \in \mathbf{S}_i} \mathcal{P}C_i((\delta, h_i), (\delta, f_i)) \quad (i = 1, 2)$$

follows immediately from Proposition 2.7 (1), and then, we observe that

$$\delta = \bigvee_{(\delta, h_i) \in \mathbf{S}_i, (i=1,2)} \mathcal{P}C_1((\delta, h_1), (\delta, f_1)) \wedge \mathcal{P}C_2((\delta, h_2), (\delta, f_2)),$$

by means of Lemma 2.1 (1). Since for each $(\delta, h_i) \in \mathbf{S}_i$ ($i = 1, 2$), by Lemma 2.1 (2)

$$\begin{aligned} \mathcal{P}(C_1 \times C_2)((\delta, h_1 \times h_2), (\delta, f_1 \times f_2)) &= \bigwedge_{(x_1, x_2) \in C_1 \times C_2} \delta * ((h_1(x_1) \wedge h_2(x_2)) \rightarrow (f_1(x_1) \wedge f_2(x_2))) \\ &\geq \bigwedge_{(x_1, x_2) \in C_1 \times C_2} \delta * [(h_1(x_1) \rightarrow f_1(x_1)) \wedge (h_2(x_2) \rightarrow f_2(x_2))] \\ &\geq \left(\bigwedge_{x_1 \in C_1} \delta * (h_1(x_1) \rightarrow f_1(x_1)) \right) \wedge \left(\bigwedge_{x_2 \in C_2} \delta * (h_2(x_2) \rightarrow f_2(x_2)) \right) \\ &= \mathcal{P}C_1((\delta, h_1), (\delta, f_1)) \wedge \mathcal{P}C_2((\delta, h_2), (\delta, f_2)), \end{aligned}$$

we use the above observation together with the condition (TB2) of satisfying by \mathbb{B}_i ($i = 1, 2$) to conclude that

$$\delta = \bigvee_{(\delta, h_i) \in \mathbb{S}_i, (i=1,2)} \mathcal{P}(C_1 \times C_2)((\delta, h_1 \times h_2), (\delta, f_1 \times f_2))$$

since for each $(\delta, f_1), (\delta, f_2) \in \mathbb{S}$,

$$\delta = \bigvee_{(\delta, h_1) \in \mathbb{S}_1} \mathcal{P}C_1((\delta, h_1), (\delta, f_1)) \text{ and } \delta = \bigvee_{(\delta, h_2) \in \mathbb{S}_2} \mathcal{P}C_2((\delta, h_2), (\delta, f_2)).$$

Certainly, $\bigvee_{(\delta, h_i) \in \mathbb{S}_i, (i=1,2)} \mathcal{P}(C_1 \times C_2)((\delta, h_1 \times h_2), (\delta, g_1 \times g_2)) = \delta$. From all above, (TB2) of satisfying by $\mathbb{S}_1 \times \mathbb{S}_2$ follows from

$$\begin{aligned} \delta &= \bigvee_{(\delta, h_i) \in \mathbb{S}_i, (i=1,2)} \mathcal{P}(C_1 \times C_2)((\delta, h_1 \times h_2), (\delta, f_1 \times f_2)) \wedge \mathcal{P}(C_1 \times C_2)((\delta, h_1 \times h_2), (\delta, g_1 \times g_2)) \\ &= \bigvee_{(\delta, h_i) \in \mathbb{S}_i, (i=1,2)} \mathcal{P}(C_1 \times C_2)((\delta, h_1 \times h_2), (\delta, (f_1 \times f_2) \wedge (g_1 \times g_2))) \quad (\text{By Proposition 2.7(3)}) \\ &= \bigvee_{(\delta, h_i) \in \mathbb{S}_i, (i=1,2)} \mathcal{P}(C_1 \times C_2)((\delta, h_1 \times h_2), (\delta, f \wedge g)) \end{aligned}$$

For any $(\delta, f_1) \in \mathbb{S}_1$ and $(\delta, f_2) \in \mathbb{S}_2$, we have

$$\delta = \bigvee_{x_i \in (C_i)^\delta, (i=1,2)} f_1(x_1) \wedge f_2(x_2) = \bigvee_{x_i \in (C_i)^\delta, (i=1,2)} f_1 \times f_2((x_1, x_2))$$

since $\delta = \bigvee_{x_i \in (C_i)^\delta} f_i(x_i)$ for the $(\delta, f_i) \in \mathbb{S}$ ($i = 1, 2$), that is to say that the condition (TB1) is satisfied by $\mathbb{S}_1 \times \mathbb{S}_2$.

Finally, the conclusion that the binary operation $(-)\times(-)$ of limited L -filters preserves the partial order \leq_F in each argument, follows from the definition of the product of two limited \top -filters. \square

We will show two lemmas in preparations for the construction of function spaces in Section 6.

Lemma 3.7. Let $\varphi : A \rightarrow B$ and $\psi : C \rightarrow D$ be degree-preserving maps, $(\delta, \mathbb{F}_1) \in F_\top(A)$, $(\delta, \mathbb{F}_2) \in F_\top(C)$. Then

$$(\delta, (\varphi \times \psi)^\Rightarrow(\mathbb{F}_1 \times \mathbb{F}_2)) = (\delta, \varphi^\Rightarrow(\mathbb{F}_1) \times \psi^\Rightarrow(\mathbb{F}_2)),$$

here the degree-preserving map $\varphi \times \psi : A \times C \rightarrow B \times D$ is defined by sending each $(x, y) \in (A \times C)$ to $(\varphi(x), \psi(y))$.

Proof. First of all, we observe that for $(\delta, f) \in \mathcal{P}A$, $(\delta, g) \in \mathcal{P}C$, it holds that

$$(\delta, (\varphi \times \psi)^\rightarrow(f \times g)) = (\delta, \varphi^\rightarrow(f) \times \psi^\rightarrow(g))$$

By means of the formula above, the claimed equality

$$(\delta, (\varphi \times \psi)^\Rightarrow(\mathbb{F}_1 \times \mathbb{F}_2)) = (\delta, \varphi^\Rightarrow(\mathbb{F}_1) \times \psi^\Rightarrow(\mathbb{F}_2))$$

is true since for each $(\delta, h) \in (\mathcal{P}(B \times D))^\delta$,

$$\begin{aligned} (\delta, h) \in (\varphi \times \psi)^\Rightarrow(\mathbb{F}_1 \times \mathbb{F}_2) &\Leftrightarrow \delta = \bigvee_{(\delta, f_i) \in \mathbb{F}_i, (i=1,2)} \mathcal{P}(B \times D)((\delta, (\varphi \times \psi)^\rightarrow(f_1 \times f_2)), (\delta, h)) \\ &\Leftrightarrow \delta = \bigvee_{(\delta, f_i) \in \mathbb{F}_i, (i=1,2)} \mathcal{P}(B \times D)((\delta, \varphi^\rightarrow(f_1) \times \psi^\rightarrow(f_2)), (\delta, h)), \end{aligned}$$

and the last is equivalent to $(\delta, h) \in \varphi^\Rightarrow(\mathbb{F}_1) \times \psi^\Rightarrow(\mathbb{F}_2)$ by Lemma 3.6 and the definition of $\varphi^\Rightarrow(\mathbb{F}) \times \psi^\Rightarrow(\mathbb{G})$. \square

