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Strongly (p, q)-Summable Sequences

Rachid Soualmia[a], Dahmane Achour[a], Elhadj Dahia[b,a]

[a]*Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces, University of M'sila, 28000, Algeria.*
[b]*Ecole normale supérieure de Bousaada, 28001 Bousaada, Algeria.*

Abstract. In this paper we provide a detailed study of the Banach space of strongly (p, q)-summable sequences. We prove that this space is a topological dual of a class of mixed (s, p)-summable sequences, showing in this way new properties of this space. We apply these results to obtain the characterization of the adjoints of (r, p, q)-summing operators.

1. Introduction

In 1973, the Banach space of strongly p-summable sequences was defined by Cohen [4]. He used this space to study and characterize the class of strongly p-summing operators. After this, in 1976, Apiola studied the duality relations between the space of strongly p-summable sequences, the absolutely p-summable sequences and weakly p-summable sequences (see [1, Section 2]) and applied these relations to characterize the adjoints of absolutely (p, q)-summing and Cohen (p, q)-nuclear operators. The 1982 paper by Roshdi Khalil [7] is another cornerstone in this line of thought. He introduced there the Banach space of strongly (p, q)-summable sequences, extending the space of strongly p-summable sequences in a natural way, and found his dual. In 2002, Arregui and Blasco published the paper [2], describing some properties of this space but under the name of (p, q)-summing sequences. In the famous book [9] we find another interesting sequence space: the space of mixed (s, p)-summable sequences (see also [8]).

In this work, we continue the study of the Banach space of strongly (p, q)-summable sequences. We shall begin by showing that this space coincides with the one of (p, q)-summing sequences (presented by Arregui and Blasco). We investigate the duality between the space of strongly (p, q)-summable sequences and the space of mixed (s, p)-summable sequences, obtaining in this way some relevant properties of this space. Also, we give an application to (r, p, q)-summing operators introduced by Pietsch in [9].

The paper is organized as follows. After this introduction, in Section 2 we recall some notation and basic facts on some classes of vector-valued sequences. In Section 3 we focus in the study of strongly (p, q)-summable sequences, and we show our main result: the space of mixed (s, p)-summable sequences is a predual of the space of strongly (q^*, s^*)-summable sequences. Also, we compare this space with the

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Email addresses: rachid.soualmia@univ-msila.dz (Rachid Soualmia), dahmane.achour@univ-msila.dz (Dahmane Achour), hajdahia@gmail.com (Elhadj Dahia)

spaces of absolutely p-summable sequences and strongly p-summable sequences, and we prove an inclusion theorem. Finally, Section 4 is devoted to characterize operators that belong to the space of (r, p, q)-summing operators by defining the associated operator between adequate sequence spaces.

2. Notation and preliminaries

Throughout this paper we use standard Banach space notation. Let X be a Banach space over the scalar field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$), B_X is the closed unit ball of X and X^* is the topological dual of X. Let $1 \leq p \leq \infty$, we write p^* for the real number satisfying $1/p + 1/p^* = 1$. The symbol $X^{\mathbb{N}}$ will denote the sequences with values in X.

Let $\ell_p(X)$ the Banach space of all absolutely p-summable sequences $(x_n)_n$ in X with the norm

$$\left\|(x_n)_n\right\|_{\ell_p(X)} = \left(\sum_{n\geq 1} \|x_n\|^p\right)^{\frac{1}{p}},$$

and we have the isometric isomorphism identification $\ell_p(X)^* = \ell_{p^*}(X^*)$.

We denote by $\ell_{p,\omega}(X)$ the Banach space of all weakly p-summable sequences $(x_n)_n$ in X with the norm

$$\left\|(x_n)_n\right\|_{\ell_{p,\omega}(X)} = \sup_{\|x^*\|_{X^*}\leq 1} \left(\sum_{n\geq 1} |x^*(x_n)|^p\right)^{\frac{1}{p}}.$$

If $p = \infty$ we are restricted to the case of bounded sequences and in $\ell_\infty(X)$ we use the sup norm. If we take $X = \mathbb{K}$, then the spaces $\ell_p(\mathbb{K})$ and $\ell_{p,\omega}(\mathbb{K})$ coincides and we denote $\ell_p(\mathbb{K})$ by ℓ_p. If $1 \leq p \leq s \leq \infty$, we consider the real number r satisfying $1/r + 1/s = 1/p$.

A sequence $(x_n)_n \in X^{\mathbb{N}}$ is said to be mixed (s, p)-summable if there exists a sequence $\tau = (\tau_n)_n \in \ell_r$ and a sequence $x^0 = (x_n^0)_n \in \ell_{s,\omega}(X)$ such that for all $n \in \mathbb{N}$ we have

$$x_n = \tau_n \cdot x_n^0. \tag{1}$$

We denote by $\ell_{m(s,p)}(X)$ the Banach space of all mixed (s, p)-summable sequences of elements of X with the norm

$$\left\|(x_n)_n\right\|_{\ell_{m(s,p)}(X)} = \inf \|(\tau_n)_n\|_{\ell_r} \left\|\left(x_n^0\right)_n\right\|_{\ell_{s,\omega}(X)},$$

where the infimum is taken over all possible representations of x in the form (1).

Note that if $1 \leq p, s_1, s_2 \leq \infty$ such that $s_1 \leq s_2$, then

$$\ell_{m(s_1,p)}(X) \subset \ell_{m(s_2,p)}(X), \tag{2}$$

with $\|(x_n)_n\|_{\ell_{m(s_2,p)}(X)} \leq \|(x_n)_n\|_{\ell_{m(s_1,p)}(X)}$, for all $(x_n)_n \in \ell_{m(s_1,p)}(X)$.

