



## The Adjoint Trajectory of Robot end Effector using the Curvature Theory of Ruled Surface

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**Abstract.** The ruled surface is formed by the movement of a director based on a curve. The point  $P$  not on the director vector at fixed frame  $o-ijk$  draws a curve. However, each position of this point on the curve always corresponds to position of director on the ruled surface, or this point is adjoint to director vector. Thus, the curve is adjoint to the ruled surface. In this study, we expressed the adjoint trajectory of robot end effector. We can change the trajectory of the robot movement by defining the adjoint trajectory when it may not be physically achievable and not re-computation of the robot trajectory. We investigated the angular acceleration and angular velocity of adjoint trajectory of the robot end effector. Also, we obtained the condition that moving point is a fixed point.

### 1. Introduction

The curvature theory examines the distribution of velocity and acceleration and the motion of a rigid body. Therefore, the curvature theory is useful to determine the differential properties of the motion of a robot end effector. In [1], the authors showed that the differential properties of the ruled surface generate the linear and angular motion properties of the robot end effector for robot path planning. In [11], the authors showed that relate to the motion of body carrying the line that generates the curvature theory of line trajectories seeks to characterize the shape of the trajectory ruled surface. Ryuh and Pennock [4], made use of the curvature theory of a ruled surface to study the instantaneous motion properties of a robotic device. Also, Ryuh [5] proposed a method which geometric modeling technique is related to robot trajectory planning which is based on the curvature theory of a ruled surface. Ryuh et al. [6] studied a precision control method of a robot path generation based on the dual curvature theory of a ruled surface. Güler and Kasap [9] obtained new technique for robot trajectory which ensures the calculation of robot's next path using the curvature theory. In [12] the authors obtained Dual unit spherical Bézier-like curves on the dual unit sphere (DUS) by a novel method with respect to the control points. Ding H. et al. [8] investigated the solution of the problem of large rotation angle for the algorithm combining A\* algorithm and Bezier curve. Also, the curvature theory of ruled surfaces has been an interesting research topic in Minkowski space, [2,7,13].

A robot end effector motion is defined as a robot trajectory, and the robot's trajectory can be determined by a ruled surface and spin angle. The robot trajectory consists of the velocity and acceleration of the

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positions of the end effector at a fixed point, and a series of orientations, angular velocity and angular acceleration of the end effector. The direction of the end effector is determined by a coordinate frame connected to the end effector. This coordinate frame is called the tool frame. In addition, TCP which the central point of the striction curve is taken as the origin of this frame. Each vector of the tool frame forms three ruled surfaces that share a common direction followed by the center point. It is enough to use one of them to represent the robot trajectory. The point P not on the director vector at fixed frame o-ijk draws a  $\Gamma_P$  curve. However, each position of point P on curve  $\Gamma_P$  always corresponds to a position of the director on the ruled surface or point P is adjoint to director vector. Multiple coordinate systems can be placed at a point in three-dimensional space. A point can be represented by a vector defined according to the center of this coordinate system. Thus, the position of the robot is determined. In this study, we investigated the differential properties of the robot end effector at a moving point that was not on the director vector. We determined the coordinates of this point according to the tool frame. We calculated the angular acceleration and angular velocity of trajectory adjoint of the robot end effector depending on the curvature functions of the ruled surface. We also obtained the condition that the moving point P is a fixed point.

## 2. Preliminaries

A parametric representation of a ruled surface is

$$X(\varphi, v) = \alpha(\varphi) + v\bar{R}(\varphi), \tag{2.1}$$

where  $\alpha(\varphi)$  represents a space curve which is called the directrix of the ruled surface and  $\bar{R}(\varphi)$  is a curve on the surface of sphere called the spherical indicatrix or director vector. The standard parameter may be based on the ruling as

$$s(\varphi) = \int_0^\varphi |d\bar{R}/d\varphi| d\varphi \tag{2.2}$$

where  $R = |d\bar{R}/d\varphi|$  may be considered to be the speed of  $\bar{R}(\varphi)$ . If  $R \neq 0$  then Eqn. (2.2) can be inverted to yield  $\varphi(s)$  allowing the definition of  $\bar{R}(\varphi(s)) = \bar{R}(s)$ .  $\bar{R}(s)$  has unit speed, that is its tangent vector is of unit magnitude, where "·" is the standard inner product.