Lemma 3.8. *If $p_C : C \times D \rightarrow C$ and $p_D : C \times D \rightarrow D$ be the projections, then for any $(\delta, \mathbb{F}) \in F_{\top}(C)$, $(\delta, \mathbb{G}) \in F_{\top}(D)$ and $(\delta, \mathbb{K}) \in F_{\top}(C \times D)$, (1) $(\delta, p_C^{\rightarrow}(\mathbb{F} \times \mathbb{G})) = (\delta, \mathbb{F})$ and $(\delta, p_D^{\rightarrow}(\mathbb{F} \times \mathbb{G})) = (\delta, \mathbb{G})$; (2) $(\delta, p_C^{\rightarrow}(\mathbb{K}) \times p_D^{\rightarrow}(\mathbb{K})) \leq_F (\delta, \mathbb{K})$.*

Proof. In the proof, we only show the first equality of (1) for example. In general, for each $(\delta, f) \in \mathbb{F}$ and each $(\delta, g) \in \mathbb{G}$, $(\delta, p_C^{\rightarrow}(f \times g)) = (\delta, f)$ holds since for all $x \in C^{\delta}$, here $\delta \in L$, we have

$$p_C^{\rightarrow}(f \times g)(x) = \bigvee_{p_C(u,v)=x} f \times g(u,v) = f(x) \wedge \bigvee_{v \in D^{\delta}} g(v) = f(x) \wedge \delta = f(x),$$

where $\bigvee_{v \in D^{\delta}} g(v) = \delta$ owing to $(\delta, g) \in \mathbb{G}$. Thus for each $(\delta, h) \in (\mathcal{P}C)^{\delta}$,

$$(\delta, h) \in \mathbb{F} \Leftrightarrow \delta = \bigvee_{(\delta, f) \in \mathbb{F}} \mathcal{P}C((\delta, f), (\delta, h)) \Leftrightarrow \delta = \bigvee_{(\delta, f) \in \mathbb{F}, (\delta, g) \in \mathbb{G}} \mathcal{P}C((\delta, p_C^{\rightarrow}(f \times g)), (\delta, h)) \Leftrightarrow (\delta, h) \in p_C^{\rightarrow}(\mathbb{F} \times \mathbb{G}).$$

Consequently, $(\delta, p_C^{\rightarrow}(\mathbb{F} \times \mathbb{G})) = (\delta, \mathbb{F})$. \square

4. Top L-convergence structures.

In the section, a concept of top L-convergence spaces based on L-sets other than crisp sets, is proposed by using limited L-filters. Then for the purpose of the construction of our function spaces based on L-sets in the last section, we show that the concrete category of top L-convergence spaces and degree-preserving maps is topological over the slice category $\mathbf{Set} \downarrow L$, and give the terminal objects in the concrete category of top L-convergence spaces.

In order to introduce the concept of top L-convergence spaces, we construct an L-set $D(C)$ for each L-set C , which is defined by

$$D(C)_{\text{dom}} := \bigcup_{\delta \in L} \{S \mid S \subseteq (C)^{\delta}_{\text{dom}}\},$$

and $D(C)(S) = \delta$ when $S \subseteq (C)^{\delta}_{\text{dom}}$ with some $\delta \in L$. Further there exists an partial order, written as \subseteq still, on $D(C)_{\text{dom}}$ determined by for any two $S_1, S_2 \in D(C)_{\text{dom}}$,

$$S_1 \subseteq S_2 \text{ whenever there is a } \delta \in L \text{ such that } S_1, S_2 \subseteq C^{\delta}_{\text{dom}} \text{ with } S_1 \subseteq S_2.$$

In the following, we will still write $x \in A$ instead of $x \in A_{\text{dom}}$ for an element x in the domain set A_{dom} of any L-set so that the notations are simplified.

Definition 4.1. *Let C be an L-set with the nonempty domain set. A degree preserving map*

$$\text{lim} : F_{\top}(C) \rightarrow D(C),$$

is called a top L-convergence structure on C if it sends each δ -limited L-filter $(\delta, \mathbb{F}) \in F_{\top}(C)$ to a subset $\text{lim}(\delta, \mathbb{F})$ of C^{δ} and satisfies the following conditions:

- (TC1) $x \in \text{lim}(\delta, [x]_{\delta})$ for each $x \in C^{\delta}$ and $\delta \in L$.
- (TC2) $(\delta, \mathbb{F}) \leq_F (\delta, \mathbb{G})$ means $\text{lim}(\delta, \mathbb{F}) \subseteq \text{lim}(\delta, \mathbb{G})$ for $(\delta, \mathbb{F}), (\delta, \mathbb{G}) \in F_{\top}(C)$.

Then the pair (C, lim) is called a top L-convergence space. We write $\mathbb{F} \rightarrow_{\text{lim}} x$ or just $\mathbb{F} \rightarrow x$ for $x \in \text{lim}(\delta, \mathbb{F})$, and say that \mathbb{F} is convergent to x also.

Remark 4.2. *Our initial idea of introducing top L-convergence structures on an L-set C is that a point x and a filter (δ, \mathbb{F}) should have the the same degree to which they belong, that is to say $F_{\top}(C)(\delta, \mathbb{F}) = \delta = C(x)$, whenever the filter is convergent to the point x with respect to the convergence structure. According to Definition 4.1, the concept of top L-convergence structure, already, make us to realize our initial idea.*

(2) *Let X be a classical set and \top_X denote the L-set such that $\top_X(x) = \top$ for all $x \in X$. Then X is the domain set of the \top -saturated L-subset $(\top_X)^{\top}$ of \top_X , and L^X , called the L-power set of X in the fuzzy community, is the*

domain set of the \top -saturated L -subset $(\mathcal{P}\top_X)^\top$. Thus the set of all \top -filters on X becomes the domain set of the \top -saturated L -subset $F_\top^\top(\top_X)$ of $F_\top(\top_X)$. Under these notions, a \top -convergence on X in the terminology of [12] could be understood as the restriction $\lim |F_\top^\top(\top_X)$ of a top L -convergence structure \lim on \top_X . Hence we could say \top -convergence in [12] is a type of top L -convergence structures defined partly, in other words, each \top -convergence on a set X is a part of a top L -convergence structure on the L -set \top_X with the mark \top .

We are going to introduce the notion of the category of top L -convergence spaces. A degree-preserving map $\varphi : (C, \lim^C) \rightarrow (D, \lim^D)$ between top L -convergence spaces is said to be continuous if it fulfills that for an $x \in C^\alpha$ and a limited L -filter (α, \mathbb{F}) , $(\alpha, \mathbb{F}) \rightarrow x$ means $(\alpha, \varphi^\Rightarrow(\mathbb{F})) \rightarrow \varphi(x)$. All top L -convergence spaces together with continuous degree-preserving maps form a category over $\mathbf{Set}\downarrow L$, denoted by \top - L - \mathbf{Conv} .

Now we offer some examples of top L -convergence structures on L -sets. Of course, part of these examples will be useful to the following sections.

Example 4.3. (1) Define a degree-preserving map $\lim_{dis} : F_\top(C) \rightarrow D(C)$ on an L -set C by for each δ -limited L -filter (δ, \mathbb{F}) and an $x \in C^\delta$,

$$x \in \lim_{dis}(\delta, \mathbb{F}) \text{ if and only if } (\delta, [x]_\delta) \leq (\delta, \mathbb{F}),$$

or equivalently $(\delta, [x]_\delta) = (\delta, \mathbb{F})$ since $(\delta, [x]_\delta)$ ultrafilter here (See Example 3.3 (1)), which is a top L -convergence structure on C , called the discrete structure. Thus every degree-preserving map $\varphi : (C, \lim_{dis}) \rightarrow (D, \lim)$ is continuous for any space (D, \lim) since every degree-limited L -filter under a degree-preserving map is a degree-limited L -filter (See Example 3.3 (2) (i)).