If $s = p$ we have

$$\ell_{m(p,p)}(X) = \ell_{p,\omega}(X), \tag{3}$$

with $\|\cdot\|_{\ell_{m(p,p)}(X)} = \|\cdot\|_{\ell_{p,\omega}(X)}$ and for $s = +\infty$ we obtain

$$\ell_{m(\infty,p)}(X) = \ell_p(X), \tag{4}$$

with $\|\cdot\|_{\ell_{m(\infty,p)}(X)} = \|\cdot\|_{\ell_p(X)}$.

The space of strongly p-summable sequences $(1 < p < +\infty)$ was introduced by Cohen in [4] in order to give a characterization of the class of strongly p-summing linear operators.

A sequence $(x_n)_n \in X^{\mathbb{N}}$ is strongly p-summable if the series $\sum_{n=1}^{\infty} x_n^*(x_n)$ converges for all $(x_n^*)_n \in \ell_{p^*,\omega}(X^*)$. We denote by $\ell_p \langle X \rangle$ the space of strongly p-summable sequences in X which is a Banach space (see [5, Proposition 2.1.8]) with the norm

$$\left\|(x_n)_n\right\|_{\ell_p\langle X\rangle} := \sup_{\left\|(x_n^*)_n\right\|_{\ell_{p^*,\omega}(X^*)}\leq 1} \left|\sum_{n\geq 1} x_n^*(x_n)\right|. \tag{5}$$

If $p = 1$ we have $\ell_1 \langle X \rangle = \ell_1(X)$ with $\|\cdot\|_{\ell_1\langle X\rangle} = \|\cdot\|_{\ell_1(X)}$.

The relationships between the various sequence spaces are given by

$$\ell_p \langle X \rangle \subset \ell_p (X) \subset \ell_{m(s,p)} (X) \subset \ell_{p,\omega} (X),$$

with

$$\left\|(x_n)_n\right\|_{\ell_{p,\omega}(X)} \leq \left\|(x_n)_n\right\|_{\ell_{m(s,p)}(X)} \leq \left\|(x_n)_n\right\|_{\ell_p(X)} \leq \left\|(x_n)_n\right\|_{\ell_p\langle X\rangle},$$

for all $(x_n)_n \in \ell_p \langle X \rangle$.

Further, Apiola, in [1], shows the duality identifications

$$\ell_p \langle X \rangle^* = \ell_{p^*,\omega}(X^*) \quad \text{and} \quad \ell_{p,\omega}(X)^* = \ell_{p^*} \langle X^* \rangle.$$

3. Strongly (p, q)-summable sequences

Roshdi Khalil in [7] introduced the Banach space of strongly (p, q)-summable sequences, $\ell_{p,q} \langle X \rangle$ $(1 \leq p, q \leq +\infty)$, naturally extending the space of strongly p-summable sequences which described as follows.

A sequence $(x_n)_n$ in X is strongly (p, q)-summable if $\sum_n \left|x_n^*(x_n)\right|^p < +\infty$ for all $(x_n^*)_n \in \ell_{q^*,\omega}(X^*)$. The norm of $(x_n)_n$ is given by

$$\|(x_n)_n\|_{\ell_{p,q}\langle X\rangle} := \sup_{\|(x_n^*)_n\|_{\ell_{q^*,\omega}(X^*)}\leq 1} \left(\sum_{n\geq 1} \left|x_n^*(x_n)\right|^p\right)^{\frac{1}{p}}.$$

For $p = 1$ we have

$$\ell_{1,q} \langle X \rangle \equiv \ell_q \langle X \rangle, \tag{6}$$

with $\|\cdot\|_{\ell_{1,q}\langle X\rangle} = \|\cdot\|_{\ell_q\langle X\rangle}$.

Arregui and Blasco in [2] introduced and studied the Banach space, $\ell_{\pi_{p,q}}(X)$, of (p, q)-summing sequences $(1 \leq p, q < \infty)$, to be the space of all sequence in X such that for some constant $C \geq 0$ we have

$$\left(\sum_{i=1}^{n} |x_i^*(x_i)|^p\right)^{\frac{1}{p}} \leq C \sup_{x\in B_X} \left(\sum_{i=1}^{n} |x_i^*(x)|^q\right)^{\frac{1}{q}}.$$

The smallest constant C such that the above inequality holds is the norm of $(x_n)_n \in \ell_{\pi_{p,q}}(X)$, and is denoted by $\pi_{p,q}((x_n)_n)$.

In the following proposition we show that the spaces $\ell_{\pi_{p,q^*}}(X)$ and $\ell_{p,q} \langle X \rangle$ are coincides. The proof is straightforward using the closed graph theorem and will be omitted.

Proposition 3.1. *The sequence $(x_n)_n \in X^{\mathbb{N}}$ is (p, q^*)-summing sequence if and only if it is strongly (p, q)-summable sequence. Moreover, we have*

$$\|(x_n)_n\|_{\ell_{p,q}\langle X\rangle} = \pi_{p,q^*}((x_n)_n).$$

The following theorem asserts that the topological dual of $\ell_{p,q}\langle X\rangle$ is the product space $\ell_{p^*} \cdot \ell_{q^*,\omega}(X^*)$, i.e. the set of all elements of the form $x.y$ such that $x \in \ell_{p^*}$ and $y \in \ell_{q^*,\omega}(X^*)$ (see [7, Theorem 1.3]). Pietsch in [9, Page 225] mentioned that this set is exactly the Banach space $\ell_{m(q^*,s^*)}(X^*)$ such that $\frac{1}{s^*} = \frac{1}{p^*} + \frac{1}{q^*}$.