$\{e, t, g\}$  is called the generator trihedron of the ruled surface  $X$  such that  $e = \bar{R}/|\bar{R}|$ ,  $t = \bar{R}'$  and  $g = e \times t$  are the unit vector in  $\bar{R}$  direction, the central normal and the asymptotic normal direction of  $X$ , respectively.

The angular variation of the frame  $\{e, t, g\}$  is obtained by computing  $dt/ds$  and  $dg/ds$  in terms of  $e, t, g$ . Since  $e \cdot t = 0$  we have  $dt/ds \cdot e = -t \cdot de/ds = -1$ . Defining  $dt/ds \cdot g$  as the function  $\gamma$  we obtain the generator trihedron equations of  $\bar{R}(\varphi)$  by [11]:

$$\begin{bmatrix} de/ds \\ dt/ds \\ dg/ds \end{bmatrix} = \begin{bmatrix} 0 & 1/R & 0 \\ -1/R & 0 & \gamma/R \\ 0 & -\gamma/R & 0 \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix}, \tag{2.3}$$

where  $\gamma$  is the geodesic curvature of  $\bar{R}(\varphi)$  and the formula for  $\gamma$  is obtained in terms of  $\bar{R}(\varphi)$  and its derivatives with respect to  $\varphi$  is by [10]

$$\gamma = \frac{e \times de/d\varphi \cdot d^2e/d\varphi^2}{|de/d\varphi|^3}. \tag{2.4}$$

The striction curve  $\beta(s)$  of ruled surface  $X$  is obtained as

$$\beta(s) = \alpha(s) - \mu(s)e(s), \tag{2.5}$$

where

$$X(s, v) = \alpha(s) + v\bar{R}(s), \mu = \alpha'(s) \cdot \bar{R}'(s). \tag{2.6}$$

$d\beta/ds$  expanded in terms of the frame  $e, t, g$  is

$$\frac{d\beta}{ds} = \left(\frac{d\beta}{ds} \cdot e\right)e + \left(\frac{d\beta}{ds} \cdot t\right)t + \left(\frac{d\beta}{ds} \cdot g\right)g. \tag{2.7}$$

Using Eqn. (2.5), we obtain

$$\frac{d\beta}{ds} = \Gamma(s)e(s) + \Delta(s)g(s) \tag{2.8}$$

where

$$\Gamma(s) = (1/R)d\alpha/ds \cdot e - Rd\mu/ds, \Delta(s) = (1/R)d\alpha/ds \cdot e \times de/ds. \tag{2.9}$$

The functions  $\gamma, \Gamma$  and  $\Delta$  are defined to be the curvature functions of the ruled surface  $X$ . The ruled surface  $X$  is completely defined together with this functions, [11].

The Frenet frame  $\{t, n, b\}$  along  $\alpha(s)$  is defined, where  $t(s) = \alpha'(s)$ ,  $n(s) = \alpha''(s)$  and  $b(s) = t(s) \times n(s)$  are the unit tangent, principal normal and binormal vector fields of the curve, respectively. The derivative formulas of the Frenet frame are governed by the following relations [10]:

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{2.10}$$

where  $\kappa$  and  $\tau$  are the curvature and torsion of  $\alpha$ , respectively.