(2) Define a map $\lim_{ind} : F_\top(C) \rightarrow D(C)$ by $\lim_{ind}(\delta, \mathbb{F}) = C^\delta$ for each δ -limited L -filter (δ, \mathbb{F}) , which is an example of top L -convergence structures so that a degree-preserving map $\varphi : (D, \lim) \rightarrow (C, \lim_{ind})$ is continuous for any space (D, \lim) , and hence called the indiscrete structure.

For the category \top - L - \mathbf{Conv} of top L -convergence spaces, we have

Theorem 4.4. The category \top - L - \mathbf{Conv} of top L -convergence spaces is topological over $\mathbf{Set}\downarrow L$ in the sense that for any L -set A , any family $\{(A_j, \lim_j)\}_{j \in J}$ of top L -convergence spaces and any family $\{\varphi_j : A \rightarrow A_j\}_{j \in J}$ of degree-preserving maps, indexed by a class J , there exists a unique top L -convergence structure \lim on A which is initial with respect to $\{\varphi_j : A \rightarrow (A_j, \lim_j)\}_{j \in J}$, i.e., for a space (C, \lim^C) , a degree-preserving map $\psi : (C, \lim^C) \rightarrow (A, \lim)$ is a continuous if and only if for every $j \in J$, the composite $\varphi_j \circ \psi : (C, \lim^C) \rightarrow (A_j, \lim_j)$ is continuous.

Proof. For $\{\varphi_j : A \rightarrow (A_j, \lim_j)\}_{j \in J}$, we only list the unique structure map \lim on A about it, determined by

$$(\delta, \mathbb{F}) \rightarrow_{\lim} x \text{ if and only if } (\delta, \varphi_j^\Rightarrow(\mathbb{F})) \rightarrow_{\lim_j} \varphi_j(x) \text{ for all } j \in J,$$

for each limited L -filter (δ, \mathbb{F}) together with an $x \in A^\delta$. \square

Following Theorem 4.4, we introduce the products and subspaces respectively. Firstly, for given (C, \lim^C) and (D, \lim^D) , $C \times D$ with the unique top L -convergence structure on with respect to

$$\{p_C : C \times D \rightarrow (C, \lim^C); p_D : C \times D \rightarrow (D, \lim^D)\}$$

is called the product of (C, \lim^C) and (D, \lim^D) , denoted by $(C \times D, \lim^C \times \lim^D)$ explicitly. Of course, for any $(\delta, \mathbb{F}) \in F_\top(C \times D)$ and $(x, y) \in C^\delta \times D^\delta$, $(x, y) \in (\lim^C \times \lim^D)(\delta, \mathbb{F})$ if and only if $x \in \lim^C(\delta, p_C^\Rightarrow(\mathbb{F}))$ and $y \in \lim^D(\delta, p_D^\Rightarrow(\mathbb{F}))$.

(2) If the domain set of a saturated L -subset C^δ of C is not empty for some $\delta \in L$. Then the C^δ with the unique top L -convergence structure with respect to the inclusion degree-preserving map $i_\delta : C^\delta \rightarrow (C, \lim^C)$ is called the subspace of (C, \lim^C) , which is denoted by $(C^\delta, \lim^C|_{C^\delta})$.

Finally, we note a fact on the terminal object in \top - L - \mathbf{Conv} . In details, we have

Corollary 4.5. *In the category \top -L-Conv of top L-convergence spaces and degree-preserving maps, (id_L, \lim_{ind}) is the terminal object in the sense that there is exactly one continuous degree-preserving map from any top L-convergence space to the (id_L, \lim_{ind}) .*

Proof. Let (C, \lim) be any top L-convergence space and \lim_{ind} be the indiscrete top L-convergence structure on id_L (See Example 4.3 (2)). Since every degree-preserving map from (C, \lim) to (id_L, \lim_{ind}) is continuous, the only one degree-preserving map C from (C, \lim) to (id_L, \lim_{ind}) becomes the unique continuous degree-preserving map. \square

5. Two kind of nontrivial Examples.

Our object of the section have twofold: one is to explore that there exist top L-convergence structures induced by a kind of lattice-valued topological spaces, indeed. Another is to explore that there exists a kind of top L-convergence structures induced by classical convergence spaces.

5.1. Examples from stratified L-topological spaces

First of all, we introduce stratified L-topology in [16] for case of L-sets as follows.

Definition 5.1. *Let A be an L-set. A saturated L-subset \mathbb{T} of $\mathcal{P}A$ is said to be a stratified L-topology on A if it satisfies the following axioms:*

- (O0) $(\delta, \perp_A) \in \mathbb{T}$ for all $\delta \in L$, here $\perp_A(x) = \perp$ for all $x \in A$,
- (O1) $(\delta, \delta \wedge A) \in \mathbb{T}$ for all $\delta \in L$,
- (O2) $(\delta, \bigvee_{j \in J} h_j) \in \mathbb{T}$ for any family $\{(\delta, h_j) \mid j \in J\} \subseteq \mathbb{T}$ with $\delta \in L$,
- (O3) $(\delta, g \wedge h) \in \mathbb{T}$ for any $(\delta, g), (\delta, h) \in \mathbb{T}$ with $\delta \in L$,
- (Os) $(\sigma, \alpha \circ_q h) = (\sigma, \alpha * (\delta \rightarrow h)) \in \mathbb{T}$ for $(\delta, h) \in \mathbb{T}$ and $\alpha \in \mathcal{D}(L)(\delta, \sigma)$, here for all $x \in A$, $(\alpha * (\delta \rightarrow h))(x) = \alpha * (\delta \rightarrow h(x))$.

Then the pair (A, \mathbb{T}) is called a stratified L-topological space, and a δ -limited L-subset (δ, h) is said to be open w.r.t. \mathbb{T} whenever $(\delta, h) \in \mathbb{T}$.

As usual, we need to construct limited L-neighborhood systems in a stratified L-topological space (A, \mathbb{T}) so that top L-convergence spaces could be obtained from stratified L-topological spaces. In fact, for an $x \in A$, let a pair $(A(x), \mathbb{U}_{\mathbb{T}}^x)$ is determined by

$$\mathbb{U}_{\mathbb{T}}^x := \{(A(x), f) \in (\mathcal{P}A)^{A(x)} \mid A(x) = \bigvee \{g(x) \mid (A(x), g) \in \mathbb{T}^{A(x)}, (A(x), g) \leq (A(x), f)\}\},$$

or equivalently,

$$\mathbb{U}_{\mathbb{T}}^x = \{(A(x), f) \in (\mathcal{P}A)^{A(x)} \mid A(x) = \bigvee_{(A(x), g) \in \mathbb{T}^{A(x)}} \mathcal{P}A((A(x), g), (A(x), f)) \circ_q g(x)\}$$

since for each $(A(x), f) \in (\mathcal{P}A)^{A(x)}$, the pair $(A(x), h)$, here

$$h := \bigvee_{(A(x), g) \in \mathbb{T}^{A(x)}} \mathcal{P}A((A(x), g), (A(x), f)) \circ_q g$$

is open w.r.t. \mathbb{T} (Cf. (O2) and (Os)) and $(A(x), h) \leq (A(x), f)$.

Thus we get a system of $\{(A(x), \mathbb{U}_{\mathbb{T}}^x)\}_{x \in A}$ in (A, \mathbb{T}) and each pair $(A(x), \mathbb{U}_{\mathbb{T}}^x)$ of the system is an example of limited L-filters on A, which could be confirmed by the following proposition.