Theorem 3.2. *Let $1 \le p, q, s \le +\infty$ such that $\frac{1}{s^*} = \frac{1}{p^*} + \frac{1}{q^*}$. The space $\ell_{m(q^*,s^*)}(X^*)$ is isometrically isomorphic to $(\ell_{p,q}\langle X\rangle)^*$ through the mapping ψ given by*

$$\psi((x_n^*)_n)((x_n)_n) = \sum_{n\ge1} x_n^*(x_n),$$

for every $(x_n^*)_n \in \ell_{m(q^*,s^*)}(X^*)$ and $(x_n)_n \in \ell_{p,q}\langle X\rangle$.

Remark 3.3. *The duality identification $(\ell_{p,q}\langle X\rangle)^* \equiv \ell_{m(q^*,s^*)}(X^*)$ yields a new formula for the norm $\|\cdot\|_{\ell_{p,q}\langle X\rangle}$,*

$$\|(x_n)_n\|_{\ell_{p,q}\langle X\rangle} = \sup_{\|(x_n^*)_n\|_{\ell_{m(q^*,s^*)}(X^*)}\le1} \left|\sum_{n\ge1} x_n^*(x_n)\right|. \tag{7}$$

Consequently, we obtain a special case of the strongly (p, q)-summable sequences.

Corollary 3.4. *If $q = 1$ then $\ell_{p,1}\langle X\rangle = \ell_p(X)$ with $\|\cdot\|_{\ell_{p,1}\langle X\rangle} = \|\cdot\|_{\ell_p(X)}$.*

Proof. For all $(x_n)_n \in \ell_p(X)$, by (4) we have

$$\begin{aligned}\|(x_n)_n\|_{\ell_{p,1}\langle X\rangle} &= \sup_{\|(x_n^*)_n\|_{\ell_{m(\infty,p^*)}(X^*)}\le1} \left|\sum_{n\ge1} x_n^*(x_n)\right| \\ &= \sup_{\|(x_n^*)_n\|_{\ell_{p^*}(X^*)}\le1} \left|\sum_{n\ge1} x_n^*(x_n)\right| \\ &= \|(x_n)_n\|_{\ell_p(X)} < \infty.\end{aligned}$$

□

We can use (2) and (7) to establish useful inclusion relations between $\ell_{p,q}\langle X\rangle$.

Proposition 3.5. *Let $1 \le p_1, p_2, q_1, q_2, s \le \infty$ such that $1 + \frac{1}{s} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, if $q_1 \le q_2$ then $p_2 \le p_1$ and we have $\ell_{p_2,q_2}\langle X\rangle \subset \ell_{p_1,q_1}\langle X\rangle$. In this case we have $\|(x_n)_n\|_{\ell_{p_1,q_1}\langle X\rangle} \le \|(x_n)_n\|_{\ell_{p_2,q_2}\langle X\rangle}$, for all $(x_n)_n \in \ell_{p_2,q_2}\langle X\rangle$.*

In the following proposition we prove a relationship between the space of absolutely p-summable sequences, strongly p-summable sequences and strongly (p, q)-summable sequences.

Proposition 3.6. *Let $1 \le p, q \le +\infty$, we have the inclusions $\ell_p(X) \subset \ell_{p,q}\langle X\rangle$ and $\ell_q\langle X\rangle \subset \ell_{p,q}\langle X\rangle$. In addition $\|\cdot\|_{\ell_{p,q}\langle X\rangle} \le \|\cdot\|_{\ell_p(X)}$ and $\|\cdot\|_{\ell_{p,q}\langle X\rangle} \le \|\cdot\|_{\ell_q\langle X\rangle}$.*

Proof. If $(x_n)_n \in \ell_p(X)$ we have

$$\begin{aligned}\|(x_n)_n\|_{\ell_{p,q}\langle X\rangle} &\le \sup_{\|(x_n^*)_n\|_{\ell_{\infty,\omega}(X^*)}\le1} \left\|(x_n^*(x_n))_n\right\|_{\ell_p} \\ &= \|(x_n)_n\|_{\ell_{p,1}\langle X\rangle} = \|(x_n)_n\|_{\ell_p(X)} < \infty.\end{aligned}$$

Similarly, if $(x_n)_n \in \ell_q \langle X \rangle$,

$$\|(x_n)_n\|_{\ell_{p,q}\langle X\rangle} \leq \sup_{\|(x_n^*)_n\|_{\ell_{q^*,\omega}(X^*)}\leq 1} \left\|(x_n^*(x_n))_n\right\|_{\ell_1}$$
$$= \|(x_n)_n\|_{\ell_{1,q}\langle X\rangle} = \|(x_n)_n\|_{\ell_q\langle X\rangle} < \infty.$$

□

In order to give the proof of the main theorem we need the following results.

Lemma 3.7. *Let $(x_n)_n \in \ell_{p,q} \langle X \rangle$. Then,*

$$\|(x_n)_n\|_{\ell_{p,q}\langle X\rangle} = \sup_{\|(\alpha_n)_n\|_{\ell_{p^*}}\leq 1} \left\|(\alpha_n x_n)_n\right\|_{\ell_q\langle X\rangle}. \tag{8}$$

Proof. Let $(x_n)_n \in \ell_{p,q} \langle X \rangle$, by using the duality between the spaces ℓ_p and ℓ_{p^*} we obtain

$$\|(x_n)_n\|_{\ell_{p,q}\langle X\rangle} = \sup_{\|(x_n^*)_n\|_{\ell_{q^*,\omega}}\leq 1} \left\|(x_n^*(x_n))_n\right\|_{\ell_p}$$
$$= \sup_{\|(x_n^*)_n\|_{\ell_{q^*,\omega}}\leq 1} \sup_{\|(\alpha_n)_n\|_{\ell_{p^*}}\leq 1} \left|\sum_{n\geq 1} \alpha_n x_n^*(x_n)\right|$$
$$= \sup_{\|(\alpha_n)_n\|_{\ell_{p^*}}\leq 1} \|(\alpha_n x_n)_n\|_{\ell_q\langle X\rangle}.$$