Let  $\eta$  be the angle between the vectors  $e$  and  $b$ , (Fig 1.). Here, the generator trihedron and the Frenet trihedron have the central normal vector in common. Then, the relations between these frames can be given as follows, [5]

$$\begin{bmatrix} e \\ t \\ g \end{bmatrix} = \begin{bmatrix} 0 & -\sin \eta & \cos \eta \\ 1 & 0 & 0 \\ 0 & \cos \eta & \sin \eta \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{2.11}$$

where  $t = \bar{R}' = Rr'$ ,  $n = (1/\kappa)t'$ ,  $b = t \times n$  and  $\kappa = |t'|$  the torsion of the curve  $\alpha$  is equal to the angular velocity of the angle of rotation of the central normal vector. So,[5]

$$\eta' = -\tau. \tag{2.12}$$

Path of a robot may be represented by a TCP and tool frame of end effector. The tool frame is represented by three mutually perpendicular unit vectors  $\{O, A, N\}$ , where  $O$  is the orientation vector,  $A$  is the approach vector and  $N$  is the normal vector. The path of the tool center point and the vector  $O$  are called director and ruling, respectively. Then, the surface frame  $\{O, S_n, S_b\}$  of the ruled surface  $X$  may be determined as follows

$$S_n = \frac{X_s \times X_v}{|X_s \times X_v|} \Big|_{v=0}, S_b = O \times S_n, \tag{2.13}$$

where  $S_n$  is the unit normal vector of ruled surface  $X$  and  $S_b$  is the unit binormal vector of the surface. Let  $\zeta$  be the angle between the surface binormal vector  $S_b$  and the approach vector  $A$ . Then, the relations between tool trihedron and surface trihedron can be given as follows, [5]

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \zeta & \sin \zeta \\ 0 & -\sin \zeta & \cos \zeta \end{bmatrix} \begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix}. \tag{2.14}$$

Let  $\sigma$  be the angle between the vectors  $S_n$  and  $t$ . Then, the relations between tool trihedron and generator trihedron can be given as follows, [5]

$$\begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \sigma & \sin \sigma \\ 0 & -\sin \sigma & \cos \sigma \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix}. \tag{2.15}$$

From Eqns. (2.14) and (2.15), the relations between tool trihedron and generator trihedron can be written in the matrix form as, [5]

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Sigma & \sin \Sigma \\ 0 & -\sin \Sigma & \cos \Sigma \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix}. \tag{2.16}$$

where  $\Sigma = \zeta + \sigma$ . Differentiating Eqn. (2.16) and substituting Eqn. (2.16), the first order angular variation of the tool trihedron may be expressed in the matrix form as, [5]

$$\frac{d}{ds} \begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{R} \cos \Sigma & -\frac{1}{R} \sin \Sigma \\ -\frac{1}{R} \cos \Sigma & 0 & \Sigma' + \frac{\gamma}{R} \\ \frac{1}{R} \sin \Sigma & -(\Sigma' + \frac{\gamma}{R}) & 0 \end{bmatrix} \begin{bmatrix} O \\ A \\ N \end{bmatrix}. \tag{2.17}$$

### 3. The Adjoint Trajectory of Robot end Effector using the Curvature Theory of Ruled Surface

The parametric representation of ruled surface is

$$X(s, v) = \alpha(s) + v\bar{R}(s), \tag{3.1}$$

where  $\alpha(s)$  is called the directrix of ruled surface and  $\bar{R}(s)$  is director vector which is called the spherical indicatrix. By using Eqns. (2.8) and (2.16) the tangent of striction curve of this ruled surface according to tool trihedron is

$$\beta'(s) = \Gamma(s)O(s) + \Delta(s)\sin \Sigma(s)A(s) + \Delta(s)\cos \Sigma(s)N(s). \tag{3.2}$$