Proposition 5.2. *For every stratified L-topology \mathbb{T} on an L-set A, each $(A(x), \mathbb{U}_{\mathbb{T}}^x)$ in the system $\{(A(x), \mathbb{U}_{\mathbb{T}}^x)\}_{x \in A}$ with respect to \mathbb{T} satisfies the following conditions:*

- (N1) $f(x) = A(x)$ for each pair $(A(x), f) \in \mathbb{U}_{\mathbb{T}}^x$.

(N2) $(A(x), \mathbb{U}_{\mathbb{T}}^x)$ is a limited L -filter.

Proof. For convenience here, we write δ_x for $A(x)$ with $x \in A$. Then the claim $f(x) = \delta_x$ for $(\delta_x, f) \in \mathbb{U}_{\mathbb{T}}^x$ follows from

$$\delta_x = \bigvee \{g(x) \mid (\delta_x, g) \in \mathbb{T}^{\delta_x} \text{ with } (\delta_x, g) \leq (\delta_x, f)\} \leq f(x) \leq \delta_x.$$

In order to confirm the claim (N2), we have to check that (T1)-(T3) of Definition 3.1. For (T2), we recall that

$$\delta_x = \bigvee_{(\delta_x, h) \in \mathbb{T}^{\delta_x}} \mathcal{P}A((\delta_x, h), (\delta_x, g)) \circ_q h(x)$$

holds for any $(\delta_x, g) \in \mathbb{U}_{\mathbb{T}}^x$. From this, for a $(\delta_x, f) \in (\mathcal{P}A)^{\delta_x}$ with $\delta_x = \bigvee_{(\delta_x, g) \in \mathbb{U}_{\mathbb{T}}^x} \mathcal{P}A((\delta_x, g), (\delta_x, f))$, it holds that for every $(\delta_x, g) \in \mathbb{U}_{\mathbb{T}}^x$,

$$\begin{aligned} \bigvee_{(\delta_x, h) \in \mathbb{T}^{\delta_x}} \mathcal{P}A((\delta_x, h), (\delta_x, f)) \circ_q h(x) &= \left(\bigvee_{(\delta_x, h) \in \mathbb{T}^{\delta_x}} \mathcal{P}A((\delta_x, h), (\delta_x, f)) \circ_q h(x) \right) \not\leq \delta_x \quad (\text{By Lemma 2.3 (1)}) \\ &\geq \bigwedge_{(\delta_x, h) \in \mathbb{T}^{\delta_x}} \left(\mathcal{P}A((\delta_x, h), (\delta_x, f)) \circ_q h(x) \not\leq \mathcal{P}A((\delta_x, h), (\delta_x, g)) \circ_q h(x) \right) \\ &\quad (\text{by } \delta_x = \bigvee_{(\delta_x, h) \in \mathbb{T}^{\delta_x}} \mathcal{P}A((\delta_x, h), (\delta_x, g)) \circ_q h(x) \text{ and Lemma 2.3 (3)}) \\ &\geq \mathcal{P}A((\delta_x, g), (\delta_x, f)), \end{aligned}$$

here the last is from Prop. 2.7 (2). Thus we obtain

$$\delta_x = \bigvee_{(\delta_x, g) \in \mathbb{U}_{\mathbb{T}}^x} \mathcal{P}A((\delta_x, g), (\delta_x, f)) \leq \bigvee_{(\delta_x, h) \in \mathbb{T}^{\delta_x}} \mathcal{P}A((\delta_x, h), (\delta_x, f)) \circ_q h(x)$$

i.e., $(\delta_x, f) \in \mathbb{U}_{\mathbb{T}}^x$ by the definition of $\mathbb{U}_{\mathbb{T}}^x$, and so (T2) is satisfied by $\mathbb{U}_{\mathbb{T}}^x$.

Next, let us demonstrate that the condition (T1) is satisfied by $(\delta_x, \mathbb{U}_{\mathbb{T}}^x)$. For this, take an element $(\delta_x, f) \in \mathbb{U}_{\mathbb{T}}^x$. Then

$$\bigvee_{y \in A^{\delta_x}} f(y) \geq f(x) = \delta_x,$$

which already means $\delta_x = \bigvee_{y \in A^{\delta_x}} f(y)$.

Finally, we are going to show $\mathbb{U}_{\mathbb{T}}^x$ fulfils the condition (T3) also. For this, let $(\delta_x, f_1), (\delta_x, f_2) \in \mathbb{U}_{\mathbb{T}}^x$. And hence we have

$$\bigvee \{g(x) \mid (\delta_x, g) \in \mathbb{T}^{\delta_x} \text{ with } (\delta_x, g) \leq (\delta_x, f_i)\} = \delta_x \quad (i = 1, 2).$$

Thus the conclusion of $(\delta_x, f_1 \wedge f_2) \in \mathbb{U}_{\mathbb{T}}^x$ follows from

$$\begin{aligned} \delta_x &= \delta_x \wedge \delta_x \\ &= \left(\bigvee \{g_1(x) \mid (\delta_x, g_1) \in \mathbb{T}^{\delta_x}, (\delta_x, g_1) \leq (\delta_x, f_1)\} \right) \wedge \left(\bigvee \{g_2(x) \mid (\delta_x, g_2) \in \mathbb{T}^{\delta_x}, (\delta_x, g_2) \leq (\delta_x, f_2)\} \right) \\ &\stackrel{(O2)}{\leq} \bigvee \{g(x) \mid (\delta_x, g) \in \mathbb{T}^{\delta_x} \text{ with } (\delta_x, g) \leq (\delta_x, f_1 \wedge f_2)\} \\ &\leq \delta_x. \end{aligned}$$

□

Following Proposition 5.2, for a stratified L -topological space (A, \mathbb{T}) , $\mathbb{U}_{\mathbb{T}} : A \rightarrow F_+(A)$ given by for each $x \in A$, $\mathbb{U}_{\mathbb{T}}(x) = (A(x), \mathbb{U}_{\mathbb{T}}^x)$, is a degree-preserving map, which is called Top L -neighborhood system of \mathbb{T} .

By means of Top L -neighborhood system $\mathbb{U}_{\mathbb{T}}$ of a stratified L -topology \mathbb{T} on an L -set A , it is possible to capture a top L -convergence structure on A as showed in following example.

Example 5.3. Let $\mathbb{U}_{\mathbb{T}}$ be the Top L -neighborhood system of a stratified L -topology \mathbb{T} on an L -set A . The structure map $\lim_{\mathbb{T}} : F_{\mathbb{T}}(A) \rightarrow D(A)$ defined by $\lim_{\mathbb{T}}(\delta, \mathbb{F}) = \{x \in A^\delta \mid A(x) = \delta, \mathbb{U}_{\mathbb{T}}^x \subseteq \mathbb{F}\}$ is a top L -convergence structure on A , called top L -convergence structure of \mathbb{T} .