□

Lemma 3.8. *[3, Page 526]. For all $(x_n^*)_n \in \ell_p \langle X^* \rangle$ we have*

$$\left\|(x_n^*)_n\right\|_{\ell_p\langle X^*\rangle} = \sup_{\|(x_n)_n\|_{\ell_{p^*,\omega}(X)}\leq 1} \left\|(x_n^*(x_n))_n\right\|_{\ell_1}. \tag{9}$$

Proposition 3.9. *For each $(x_n^*)_n \in \ell_{p,q} \langle X^* \rangle$, we have*

$$\left\|(x_n^*)_n\right\|_{\ell_{p,q}\langle X^*\rangle} = \sup_{\|(x_n)_n\|_{\ell_{q^*,\omega}}\leq 1} \left\|(x_n^*(x_n))_n\right\|_{\ell_p}. \tag{10}$$

Proof. Let $(x_n^*)_n \in \ell_{p,q} \langle X^* \rangle$. By (8) and (9) we get

$$\left\|(x_n^*)_n\right\|_{\ell_{p,q}\langle X^*\rangle} = \sup_{\|(\alpha_n)_n\|_{\ell_{p^*}}\leq 1} \left\|(\alpha_n x_n^*)_n\right\|_{\ell_q\langle X^*\rangle}$$
$$= \sup_{\|(\alpha_n)_n\|_{\ell_{p^*}}\leq 1} \sup_{\|(x_n)_n\|_{\ell_{q^*,\omega}}\leq 1} \left\|(\alpha_n x_n^*(x_n))_n\right\|_{\ell_1}$$
$$= \sup_{\|(x_n)_n\|_{\ell_{q^*,\omega}}\leq 1} \sup_{\|(\alpha_n)_n\|_{\ell_{p^*}}\leq 1} \left\|(\alpha_n x_n^*(x_n))_n\right\|_{\ell_1}$$
$$= \sup_{\|(x_n)_n\|_{\ell_{q^*,\omega}}\leq 1} \left\|(x_n^*(x_n))_n\right\|_{\ell_p}.$$

□

Now we are ready to prove the main theorem. This result asserts that the space of mixed (s,p)-summable sequences is a predual of $\ell_{q^*,s^*} \langle X^* \rangle$.

Theorem 3.10. *If $1 \leq p, q, s \leq +\infty$ such that $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$ then we have the isometric isomorphic identification*

$$\left(\ell_{m(s,p)}(X)\right)^* \equiv \ell_{q^*,s^*}\langle X^*\rangle .$$

through the mapping $T : \ell_{q^*,s^*}\langle X^*\rangle \longrightarrow \left(\ell_{m(s,p)}(X)\right)^*$ defined by

$$T((x_n^*)_n)((x_n)_n) = \sum_{n\geq 1} x_n^*(x_n),$$

for all $(x_n^*)_n \in \ell_{q^*,s^*}\langle X^*\rangle$ and $(x_n)_n \in \ell_{m(s,p)}(X)$.

Proof. First note that $\frac{1}{q^*} + \frac{1}{s^*} = 1 + \frac{1}{p^*}$. It is easy to see that the correspondence T is linear. We take $(x_n^*)_n \in \ell_{q^*,s^*}\langle X^*\rangle$ and let $(x_n)_n = (\tau_n x_n^0)_n \in \ell_{m(s,p)}(X)$ where $(\tau_n)_n \in \ell_q$ and $(x_n^0)_n \in \ell_{s,\omega}(X)$. Hence, by Hölder's inequality it follows that

$$\begin{aligned}
\left|\sum_{n\geq 1} x_n^*(x_n)\right| &\leq \sum_{n\geq 1} |\tau_n| \left|x_n^*\left(x_n^0\right)\right| \\
&\leq \|(\tau_n)_n\|_{\ell_q} \left\|\left(x_n^*\left(x_n^0\right)\right)_n\right\|_{\ell_{q^*}} \\
&\leq \|(\tau_n)_n\|_{\ell_q} \left\|\left(x_n^0\right)_n\right\|_{\ell_{s,\omega}(X)} \sup_{\|(z_n)_n\|_{\ell_{s,\omega}(X)}\leq 1} \left\|(x_n^*(z_n))_n\right\|_{\ell_{q^*}} \\
&= \|(\tau_n)_n\|_{\ell_q} \left\|\left(x_n^0\right)_n\right\|_{\ell_{s,\omega}(X)} \left\|(x_n^*)_n\right\|_{\ell_{q^*,s^*}\langle X^*\rangle} .
\end{aligned}$$

Since this holds for all possible factorization of the form $x_n = \tau_n x_n^0$, it follows that,

$$\left|T((x_n^*)_n)((x_n)_n)\right| \leq \|(x_n)_n\|_{\ell_{m(s,p)}(X)} \left\|(x_n^*)_n\right\|_{\ell_{q^*,s^*}\langle X^*\rangle} .$$

Since $(x_n)_n$ is arbitrary it follows that

$$\left\|T((x_n^*)_n)\right\| \leq \left\|(x_n^*)_n\right\|_{\ell_{q^*,s^*}\langle X^*\rangle} .$$