Let the point P be not on the director vector. This point draws a curve  $\Gamma_p$ . Let  $r_p$  be the trajectory of the curve  $\Gamma_p$ . This curve is adjoint to the surface  $X(s, v)$ . So, we write parameter form of this curve depending on the generator curve trihedron  $\{e, t, g\}$ , (Fig 1.),

$$\Gamma_p : r_p(s) = \beta(s) + x_1(s)e(s) + x_2(s)t(s) + x_3(s)g(s), \tag{3.3}$$

where  $x_1, x_2$  and  $x_3$  are coordinates of point P according to the generator trihedron. By using Eqns.(2.16) and (3.3), the parameter form of curve  $\Gamma_p$  with respect to tool trihedron is

$$r_p(s) = \beta + x_1 O + (x_2 \cos \Sigma + x_3 \sin \Sigma) A + (x_3 \cos \Sigma - x_2 \sin \Sigma) N. \tag{3.4}$$

Differentiating Eqn.(3.4) and substituting Eqns.(3.2) and (2.17), into the result, the first order positional variation of the moving point P with respect to the tool trihedron is

$$r'_p(s) = \left(\Gamma + x'_1 - \frac{1}{R}x_2\right) O + (\Omega_1 \sin \Sigma + \Omega_2 \cos \Sigma) A + (\Omega_1 \cos \Sigma - \Omega_2 \sin \Sigma) N \tag{3.5}$$

$$\begin{aligned} \Omega_1 &= \Delta + x'_3 + \frac{\gamma x_2}{R} \\ \Omega_2 &= x'_2 + \frac{x_1 - \gamma x_3}{R} \end{aligned} \tag{3.6}$$

Differentiating Eqn.(3.5) and substituting Eqns. (2.17) and (3.6), into the result, the second order positional variation of the moving point P is

$$\begin{aligned} r''_p(s) &= \left[\Gamma' + x''_1 + \frac{1}{R^2}(\gamma x_3 - x_1 - 2Rx'_2)\right] O \\ &+ \left[ \begin{aligned} &\frac{1}{R^2}(\Gamma R + 2Rx'_1 - R(\gamma x_3)' - \gamma(\Delta + x'_3) - x_2(1 + \gamma^2)) \cos \Sigma \\ &+ \left(\frac{1}{R^2}(\gamma x_1 - \gamma^2 x_3 + R(\gamma x_2)' + R\gamma x'_2) + \Delta' + x''_3\right) \sin \Sigma \end{aligned} \right] A \\ &+ \left[ \left[ \begin{aligned} &\frac{1}{R^2}(x_2(1 + \gamma^2) + R(\Delta\gamma - \Gamma + \gamma x'_3 - x'_1 + (\gamma x_3)')) \sin \Sigma \\ &+ \left(\frac{1}{R^2}(\gamma x_1 - \gamma^2 x_3 + R(\gamma x_2)' + R\gamma x'_2) + \Delta' + x''_3\right) \cos \Sigma \end{aligned} \right] \right] N. \end{aligned} \tag{3.7}$$

**Corollary 3.1** *The angular velocity of the robot end effector of connected at the point P depending on the tool trihedron can be given as*

$$r'_p(s) = \left(\Gamma + x'_1 - \frac{1}{R}x_2\right) O + (\Omega_1 \sin \Sigma + \Omega_2 \cos \Sigma) A + (\Omega_1 \cos \Sigma - \Omega_2 \sin \Sigma) N$$

$$\begin{aligned} \Omega_1 &= \Delta + x'_3 + \frac{\gamma x_2}{R} \\ \Omega_2 &= x'_2 + \frac{x_1 - \gamma x_3}{R}. \end{aligned}$$

**Corollary 3.2** *The angular acceleration of the robot end effector of connected at the point P depending on the tool trihedron can be given as*