Proof. (TC1) follows from the condition (N1) of Proposition 5.2. In fact, by (N1), $(A(x), \mathbb{U}_{\mathbb{T}}^x \leq_F (A(x), [x]_{A(x)})$ holds for every $x \in A$. So, $x \in \lim_{\mathbb{T}}(A(x), [x]_{A(x)})$ is true for all $x \in A$, in other words, (TC1) is satisfied by $\lim_{\mathbb{T}}$. Directly, (TC2) can be verified by the definition of $\lim_{\mathbb{T}}$. Therefore we conclude that $\lim_{\mathbb{T}}$ is a top L -convergence structure on A , as desired. \square

5.2. Examples from classical convergence spaces.

Now, we want to introduce another example of top L -convergence structures induced by a family of convergence structures [31]. Firstly, we need to give a method of inducing limited L -filters on an L -set from classical filters. In details, for an L -set C and $C^\delta \neq \emptyset$ with some $\delta \in L$. If we write $F(C^\delta)$ for the set of filters on C^δ , then for each filter $\mathcal{F} \in F(C^\delta)$, a pair $(\delta, \mathbb{F}_{\mathcal{F}})$ could be defined by

$$\mathbb{F}_{\mathcal{F}} := \left\{ (\delta, f) \in (\mathcal{P}C)^\delta \mid \delta = \bigvee_{S \in \mathcal{F}} \bigwedge_{s \in S} f(s) \right\}.$$

Thus we have

Proposition 5.4. Let C be an L -set and $C^\delta \neq \emptyset$ with some $\delta \in L$. Then for each $\mathcal{F} \in F(C^\delta)$, the pair $(\delta, \mathbb{F}_{\mathcal{F}})$ is a limited L -filter on C .

Proof. To complete the proof of the proposition, our strategy is to check that the $(\delta, \mathbb{F}_{\mathcal{F}})$ fulfills the axioms (T1)-(T3) of limited L -filters as follows:

(i) (T1) is satisfied by $(\delta, \mathbb{F}_{\mathcal{F}})$ because for each $(\delta, f) \in \mathbb{F}_{\mathcal{F}}$,

$$\delta = \left(\bigvee_{S \in \mathcal{F}} \bigwedge_{s \in S} f(s) \right) \leq \bigvee_{s \in C^\delta} f(s).$$

(ii) In order to check that (T2) is satisfied by $(\delta, \mathbb{F}_{\mathcal{F}})$, we take a $(\delta, g) \in \mathcal{P}C$ with the property of

$$\delta = \bigvee_{(\delta, f) \in \mathbb{F}_{\mathcal{F}}} \mathcal{P}C((\delta, f), (\delta, g)).$$

Since $\delta = \bigvee_{S \in \mathcal{F}} \bigwedge_{s \in S} f(s)$ for each $(\delta, f) \in \mathbb{F}_{\mathcal{F}}$, we have

$$\begin{aligned} \delta &= \bigvee_{(\delta, f) \in \mathbb{F}_{\mathcal{F}}} \delta \circ_q \mathcal{P}C((\delta, f), (\delta, g)) = \bigvee_{(\delta, f) \in \mathbb{F}_{\mathcal{F}}} \left[\bigvee_{S \in \mathcal{F}} \bigwedge_{s \in S} f(s) \right] \circ_q \mathcal{P}C((\delta, f), (\delta, g)) \\ &= \bigvee_{S \in \mathcal{F}} \bigvee_{(\delta, f) \in \mathbb{F}_{\mathcal{F}}} \left[\left(\bigwedge_{s \in S} f(s) \right) \circ_q \mathcal{P}C((\delta, f), (\delta, g)) \right] = \bigvee_{S \in \mathcal{F}} \bigvee_{(\delta, f) \in \mathbb{F}_{\mathcal{F}}} \left[\left(\bigwedge_{s \in S} f(s) \right) * \left(\delta \rightarrow \mathcal{P}C((\delta, f), (\delta, g)) \right) \right] \\ &= \bigvee_{S \in \mathcal{F}} \bigvee_{(\delta, f) \in \mathbb{F}_{\mathcal{F}}} \left(\bigwedge_{s \in S} f(s) \right) * \left[\bigwedge_{s \in C} (f(s) \rightarrow g(s)) \right] \leq \bigvee_{S \in \mathcal{F}} \bigvee_{(\delta, f) \in \mathbb{F}_{\mathcal{F}}} \left(\bigwedge_{s \in S} f(s) \right) * \left[\bigwedge_{s \in S} (f(s) \rightarrow g(s)) \right] \\ &\leq \bigvee_{S \in \mathcal{F}} \bigvee_{(\delta, f) \in \mathbb{F}_{\mathcal{F}}} \left(\bigwedge_{s \in S} f(s) \right) * \left[\left(\bigwedge_{s \in S} f(s) \right) \rightarrow \left(\bigwedge_{s \in S} g(s) \right) \right] \\ &\leq \bigvee_{S \in \mathcal{F}} \bigwedge_{s \in S} g(s), \end{aligned}$$

i.e., $\delta = \bigvee_{S \in \mathcal{F}} \bigwedge_{s \in S} g(s)$. Thus $(\delta, g) \in \mathbb{F}_{\mathcal{F}}$ by the definition of $(\delta, \mathbb{F}_{\mathcal{F}})$.

(iii) To check that (T3) is satisfied by $(\delta, \mathbb{F}_{\mathcal{F}})$, we take $(\delta, f_1), (\delta, f_2) \in \mathbb{F}_{\mathcal{F}}$. Then $\delta = \bigvee_{S \in \mathcal{F}} \bigwedge_{s \in S} f_i(s)$ for $i = 1, 2$. Finally, it follows from

$$\begin{aligned} \delta &= (\delta \wedge \delta) = \left(\bigvee_{S_1 \in \mathcal{F}} \bigwedge_{s \in S_1} f_1(s) \right) \wedge \left(\bigvee_{S_2 \in \mathcal{F}} \bigwedge_{t \in S_2} f_2(t) \right) = \bigvee_{S_1, S_2 \in \mathcal{F}} \bigwedge_{t \in S_2} \bigwedge_{s \in S_1} (f_1(s) \wedge f_2(t)) \quad (\text{by Lemma 2.1 (1)}) \\ &\leq \bigvee_{S \in \mathcal{F}} \bigwedge_{s \in S} (f_1(s) \wedge f_2(s)) = \bigvee_{S \in \mathcal{F}} \bigwedge_{s \in S} (f_1 \wedge f_2)(s) \end{aligned}$$

that $(\delta, f_1 \wedge f_2) \in \mathbb{F}_{\mathcal{F}}$, that is to say $(\delta, \mathbb{F}_{\mathcal{F}})$ satisfies (T3) indeed. \square

For a limited L -filters induced by a filter, we have the corollary below and omit its routine proof.

Corollary 5.5. *Let C be an L -set and $\delta \in L$. If $x \in C^\delta$, then (δ, \mathbb{F}_x) is a limited L -filter, here $x := \{S \subseteq C^\delta \mid x \in S\}$. Further, $(\delta, \mathbb{F}_x) = (\delta, [x]_\delta)$ holds for any element $x \in C^\delta$.*

Following Proposition 5.4 and Corollary 5.5, it is the position to introduce how to obtain top L -convergence structures on an L -set from the classical convergence structures. For this object, we have

Example 5.6. *Let C be an L -set and $\Theta := \{\Theta_\delta\}_{\delta \in L}$, where each Θ_δ is a convergence structure (see [31]) on C^δ in the sense that $\Theta_\delta : F(C^\delta) \rightarrow \mathcal{P}(C^\delta)$ is a map satisfying*

- (L1) *If $x \in C^\delta$, then $x \in \Theta_\delta(x)$.*
- (L2) *If $\mathcal{F} \subseteq \mathcal{G}$, then $\Theta_\delta(\mathcal{F}) \subseteq \Theta_\delta(\mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in F(C^\delta)$.*

Then a degree-preserving map $\lim_\Theta : F_\top(C) \rightarrow D(C)$ such that for a limited L -filter (δ, \mathbb{F}) , $\lim_\Theta(\delta, \mathbb{F})$ is defined by for every $x \in C^\delta$,

$$x \in \lim_\Theta(\delta, \mathbb{F}) \text{ iff } \exists \mathcal{F} \in F(C^\delta) \text{ such that } x \in \Theta_\delta(\mathcal{F}) \text{ and } (\delta, \mathbb{F}_{\mathcal{F}}) \leq_F (\delta, \mathbb{F}).$$

Proof. The definition of \lim_Θ above together with Corollary 5.5 assures that \lim_Θ is an example of top L -convergence structures on C . \square

6. Function spaces

In the last section, our object is to construct function spaces in the concrete category \top - L -**Conv** of top L -convergence spaces over the slice category $\mathbf{Set} \downarrow L$. And then we will show our function spaces have the desired universal property, i.e., the condition (3) of Definition 2.9 is satisfied by the function spaces, so that we could conclude that the concrete category \top - L -**Conv** of top L -convergence spaces over the slice category $\mathbf{Set} \downarrow L$ is Cartesian closed.