This is implies that T is well-defined and continuous. Now consider the linear operator $S : \left(\ell_{m(s,p)}(X)\right)^* \longrightarrow \ell_{q^*,s^*}\langle X^*\rangle$ given by $S(g) = (g \circ \varphi_n)_n$ where $g \in \left(\ell_{m(s,p)}(X)\right)^*$ and $\varphi_n : X \longrightarrow \ell_{m(s,p)}(X)$ is the linear operator defined by $\varphi_n(x) = (0, \cdots, 0, x, 0, \cdots)$ with x placed in the n-th position. Using (10) and the duality between ℓ_q and ℓ_{q^*} we obtain

$$\begin{aligned}
\left\|(g\circ\varphi_n)_n\right\|_{\ell_{q^*,s^*}\langle X^*\rangle} &= \sup_{\|(x_n)_n\|_{\ell_{s,\omega}(X)}\leq 1} \left\|(g\circ\varphi_n(x_n))_n\right\|_{\ell_{q^*}} \\
&= \sup_{\|(x_n)_n\|_{\ell_{s,\omega}(X)}\leq 1} \sup_{\|(\alpha_n)_n\|_{\ell_q}\leq 1} \left|\sum_{n\geq 1} g\circ\varphi_n(\alpha_n x_n)\right| \\
&= \sup_{\|(x_n)_n\|_{\ell_{s,\omega}(X)}\leq 1} \sup_{\|(\alpha_n)_n\|_{\ell_q}\leq 1} \left|g\left((\alpha_n x_n)_n\right)\right| \\
&\leq \|g\| \sup_{\|(x_n)_n\|_{\ell_{s,\omega}(X)}\leq 1} \sup_{\|(\alpha_n)_n\|_{\ell_q}\leq 1} \|(\alpha_n x_n)_n\|_{\ell_{m(s,p)}(X)} \\
&\leq \|g\| \sup_{\|(x_n)_n\|_{\ell_{s,\omega}(X)}\leq 1} \sup_{\|(\alpha_n)_n\|_{\ell_q}\leq 1} \|(\alpha_n)_n\|_{\ell_q} \|(x_n)_n\|_{\ell_{s,\omega}(X)} \\
&\leq \|g\| < \infty.
\end{aligned}$$

This means that $(g \circ \varphi_n)_n \in \ell_{q^*,s^*}\langle X^*\rangle$ and we can conclude that S is well-defined, continuous and $\|S\| \leq 1$. On the other hand, a straightforward calculation shows that S and T are inverses. Finally, if $(x_n^*)_n \in \ell_{q^*,s^*}\langle X^*\rangle$ then

$$\left\|T((x_n^*)_n)\right\| \leq \left\|(x_n^*)_n\right\|_{\ell_{q^*,s^*}\langle X^*\rangle} = \left\|S \circ T((x_n^*)_n)\right\|_{\ell_{q^*,s^*}\langle X^*\rangle} \leq \left\|T((x_n^*)_n)\right\|.$$

□

According to the above theorem and Hahn-Banach theorem, we have the following result.

Corollary 3.11. *Let $1 \leq p, q, s \leq +\infty$ such that $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$. For every $(x_n)_n \in \ell_{m(s,p)}(X)$ we have,*

$$\|(x_n)_n\|_{\ell_{m(s,p)}(X)} = \sup_{\|(x_n^*)_n\|_{\ell_{q^*,s^*}\langle X^*\rangle} \leq 1} \left|\sum_{n\geq 1} x_n^*(x_n)\right|.$$

A direct consequence of Theorem 3.2 and Theorem 3.10 is the following.

Corollary 3.12. *We have the two isometric isomorphism identifications*

(*i*) $\ell_{p,q}\langle X\rangle^{**} \equiv \ell_{p,q}\langle X^{**}\rangle$.

(*ii*) $\ell_{m(s,p)}(X)^{**} \equiv \ell_{m(s,p)}(X^{**})$.

Using the principle of local reflexivity and previous corollary we obtain the following results.

Proposition 3.13. *Let X be a Banach space and $1 \leq p, q, s \leq +\infty$.*

1. *If $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ and $(x_n^*)_n \in \ell_{m(s,p)}(X^*)$ then*

$$\|(x_n^*)_n\|_{\ell_{m(s,p)}(X^*)} = \sup_{\|(x_n)_n\|_{\ell_{q^*,s^*}\langle X\rangle} \leq 1} \left|\sum_n x_n^*(x_n)\right|.$$

2. *If $\frac{1}{s^*} = \frac{1}{q^*} + \frac{1}{p^*}$ and $(x_n^*)_n \in \ell_{p,q}\langle X^*\rangle$ then*

$$\|(x_n^*)_n\|_{\ell_{p,q}\langle X^*\rangle} = \sup_{\|(x_n)_n\|_{\ell_{m(q^*,s^*)}(X)} \leq 1} \left|\sum_n x_n^*(x_n)\right|.$$

Proof. 1) Let $(x_n^*)_n \in \ell_{m(s,p)}(X^*)$. Since $\ell_{q^*,s^*}\langle X\rangle \subseteq \ell_{q^*,s^*}\langle X^{**}\rangle \equiv \left(\ell_{q^*,s^*}\langle X\rangle\right)^{**}$, we have

$$\|(x_n^*)_n\|_{\ell_{m(s,p)}(X^*)} = \sup_{\|(x_n^{**})_n\|_{\ell_{q^*,s^*}\langle X^{**}\rangle} \leq 1} \left|\sum_n x_n^{**}(x_n^*)\right| \geq \sup_{\|(x_n)_n\|_{\ell_{q^*,s^*}\langle X\rangle} \leq 1} \left|\sum_n x_n^*(x_n)\right|.$$