$$\begin{aligned} r''_p(s) &= \left[\Gamma' + x''_1 + \frac{1}{R^2}(\gamma x_3 - x_1 - 2Rx'_2)\right] O \\ &+ \left[ \begin{aligned} &\frac{1}{R^2}(\Gamma R + 2Rx'_1 - R(\gamma x_3)' - \gamma(\Delta + x'_3) - x_2(1 + \gamma^2)) \cos \Sigma \\ &+ \left(\frac{1}{R^2}(\gamma x_1 - \gamma^2 x_3 + R(\gamma x_2)' + R\gamma x'_2) + \Delta' + x''_3\right) \sin \Sigma \end{aligned} \right] A \\ &+ \left[ \left[ \begin{aligned} &\frac{1}{R^2}(x_2(1 + \gamma^2) + R(\Delta\gamma - \Gamma + \gamma x'_3 - x'_1 + (\gamma x_3)')) \sin \Sigma \\ &+ \left(\frac{1}{R^2}(\gamma x_1 - \gamma^2 x_3 + R(\gamma x_2)' + R\gamma x'_2) + \Delta' + x''_3\right) \cos \Sigma \end{aligned} \right] \right] N. \end{aligned}$$

**Theorem 3.3** Let  $X(s, v) = \alpha(s) + v\bar{R}(s)$  be the ruled surface and the point  $P$  be not on the director vector. If the point  $P$  is fixed, then

$$\Gamma = c\Delta, \tag{3.8}$$

where  $c = -\frac{x_1}{x_3} = \text{cons tan } t$ , and  $\Gamma, \Delta$  functions are curvature functions of the ruled surface  $X(s, v)$ .

*Proof.* If  $P$  is a fixed point, then Eqn. (3.5) is zero. Then all coefficients in Eqn. (3.5) must be zero. Therefore,

$$\Gamma + x'_1 - \frac{1}{R}x_2 = 0, \Omega_1 \sin \Sigma + \Omega_2 \cos \Sigma = 0, \Omega_1 \cos \Sigma - \Omega_2 \sin \Sigma = 0. \tag{3.9}$$

By using Eqns. (3.6) and (3.9), we have

$$\begin{cases} x'_1 = \frac{1}{R}x_2 - \Gamma \\ x'_2 = \frac{1}{R}(\gamma x_3 - x_1), \\ x'_3 = -\frac{1}{R}\gamma x_2 - \Delta. \end{cases} \tag{3.10}$$

Since point  $P$  is fixed,  $x_i = \text{cons tan } t, i = 1, 2, 3$ . So,  $\frac{dx_i}{ds} = 0$ . We can write using Eqn. (3.10)

$$\Delta = -\frac{x_1}{x_3}\Gamma,$$

and since  $x_i = \text{cons tan } t$ , we can get  $c = -\frac{x_1}{x_3} = \text{cons tan } t$ . Thus the proof is completed.  $\square$

Darboux vector of the natural trihedron is

$$W = \tau t + \kappa b \tag{3.11}$$

where  $\kappa$  and  $\tau$  are curvature and torsion of the curve  $\alpha(s)$ , respectively. By using Eqns. (2.11) and (2.16), we can write

$$W = \kappa \cos \eta O + (\tau \cos \Sigma + \kappa \sin \eta \sin \Sigma) A + (\kappa \sin \eta \cos \Sigma - \tau \sin \Sigma) N. \tag{3.12}$$

This formulation describes the angular motion of the ruled surface and useful for studying the rotational motion of a rigid body. The velocity of point  $P$  can be obtained as

$$V_P = W \times P \tag{3.13}$$

where  $W$  is Darboux vector and the moving point  $P$  according to generator trihedron is

$$P = x_1(s)e(s) + x_2(s)t(s) + x_3(s)g(s), \tag{3.14}$$

where  $x_1, x_2$  and  $x_3$  are coordinates of point  $P$  according to the generator trihedron. Using Eq. (2.16), the point  $P$  with respect to tool trihedron is

$$P = x_1O + (x_2 \cos \Sigma + x_3 \sin \Sigma) A + (x_3 \cos \Sigma - x_2 \sin \Sigma) N. \tag{3.15}$$