For the existence of function spaces with respect to two spaces (A, \lim^A) and (B, \lim^B) in the concrete category \top - L -**Conv**, we construct an L -set, denoted by $[A, B]$, such that its domain set is given by the set

$$\{(\delta, \varphi) \mid \delta \in L \text{ and } \varphi : (A^\delta, \lim^A|_{A^\delta}) \rightarrow (B, \lim^B) \text{ is continuous}\}$$

and for each (δ, φ) in the domain set, $[A, B](\delta, \varphi) = \delta$. Notice that in the base category $\mathbf{Set} \downarrow L$, there exists the evaluation map $E_{A,B} : [A, B] \times A \rightarrow B$ defined by

$$E_{A,B}((\delta, \varphi), x) = \varphi(x), \quad \forall ((\delta, \varphi), x) \in [A, B]^\delta \times A^\delta.$$

First of all, we need a lemma about $E := E_{A,B}$ in preparation.

Lemma 6.1. (Cf. [21] for a crisp set). Let A be an L -set with the nonempty domain set and $\delta \in L$. If $i_\delta : A^\delta \rightarrow A$ is the inclusion degree-preserving map and $(\delta, \varphi) \in [A, B]$, then for a δ -limited L -subset (δ, g) of A ,

$$(\delta, E^\rightarrow(\delta_{(\delta, \varphi)} \times g)) = (\delta, \varphi^\rightarrow(i_\delta^\leftarrow(g))),$$

here the limited L -subset $(\delta, \delta_{(\delta, \varphi)})$ is determined by for each $(\sigma, \psi) \in [A, B]$,

$$\delta_{(\delta, \varphi)}(\sigma, \psi) = \begin{cases} \delta, & \text{if } (\delta, \varphi) = (\sigma, \psi), \\ \perp, & \text{otherwise.} \end{cases}$$

Proof. Take any δ -limited L -subset (δ, g) of A . Then the claimed equality follows from for each $y \in B$,

$$\begin{aligned} E^\rightarrow(\delta_{(\delta, \varphi)} \times g)(y) &= \bigvee_{E((\delta, \psi), x)=y} (\delta_{(\delta, \varphi)} \times g)((\delta, \psi), x) \\ &= \bigvee_{\varphi(x)=y, x \in A^\delta} \delta_{(\delta, \varphi)}(\delta, \varphi) \wedge g(x) \quad (\text{Here, must be } x \in A^\delta) \\ &= \bigvee_{\varphi(x)=y} \delta \wedge i_\delta^\leftarrow(g)(x) = \bigvee_{\varphi(x)=y} i_\delta^\leftarrow(g)(x) \\ &= \varphi^\rightarrow(i_\delta^\leftarrow(g))(y). \end{aligned}$$

Thus the proof is completed. \square

Then a function space based on $[A, B]$ is constructed by the theorem below.

Theorem 6.2. (Construction of function spaces) Let (A, \lim^A) , (B, \lim^B) be τ - L -Conv-objects and a map

$$\lim^{A,B} : F_\tau([A, B]) \rightarrow D(C)([A, B])$$

define by for all $(\delta, \mathbb{H}) \in F_\tau([A, B])$,

$$\lim^{A,B}(\delta, \mathbb{H}) = \{(\delta, \varphi) \in [A, B]^\delta \mid \forall x \in A^\delta, \forall (\delta, \mathbb{F}) \in F_\tau(A), x \in \lim^A(\delta, \mathbb{F}) \Rightarrow \varphi(x) \in \lim^B(\delta, E^\Rightarrow(\mathbb{H} \times \mathbb{F}))\}.$$

Then $\lim^{A,B}$ is a top L -convergence structure on $[A, B]$, and the pair $([A, B], \lim^{A,B})$ will be called the function space determined by (A, \lim^A) and (B, \lim^B) .

Proof. We have to check the map $\lim^{A,B}$ satisfies the axioms (TC1) and (TC2). Since the maps

$$E^\Rightarrow : (F_\tau([A, B] \times A), \leq_F) \rightarrow (F_\tau(B), \leq_F) \quad \text{and} \quad \mathbb{H} \times (-) : (F_\tau(A), \leq_F) \rightarrow F_\tau([A, B] \times A)$$

for each $(\delta, \mathbb{H}) \in F_\tau([A, B])$ are order-preserving by Example 3.3 (2) (ii) and Lemma 3.6, respectively, the axiom (TC2) follows immediately from the definition of $\lim^{A,B}$. For the condition (TC1), we will show $(\delta, \varphi) \in \lim^{A,B}(\delta, [(\delta, \varphi)]_\delta)$ for each $(\delta, \varphi) \in [A, B]$.

First of all, we claim that $(\delta, \varphi^\rightarrow(i_\delta^\leftarrow(\mathbb{F}))) \leq_F (\delta, E^\Rightarrow([(\delta, \varphi)]_\delta \times \mathbb{F}))$ for $(\delta, \mathbb{F}) \in F_\tau(A)$, here $i_\delta : A^\delta \rightarrow A$ is the inclusion degree-preserving map. For this, we take $(\delta, f) \in \varphi^\rightarrow(i_\delta^\leftarrow(\mathbb{F}))$. Then it follows from Lemma 6.1 that

$$\begin{aligned} \delta &= \bigvee_{(\delta, g) \in \mathbb{F}} \mathcal{PB}((\delta, \varphi^\rightarrow(i_\delta^\leftarrow(g))), (\delta, f)) = \bigvee_{(\delta, g) \in \mathbb{F}} \mathcal{PB}((\delta, E^\rightarrow(\delta_{(\delta, \varphi)} \times g)), (\delta, f)) \\ &\leq \bigvee_{(\delta, h) \in [(\delta, \varphi)]_\delta \times \mathbb{F}} \mathcal{PB}((\delta, E^\rightarrow(h)), (\delta, f)), \end{aligned}$$

i.e., $\delta = \bigvee_{(\delta, h) \in [(\delta, \varphi)]_\delta \times \mathbb{F}} \mathcal{PB}((\delta, E^\rightarrow(h)), (\delta, f))$, which assures

$$(\delta, f) \in E^\Rightarrow([(\delta, \varphi)]_\delta \times \mathbb{F}).$$

Thus $(\delta, \varphi \Rightarrow (i_\delta^{\leftarrow}(\mathbb{F}))) \leq_F (\delta, E \Rightarrow ([(\delta, \varphi)]_\delta \times \mathbb{F}))$ for $(\delta, \mathbb{F}) \in F_\top(A)$.