For the reverse inequality, let E be the linear space spanned by the finite set $\left\{x_1^{**}, ..., x_N^{**}\right\} \subset X^{**}$. By the principle of local reflexivity for each $\varepsilon > 0$ there exists a bounded linear operator $u : E \longrightarrow X$ such that $\|u\| \leq 1$ and $\left|x_j^{**}(x_j^*) - x_j^*(u(x_j^{**}))\right| \leq \dfrac{\varepsilon}{N}$ for all $x_j^* \in X^*$, $j = 1, ..., N$. Then

$$\begin{aligned}\sum_{j\leq N}\left|x_j^{**}\left(x_j^*\right)\right| &\leq \varepsilon + \sum_{j\leq N}\left|x_j^*(u(x_j^{**}))\right| \\ &\leq \varepsilon + \left\|(x_n^{**})_n\right\|_{\ell_{q^*,s^*}\langle X^{**}\rangle} \sup_{\|(x_n)_n\|_{\ell_{q^*,s^*}\langle X\rangle} \leq 1} \sum_{n\geq 1}\left|x_n^*\left(x_n\right)\right|.\end{aligned}$$

Since this holds for every $N \in \mathbb{N}$ and $\varepsilon > 0$ it follows that

$$\|(x_n^*)_n\|_{\ell_{m(s,p)}(X^*)} = \sup_{\|(x_n^{**})_n\|_{\ell_{q^*,s^*}\langle X^{**}\rangle}\leq 1} \sum_{n\geq 1} \left|x_n^{**}(x_n^*)\right| \leq \sup_{\|(x_n)_n\|_{\ell_{q^*,s^*}\langle X\rangle}} \sum_{n\geq 1} \left|x_n^*(x_n)\right|.$$

Part (2) is proved in a similar way. □

Remark 3.14. *If we apply Theorem 3.2 and Theorem 3.10 for some extreme cases of parameters p, q and s, we obtain the well-known duality identifications for the sequence spaces* $\ell_q\langle X\rangle$, $\ell_p(X)$ *and* $\ell_{p,\omega}(X)$.

(i) In the Theorem 3.2 if we take $p = 1$, *then by (3) and (6) we obtain*

$$\left(\ell_q\langle X\rangle\right)^* \equiv \left(\ell_{1,q}\langle X\rangle\right)^* \equiv \ell_{m(q^*,q^*)}(X^*) \equiv \ell_{q^*,\omega}(X^*).$$

(ii) In the Theorem 3.2 if we take $p = s$, *then by (4) and Corollary 3.4 we obtain*

$$\left(\ell_p(X)\right)^* \equiv \left(\ell_{p,1}\langle X\rangle\right)^* \equiv \ell_{m(+\infty,p^*)}(X^*) \equiv \ell_{p^*}(X^*).$$

(iii) In the Theorem 3.10 if we take $s = p$, *then we obtain*

$$\left(\ell_{p,\omega}(X)\right)^* \equiv \left(\ell_{m(p,p)}(X)\right)^* \equiv \ell_{1,p^*}\langle X^*\rangle \equiv \ell_{p^*}\langle X^*\rangle.$$

In the following proposition we give the relation between the space of the strongly (q,s)-summable sequences and the spaces of the absolutely (strongly) p-summable sequences.

Proposition 3.15. *Let* $1 \leq p,q,s \leq \infty$ *such that* $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ *then*

$$\ell_p\langle X\rangle \subset \ell_{q,s}\langle X\rangle \subset \ell_p(X).$$

In this case we have

$$\|(x_n)_n\|_{\ell_p(X)} \leq \|(x_n)_n\|_{\ell_{q,s}\langle X\rangle} \leq \|(x_n)_n\|_{\ell_p\langle X\rangle},$$

for each $(x_n)_n \in \ell_p\langle X\rangle$.

Proof. Since $\frac{1}{p^*} = \frac{1}{q^*} + \frac{1}{s^*}$ we get $\ell_{p^*}(X^*) \subset \ell_{m(s^*,p^*)}(X^*) \subset \ell_{p^*,\omega}(X^*)$. Let $(x_n)_n \in \ell_p\langle X\rangle$. From the duality between $\ell_p(X)$ and $\ell_{p^*}(X^*)$ and equality (7), we obtain

$$\begin{aligned}\|(x_n)_n\|_{\ell_p(X)} &= \sup_{\|(x_n^*)_n\|_{\ell_{p^*}(X^*)}\leq 1} \left|\sum_{n\geq 1} x_n^*(x_n)\right| \\ &\leq \sup_{\|(x_n^*)_n\|_{\ell_{m(s^*,p^*)}(X^*)}\leq 1} \left|\sum_{n\geq 1} x_n^*(x_n)\right| \\ &= \|(x_n)_n\|_{\ell_{q,s}\langle X\rangle} \\ &\leq \sup_{\|(x_n^*)_n\|_{\ell_{p^*,\omega}(X^*)}\leq 1} \left|\sum_{n\geq 1} x_n^*(x_n)\right| \\ &= \|(x_n)_n\|_{\ell_p\langle X\rangle} < \infty.\end{aligned}$$

□

Regarding Proposition 3.15, let us show with an example the difference between $\ell_{q,s}\langle X\rangle$ and $\ell_p(X)$.

Example 3.16. *Let $(e_n)_n$ the unit vector basis of ℓ_2. The sequence $(x_n)_n$ defined by $x_n = \frac{1}{\sqrt{n}} e_n$ belongs to $\ell_\infty(\ell_2)$ but it is not in $\ell_{2,2}\langle \ell_2 \rangle$. In order to see this, $\|(x_n)_n\|_{\ell_\infty(\ell_2)} = \sup_n \frac{1}{\sqrt{n}} = 1$. On the other hand, since*

$$\left\|(e_n^*)_n\right\|_{\ell_{2,\omega}(\ell_2)} = \|(e_n)_n\|_{\ell_{2,\omega}(\ell_2)} = 1,$$

we have that

$$\|(x_n)_n\|_{\ell_{2,2}\langle \ell_2 \rangle} \geq \left\|(e_n^*(x_n))_n\right\|_{\ell_2} = \left(\sum_{n\geq 1} \frac{1}{n}\right)^{\frac{1}{2}} = +\infty.$$

4. Applications to (r, p, q)-summing operators

Let $\mathcal{X} \subset X^{\mathbb{N}}$ and $\mathcal{Y} \subset Y^{\mathbb{N}}$ be spaces of vector valued sequences in X and Y respectively. A linear continuous operator $T \in \mathcal{L}(X, Y)$, between Banach spaces, induces a linear operator $\widehat{T}$ mapping $\mathcal{X}$ into $Y^{\mathbb{N}}$ in the following way: $\widehat{T}((x_n)_n) = (T(x_n))_n$ for all $(x_n)_n \in \mathcal{X}$. In the sequel, if $\widehat{T}(\mathcal{X}) \subset \mathcal{Y}$, we say that T transfers $\mathcal{X}$ into $\mathcal{Y}$.