**Corollary 3.4** *The velocity of the adjoint robot end effector of connected at the point P is*

$$V_P = (\tau x_3 - \kappa x_2 \sin \eta) O + (\kappa \Pi_1 \cos \Sigma + \Pi_2 \sin \Sigma) A + (\Pi_2 \cos \Sigma - \kappa \Pi_1 \sin \Sigma) N, \tag{3.16}$$

where

$$\begin{aligned} \Pi_1 &= x_1 \sin \eta - x_3 \cos \eta \\ \Pi_2 &= \kappa x_2 \cos \eta - \tau x_1. \end{aligned} \tag{3.17}$$

Components the orientation and normal directions in Eqn. (3.16) gives the velocity about an axis of the robot end effector of connected at the point P and the component the approach direction give the velocity of spin motion about the normal direction of the robot end effector of connected at point P.

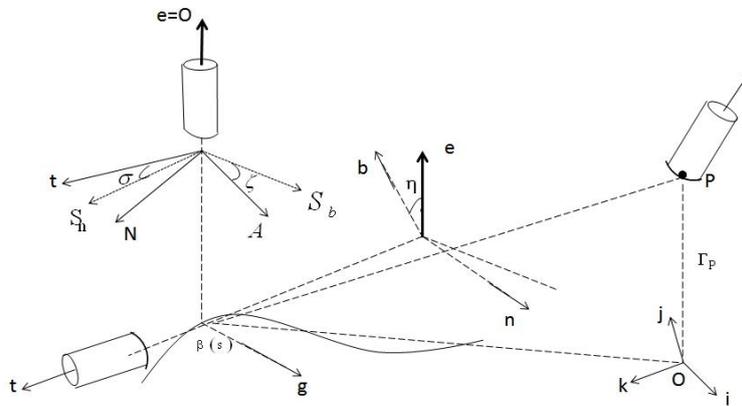


Figure 1: The adjoint trajectory of robot end effector at the point P

**Example 3.5** *Consider the ruled surface*

$$X(s, v) = \left( \frac{1}{2} \sqrt{2} v \cos^2 s + \frac{3}{5} \cos s, \frac{1}{2} \sqrt{2} v \sin^2 s + \frac{3}{5} \sin s, \frac{4}{5} s + \frac{1}{2} v \sin 2s \right), \tag{3.18}$$

where  $\alpha(s) = \left( \frac{3}{5} \cos s, \frac{3}{5} \sin s, \frac{4}{5} s \right)$  is the base curve,  $\bar{R}(s) = \left( \frac{1}{\sqrt{2}} \cos^2 s, \frac{1}{\sqrt{2}} \sin^2 s, \frac{1}{2} \sin 2s \right)$  is the director,  $-2 < s < 2, -2 < v < 2$ , (Fig. 2).

The generator trihedron  $\{e, t, g\}$  is defined by

$$\begin{aligned} e(s) &= \left( \cos^2 s, \sin^2 s, \frac{\sqrt{2}}{2} \sin 2s \right) \\ t(s) &= \left( -\frac{1}{\sqrt{2}} \sin 2s, \frac{1}{\sqrt{2}} \sin 2s, \cos 2s \right) \\ g(s) &= \left( \cos 2s \sin^2 s - \frac{1}{2} \sin^2 2s, -\cos 2s \cos^2 s + \frac{1}{2} \sin^2 2s, \frac{1}{\sqrt{2}} \sin 2s \right) \end{aligned}$$