Second, for us to show $(\delta, \varphi) \in \lim^{A,B}(\delta, [(\delta, \varphi)]_\delta)$ for each $(\delta, \varphi) \in [A, B]$, let $(\delta, \mathbb{G}) \in F_\top(A)$ and $x \in A^\delta$ with $x \in \lim^A(\delta, \mathbb{G})$. Then $(\delta, i_\delta^{\leftarrow}(\mathbb{G})) \in F_\top(A^\delta)$, and $x \in \lim^A(\delta, i_\delta^{\leftarrow} \circ i_\delta^{\leftarrow}(\mathbb{G}))$ since $(\delta, \mathbb{G}) \leq_F (\delta, i_\delta^{\leftarrow} \circ i_\delta^{\leftarrow}(\mathbb{G}))$, which means $x \in (\lim^A |A^\delta)(\delta, i_\delta^{\leftarrow}(\mathbb{G}))$. Since $\varphi : (A^\delta, \lim^A |A^\delta) \rightarrow (B, \lim^B)$ is continuous, we have $\varphi(x) \in \lim^B(\delta, \varphi \Rightarrow (i_\delta^{\leftarrow}(\mathbb{G})))$. Hence $\varphi(x) \in \lim^B(\delta, E \Rightarrow ([(\delta, \varphi)]_\delta \times \mathbb{G}))$ follows from (TC2) of satisfying by \lim^B and $(\delta, \varphi \Rightarrow (i_\delta^{\leftarrow}(\mathbb{G}))) \leq_F (\delta, E \Rightarrow ([(\delta, \varphi)]_\delta \times \mathbb{G}))$. Finally, by the definition of $\lim^{A,B}$, $(\delta, \varphi) \in \lim^{A,B}(\delta, [(\delta, \varphi)]_\delta)$ is verified. \square

In the following, we shall show that our function space given in Theorem 6.2 has the universal property. For this purpose, let A and B be L -sets and $E : [A, B] \times A \rightarrow B$ is the evaluation map. Now it is the position to show that for each pair of the spaces (A, \lim^A) and (B, \lim^B) ,

$$E : ([A, B], \lim^{A,B}) \times (A, \lim^A) \rightarrow (B, \lim^B)$$

is continuous by the following proposition.

Proposition 6.3. *The degree-preserving map*

$$E : ([A, B], \lim^{A,B}) \times (A, \lim^A) \rightarrow (B, \lim^B)$$

is continuous for each pair of spaces (A, \lim^A) and (B, \lim^B) .

Proof. For the continuity of E , we take a δ -limited L -filter (δ, \mathbb{K}) on $[A, B] \times A$ with $\delta \in L$ such that $((\delta, \varphi), x) \in (\lim^{A,B} \times \lim^A)(\delta, \mathbb{K})$, and we have to verify $E((\delta, \varphi), x) \in \lim^B(\delta, E \Rightarrow (\mathbb{K}))$ as follows.

Firstly, according to the definition of $\lim^{A,B} \times \lim^A$, we directly obtain

$$\text{both } (\delta, \varphi) \in \lim^{A,B}(\delta, p_{[A,B]}^{\Rightarrow}(\mathbb{K})) \quad \text{and} \quad x \in \lim^A(\delta, p_A^{\Rightarrow}(\mathbb{K})).$$

It follows from the definition of $\lim^{A,B}$ together with $x \in \lim^A(\delta, p_A^{\Rightarrow}(\mathbb{K}))$ that $(\delta, \varphi) \in \lim^{A,B}(\delta, p_{[A,B]}^{\Rightarrow}(\mathbb{K}))$ means that

$$\varphi(x) \in \lim^B(\delta, E \Rightarrow (p_{[A,B]}^{\Rightarrow}(\mathbb{K}) \times p_A^{\Rightarrow}(\mathbb{K}))),$$

which further means $\varphi(x) \in \lim^B(\delta, E \Rightarrow (\mathbb{K}))$ from $(\delta, p_{[A,B]}^{\Rightarrow}(\mathbb{K}) \times p_A^{\Rightarrow}(\mathbb{K})) \leq_F (\delta, \mathbb{K})$ of Lemma 3.8 (2). So, $E((\delta, \varphi), x) = \varphi(x) \in \lim^B(\delta, E \Rightarrow (\mathbb{K}))$ holds finally. \square

Let $\psi : (C \times A) \rightarrow B$ be a degree-preserving map from the product of L -sets C and A to an L -set B . For a $z \in C^\delta$ with $\delta \in L$, we obtain a degree-preserving map $\psi(z, -) : A^\delta \rightarrow B$ defined by $\psi(z, -)(x) = \psi(z, x)$ for $x \in A^\delta$, and of course, $\psi(z, -)$ is empty map whenever the domain set of A^δ is empty set. For the degree-preserving map $\psi : (C \times A) \rightarrow B$, we have the following proposition.

Proposition 6.4. *If a degree-preserving map $\psi : (C \times A, \lim^C \times \lim^A) \rightarrow (B, \lim^B)$ is continuous, then $\psi(z, -) : (A^\delta, \lim^A |A^\delta) \rightarrow (B, \lim^B)$ is continuous for each $z \in C^\delta$ with $\delta \in L$.*

Proof. Without loss of generality, we assume the domain set of A^δ is not empty. Thus in order to show the continuity of $\psi(z, -)$, here $z \in C^\delta$ with $\delta \in L$, we take any $x \in A^\delta$ and a δ -limited L -filter (δ, \mathbb{F}) such that $x \in (\lim^A |A^\delta)(\delta, \mathbb{F})$, i.e., $x \in \lim^A(\delta, i_\delta^{\leftarrow}(\mathbb{F}))$. It follows from $(\delta, i_\delta^{\leftarrow}(\mathbb{F})) = (\delta, p_A^{\Rightarrow}([z]_\delta \times i_\delta^{\leftarrow}(\mathbb{F})))$ and $(\delta, [z]_\delta) = (\delta, p_C^{\Rightarrow}([z]_\delta \times i_\delta^{\leftarrow}(\mathbb{F})))$ (See Lemma 3.8 (1)) that

$$(z, x) \in (\lim^C \times \lim^A)(\delta, [z]_\delta \times i_\delta^{\leftarrow}(\mathbb{F})).$$

Thus the continuity of ψ implies that

$$\psi(z, -)(x) = \psi(z, x) \in \lim^B(\delta, \psi \Rightarrow ([z]_\delta \times i_\delta^{\leftarrow}(\mathbb{F}))). \tag{Res1}$$

Now in the position, we need to check the inequality

$$(\delta, \psi^{-1}(f \times i_{\delta}^{-1}(g))) \geq (\delta, \psi(z, -)^{-1}(g)) \tag{Res2}$$

holds for all $(\delta, f) \in [z]_{\delta}$ and $(\delta, g) \in \mathbb{F}$, which can be proved by for each $y \in B$,

$$\begin{aligned} \psi^{-1}(f \times i_{\delta}^{-1}(g))(y) &= \bigvee \{f(u) \wedge i_{\delta}^{-1}(g)(v) \mid \psi(u, v) = y\} \\ &\geq \bigvee_{\psi(z, v)=y} (i_{\delta}^{-1}(g)(v) \wedge f(z)) = \bigvee_{\psi(z, v)=y} ((i_{\delta}^{-1}(g)(v) \wedge \delta)) \\ &= \bigvee_{\psi(z, v)=y} i_{\delta}^{-1}(g)(v) = \bigvee_{\psi(z, v)=y} \left(\bigvee_{(w=i_{\delta}(v)=v)} g(w) \right) \\ &= \bigvee_{\psi(z, w)=y} g(w) = \psi(z, -)^{-1}(g)(y). \end{aligned}$$