Throughout this section, let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$. The definition of (r, p, q)-summing operators is due to Pietsch [9, Section 17.1]

Definition 4.1. *An operator $T \in \mathcal{L}(X, Y)$ is (r, p, q)-summing, in symbols $T \in \Pi_{r,p,q}(X, Y)$, if there is $C > 0$ such that*

$$\left\|(y_i^*(T(x_i)))_{1\leq i\leq n}\right\|_{\ell_r} \leq C \|(x_i)_{1\leq i\leq n}\|_{\ell_{p,\omega}(X)} \left\|(y_i^*)_{1\leq i\leq n}\right\|_{\ell_{q,\omega}(Y^*)}, \tag{11}$$

for all $n \in \mathbb{N}$, $(x_i)_{1\leq i\leq n} \subset X$ and $(y_i^*)_{1\leq i\leq n} \subset Y^*$.

This is equivalent to say that T induces a bounded bilinear map

$$\bar{T} : \ell_{p,\omega}(X) \times \ell_{q,\omega}(Y^*) \longrightarrow \ell_r, \quad \bar{T}\left((x_n)_n, (y_n^*)_n\right) = (\langle x_n, y_n^* \rangle)_n,$$

(see [6, Page 196]). Note that $\Pi_{r,p,q}(X, Y)$ is a Banach space equipped with the norm $\pi_{r,p,q}(T)$ which is the smallest constant C satisfying the defining inequality or $\pi_{r,p,q}(T) = \left\|\bar{T}\right\|$.

As in the case of p-summing operators, the natural way of presenting the summability properties of (r, p, q)-summing operators is by defining the corresponding operator $\widehat{T}$ between $\ell_{p,\omega}(X)$ and $\ell_{r,q^*}\langle Y \rangle$.

Proposition 4.2. *The operator $T \in \mathcal{L}(X, Y)$ is (r, p, q)-summing if and only if T transfers $\ell_{p,\omega}(X)$ into $\ell_{r,q^*}\langle Y \rangle$.*

Proof. Indeed, starting from (11) and pass to the limit for n tending to ∞ we obtain

$$\|(T(x_n))_n\|_{\ell_{r,q^*}\langle Y \rangle} \leq \pi_{r,p,q}(T) \|(x_n)_n\|_{\ell_{p,\omega}(X)}, \tag{12}$$

for all $(x_n)_n \in \ell_{p,\omega}(X)$. Then it follows that $\widehat{T} : \ell_{p,\omega}(X) \longrightarrow \ell_{r,q^*}\langle Y \rangle$ is well-defined and $\widehat{T}\left(\ell_{p,\omega}(X)\right) \subset \ell_{r,q^*}\langle Y \rangle$. In addition $\widehat{T}$ is continuous with norm $\leq \pi_{r,p,q}(T)$. Suppose conversely that T transfers $\ell_{p,\omega}(X)$ into $\ell_{r,q^*}\langle Y \rangle$, an appeal to the closed graph theorem shows that $\widehat{T}$ is continuous and

$$\|(T(x_i))_{1\leq i\leq n}\|_{\ell_{r,q^*}\langle Y \rangle} \leq \left\|\widehat{T}\right\| \|(x_i)_{1\leq i\leq n}\|_{\ell_{p,\omega}(X)},$$

and so $T \in \Pi_{r,p,q}(X, Y)$ with $\pi_{r,p,q}(T) \leq \left\|\widehat{T}\right\|$. □

In the next result we give a new characterization of the (r,p,q)-summing operators by using the Banach spaces of strongly q^*-summable and mixed (p,s)-summable sequences obtaining in this way another corresponding operator $\widehat{T}$ of the (r,p,q)-summing operator T.

Theorem 4.3. *Let $p,q,r,s \geq 1$ such that $\frac{1}{s} = \frac{1}{r^*} + \frac{1}{p}$. The operator $T \in \mathcal{L}(X,Y)$ is (r,p,q)-summing if and only if there is a constant $C > 0$ such that for any $x_1, ..., x_n \in X$ we have*

$$\|(T(x_i))_{1\leq i\leq n}\|_{\ell_{q^*}\langle Y\rangle} \leq C\,\|(x_i)_{1\leq i\leq n}\|_{\ell_{m(p,s)}(X)}\,. \tag{13}$$