and if we take  $\Sigma = s$ , then the tool trihedron  $\{O, A, N\}$  is

$$\begin{aligned}
 O(s) &= \left( \cos^2 s, \sin^2 s, \frac{\sqrt{2}}{2} \sin 2s \right) \\
 A(s) &= \begin{pmatrix} \left[ -\frac{1}{\sqrt{2}} \cos s \sin 2s + \cos 2s \sin^3 s - \frac{1}{2} \sin^2 2s \sin s, \right. \\ \left. \frac{1}{\sqrt{2}} \cos s \sin 2s - \cos 2s \cos^2 s \sin s - \frac{1}{2} \sin^2 2s \sin s, \right. \\ \left. \cos 2s \cos s + \frac{1}{\sqrt{2}} \sin 2s \sin s \right] \\ N(s) &= \begin{pmatrix} \left[ \frac{1}{\sqrt{2}} \sin s \sin 2s + \cos 2s \sin^2 s \cos s - \frac{1}{2} \sin^2 2s \cos s, \right. \\ \left. -\frac{1}{\sqrt{2}} \sin 2s \cos 2s \sin s - \cos 2s \cos^3 s - \frac{1}{2} \sin^2 2s \cos s, \right. \\ \left. -\sin s \cos 2s + \frac{1}{\sqrt{2}} \cos s \sin 2s \right] \end{pmatrix}
 \end{aligned}$$

By using Eqn. (2.5), the striction curve  $\beta(s)$  of ruled surface  $X$  is obtained as

$$\beta(s) = \begin{pmatrix} \left[ \frac{3}{5} \cos s - (\cos^2 s) \left( \frac{4}{5} \cos 2s + \frac{3}{10} \sqrt{2} \cos s \sin 2s + \frac{3}{10} \sqrt{2} \sin s \sin 2s \right), \right. \\ \left. \frac{3}{5} \sin s - (\sin^2 s) \left( \frac{4}{5} \cos 2s + \frac{3}{10} \sqrt{2} \cos s \sin 2s + \frac{3}{10} \sqrt{2} \sin s \sin 2s \right), \right. \\ \left. \frac{4}{5} s - \frac{1}{2} \sqrt{2} (\sin 2s) \left( \frac{4}{5} \cos 2s + \frac{3}{10} \sqrt{2} \cos s \sin 2s + \frac{3}{10} \sqrt{2} \sin s \sin 2s \right) \right] \end{pmatrix}.$$

If we take  $x_1 = s, x_2 = 2s, x_3 = 2s + 1$ , then the curve  $\Gamma_p$  which draws of moving point P is (Red)

$$\Gamma_P : r_P(s) = \begin{pmatrix} \left[ ((\cos s)(2s + 1) - 2s \sin s) \left( \cos s \cos 2s \sin^2 s + \frac{1}{2} \sqrt{2} \sin s \sin 2s - \frac{1}{2} \cos s \sin^2 2s \right), \right. \\ \left. -((\cos s)(2s + 1) - 2s \sin s) \left( \cos 2s \cos^3 s + \frac{1}{2} \cos s \sin^2 2s + \frac{1}{2} \sqrt{2} \sin s \cos 2s \sin 2s \right), \right. \\ \left. -(\sin s \cos 2s - \frac{1}{2} \sqrt{2} \cos s \sin 2s) ((\cos s)(2s + 1) - 2s \sin s) \right] \end{pmatrix}.$$

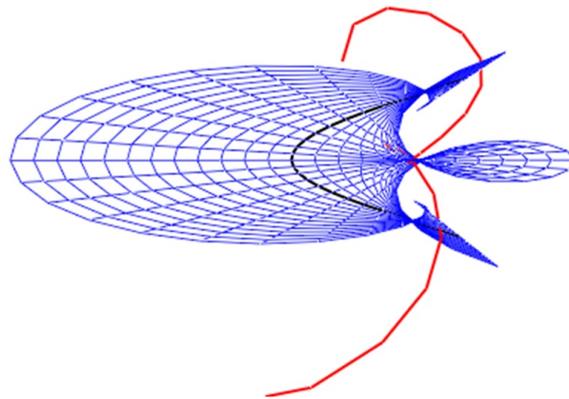


Figure 2: The trajectory of robot end effector and the adjoint trajectory at the point P

#### 4. Conclusion

In this paper, we introduce a method for finding adjoint trajectory of robot end effector. This method used to replace the trajectory of the robot movement with the adjoint trajectory when not re-computation of the robot trajectory. Also, we calculate the angular acceleration and angular velocity of adjoint trajectory. We illustrate the presented method with an example.

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