Using (Res2), we will obtain

$$(\delta, \psi^{\Rightarrow}([z]_{\delta} \times i_{\delta}^{\Rightarrow}(\mathbb{F}))) \leq_F (\delta, \psi(z, -)^{\Rightarrow}(\mathbb{F})). \tag{Res3}$$

For this purpose, we have to show $\delta = \bigvee_{(\delta, g) \in \mathbb{F}} \mathcal{PB}((\delta, \psi(z, -)^{-1}(g)), (\delta, h))$ for each $(\delta, h) \in \psi^{\Rightarrow}([z]_{\delta} \times i_{\delta}^{\Rightarrow}(\mathbb{F}))$, which, by (Res2), could be shown by

$$\delta = \bigvee_{(\delta, g) \in \mathbb{F}, (\delta, f) \in [z]_{\delta}} \mathcal{PB}((\delta, \psi^{-1}(f \times i_{\delta}^{-1}(g))), (\delta, h)) \leq \bigvee_{(\delta, g) \in \mathbb{F}} \mathcal{PB}((\delta, \psi(z, -)^{-1}(g)), (\delta, h)),$$

here the last is from Proposition 2.7 (3). Finally, using (TC2) together with (Res1) and (Res3), we observe

$$\psi(z, -)(x) = \psi(z, x) \in \lim^B(\delta, \psi(z, -)^{\Rightarrow}(\mathbb{F})),$$

which confirm that $\psi(z, -)$ is continuous. \square

For a degree-preserving map $\psi : (C \times A) \rightarrow B$, by Proposition 6.4 above, we construct a well defined degree-preserving map $\bar{\psi} : C \rightarrow [A, B]$ as follows:

$$\bar{\psi}(z) = (\delta, \psi(z, -)), \quad \text{if } z \in C^{\delta} \text{ for some } \delta \in L \tag{D1}$$

Further, the ψ can be presented by the composition $E \circ (\bar{\psi} \times \text{id}_A)$, i.e.,

$$E \circ (\bar{\psi} \times \text{id}_A) = \psi, \tag{D2}$$

and the continuity of $\bar{\psi}$ will be confirmed by the following proposition.

Proposition 6.5. *If a degree-preserving map*

$$\psi : (C \times A, \lim^C \times \lim^A) \rightarrow (B, \lim^B)$$

is continuous, then the map $\bar{\psi} : (C, \lim^C) \rightarrow ([A, B], \lim^{A,B})$ is continuous.

Proof. In order to confirm the continuity of $\bar{\psi} : (C, \lim^C) \rightarrow ([A, B], \lim^{A,B})$, we take any $z \in C^{\delta}$ with $\delta \in L$ and a δ -limited L -filter (δ, G) such that $z \in \lim^C(\delta, G)$. Then by the definition of $\bar{\psi}$ in the formula (D1), we have to verify

$$\bar{\psi}(z) = (\delta, \psi(z, -)) \in \lim^{A,B}(\delta, \bar{\psi}^{\Rightarrow}(G)).$$

For this, we further take an $x \in A^\delta$ and a δ -limited L -filter (δ, \mathbb{F}) such that $x \in \lim^A(\delta, \mathbb{F})$, and then observe that both $z \in \lim^C(\delta, p_C^\rightarrow(\mathbb{G} \times \mathbb{F}))$ and $x \in \lim^A(\delta, p_A^\rightarrow(\mathbb{G} \times \mathbb{F}))$ hold since

$$(\delta, \mathbb{G}) = (\delta, p_C^\rightarrow(\mathbb{G} \times \mathbb{F})) \text{ and } (\delta, \mathbb{F}) = (\delta, p_A^\rightarrow(\mathbb{G} \times \mathbb{F})),$$

respectively. Then we know $(z, x) \in (\lim^C \times \lim^A)(\delta, \mathbb{G} \times \mathbb{F})$ according the definition of $\lim^C \times \lim^A$. As a result of the continuity of ψ ,

$$\psi(z, -)(x) = \psi(z, x) \in \lim^B(\delta, \psi^\rightarrow(\mathbb{G} \times \mathbb{F}))$$

is obtained, which means $\psi(z, -)(x) \in \lim^B(\delta, E^\rightarrow(\overline{\psi}^\rightarrow(\mathbb{G} \times \mathbb{F})))$ holds due to Lemma 3.7 and the formula (D2), i.e., $E \circ (\overline{\psi} \times \text{id}_A) = \psi$. Finally, by the definition of $\lim^{A,B}$,

$$\overline{\psi}(z) = (\delta, \psi(z, -)) \in \lim^{A,B}(\delta, \overline{\psi}^\rightarrow(\mathbb{G})),$$

which answer the continuity of $\overline{\psi}$, as desired. \square

At the end of this section, we explore that our function space given Theorem 6.2 having the desired universal property so that the category \top - L -**Conv** is Cartesian-closed.

Theorem 6.6. *The category \top - L -**Conv** of top L -convergence spaces and continuous degree-preserving maps is Cartesian-closed.*

Proof. Since the category \top - L -**Conv** has the product of two objects and the terminal object (see Theorem 4.4 and Corollary 4.5), it suffices to show that the condition (3) of Definition 2.9 is satisfied by it. By Theorem 6.2, an object $([A, B], \lim^{A,B})$ as function space exists for any two objects (A, \lim^A) and (B, \lim^B) . Further, there exists a continuous degree-preserving map

$$E : ([A, B], \lim^{A,B}) \times (A, \lim^A) \rightarrow (B, \lim^B)$$

from Proposition 6.3, and then it follows from Proposition 6.5 and the formula (D2) before Proposition 6.5, that for every continuous degree-preserving map $\psi : (C \times A, \lim^C \times \lim^A) \rightarrow (B, \lim^B)$, there is a unique continuous $\overline{\psi} : (C, \lim^C) \rightarrow ([A, B], \lim^{A,B})$ with the equality of $E \circ (\overline{\psi} \times \text{id}_A) = \psi$. Following this together with Definition 2.9, we get the conclusion that the category \top - L -**Conv** of top L -convergence spaces and continuous degree-preserving maps is Cartesian-closed. \square

7. Concluding remarks.

In order to construct function spaces for different kinds of lattice-valued convergence spaces in the fuzzy community, there exists a common phenomenon that the domain set of function spaces is a set of suitable maps from one crisp set to another, so one could say the base category of constructing function spaces is the category **Set** of sets and maps. In this paper, we focus on the question how to construct function spaces so that the base category involved is the slice category **Set** $\downarrow L$ over a appropriate lattice L instead of the category **Set** of sets.

By this paper, we confirm that there exists a kind of lattice-valued convergence spaces, namely top L -convergence spaces, such that

- (i) it is possible to obtain the function space in the concrete category of top L -convergence spaces over the slice category **Set** $\downarrow L$ if the underlying lattice L is a commutative, integral, and divisible quantale;
- (ii) the concrete category of top L -convergence spaces over the slice category **Set** $\downarrow L$ is Cartesian closed.

In addition, the concept of top L -convergence spaces proposed in the paper seems rational since they could be obtained from both stratified L -topological topologies and classical convergence structures naturally (see Section 5). We hope our try to give new lights on discussing mathematical structures based on L -sets instead of crisp sets in our fuzzy community.

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