Proof. Suppose that $T \in \Pi_{r,p,q}(X,Y)$. Let $\left(y_i^*\right)_{1\leq i\leq n} \subset Y^*$, $(x_i)_{1\leq i\leq n} \subset X$ and $\varepsilon > 0$. Choose $(\alpha_i)_{1\leq i\leq n} \subset \mathbb{K}$ and $(z_i)_{1\leq i\leq n} \subset X$ such that $x_i = \alpha_i z_i$, $i = 1, ..., n$ and $\|(\alpha_i)_{1\leq i\leq n}\|_{\ell_{r^*}} \left\|(z_i)_{1\leq i\leq n}\right\|_{\ell_{p,\omega}(X)} \leq (1+\varepsilon)\,\|(x_i)_{1\leq i\leq n}\|_{\ell_{m(p,s)}(X)}$. By Hölder's inequality we get

$$\begin{aligned}\left|\sum_{1\leq i\leq n} y_i^*(T(x_i))\right| &= \left|\sum_{1\leq i\leq n} \alpha_i y_i^*(T(z_i))\right| \\ &\leq \|(\alpha_i)_{1\leq i\leq n}\|_{\ell_{r^*}} \left\|\left(y_i^*(T(z_i))\right)\right\|_{\ell_r} \\ &\leq \pi_{r,p,q}(T)\|(\alpha_i)_{1\leq i\leq n}\|_{\ell_{r^*}} \left\|(z_i)_{1\leq i\leq n}\right\|_{\ell_{p,\omega}(X)} \left\|\left(y_i^*\right)_{1\leq i\leq n}\right\|_{\ell_{q,\omega}(Y^*)}.\end{aligned}$$

By taking the supremum over all $\left(y_i^*\right)_{1\leq i\leq n}$ such that $\left\|\left(y_i^*\right)_{1\leq i\leq n}\right\|_{\ell_{q,\omega}(Y^*)} \leq 1$ we obtain

$$\|(T(x_i))_{1\leq i\leq n}\|_{\ell_{q^*}\langle Y\rangle} \leq \pi_{r,p,q}(T)(1+\varepsilon)\,\|(x_i)_{1\leq i\leq n}\|_{\ell_{m(p,s)}(X)}\,.$$

Since this holds for every $\varepsilon > 0$, we obtain (13).

Suppose conversely that the operator T satisfies the condition (13). For all $\left(y_i^*\right)_{1\leq i\leq n} \subset Y^*$, $(x_i)_{1\leq i\leq n} \subset X$ and $(\alpha_i)_{1\leq i\leq n} \subset \mathbb{K}$ we have

$$\begin{aligned}\left|\sum_{1\leq i\leq n} \alpha_i y_i^*(T(x_i))\right| &= \left|\sum_{1\leq i\leq n} y_i^*(T(\alpha_i x_i))\right| \\ &\leq \left\|\left(y_i^*\right)_{1\leq i\leq n}\right\|_{\ell_{q,\omega}(Y^*)} \|(T(\alpha_i x_i))_{1\leq i\leq n}\|_{\ell_{q^*}\langle Y\rangle} \\ &\leq C\left\|\left(y_i^*\right)_{1\leq i\leq n}\right\|_{\ell_{q,\omega}(Y^*)} \|(\alpha_i x_i)_{1\leq i\leq n}\|_{\ell_{m(p,s)}(X)} \\ &\leq C\left\|\left(y_i^*\right)_{1\leq i\leq n}\right\|_{\ell_{q,\omega}(Y^*)} \|(\alpha_i)_{1\leq i\leq n}\|_{\ell_{r^*}} \left\|(x_i)_{1\leq i\leq n}\right\|_{\ell_{p,\omega}(X)}.\end{aligned}$$

Taking the supremum over all $(\alpha_i)_{1\leq i\leq n} \subset \mathbb{K}$ such that $\left\|(\alpha_i)_{1\leq i\leq n}\right\|_{\ell_{r^*}} \leq 1$ we get

$$\left\|\left(y_i^*(T(x_i))\right)_{1\leq i\leq n}\right\|_{\ell_r} \leq C\left\|(x_i)_{1\leq i\leq n}\right\|_{\ell_{p,\omega}(X)} \left\|\left(y_i^*\right)_{1\leq i\leq n}\right\|_{\ell_{q,\omega}(Y^*)}.$$

□

The next corollary and its proof are similar to Proposition 4.2 except that (13) is used instead of (12).

Corollary 4.4. *$T \in \Pi_{r,p,q}(X,Y)$ if and only if T transfers $\ell_{m(p,s)}(X)$ into $\ell_{q^*}\langle Y\rangle$. In addition we have $\pi_{r,p,q}(T) = \left\|\widehat{T}\right\|$.*

Although the following result is essentially already known (it was proved by Pietsch, see [9, Theorem 17.1.5]), we write a new direct proof that highlights the role of the dual space of $\ell_{m(s,p)}(X)$ and $\ell_{p,q}\langle X\rangle$.

By using the above corollary, Proposition 4.2, the identifications $\left(\ell_{m(p,s)}(X)\right)^* \equiv \ell_{r,p^*}\langle X^*\rangle$ and $\left(\ell_{q^*}\langle Y\rangle\right)^* \equiv \ell_{q,\omega}(Y^*)$ and taking into account that the adjoint of the operator $\widehat{T} : \ell_{m(p,s)}(X) \longrightarrow \ell_{q^*}\langle Y\rangle$ can be identified with the operator

$$\widehat{T^*} : \ell_{q,\omega}(Y^*) \longrightarrow \ell_{r,p^*}\langle X^*\rangle\,; \quad \widehat{T^*}((y_i^*)_i) = (T^*(y_i^*))_i,$$

we have the following.

Theorem 4.5. *The operator T belongs to $\Pi_{r,p,q}(X,Y)$ if and only if T^* belongs to $\Pi_{r,q,p}(Y^*,X^*)$. Furthermore, $\pi_{r,p,q}(T) = \pi_{r,q,p}(T^*)$.*

It is easy to prove the following result.

Corollary 4.6. *The operator T belongs to $\Pi_{r,p,q}(X,Y)$ if and only if its bi-adjoint T^{**} belongs to $\Pi_{r,p,q}(X^{**},Y^{**})$. In addition, $\pi_{r,p,q}(T) = \pi_{r,p,q}(T^{**})$.*

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