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The Natural Operators Similar to the Twisted Courant Bracket on Couples of Vector Fields and p-Forms

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Abstract. Given natural numbers m and p with $m \geq p+2 \geq 3$, all $\mathcal{M}f_m$-natural operators A sending closed $(p+2)$-forms H on m-manifolds M into $\mathbf{R}$-bilinear operators A_H transforming pairs of couples of vector fields and p-forms on M into couples of vector fields and p-forms on M are found. If $m \geq p+2 \geq 3$, all $\mathcal{M}f_m$-natural operators A (as above) such that A_H satisfies the Jacobi identity in Leibniz form are extracted, and that the twisted Courant bracket $[-,-]_H$ is the unique $\mathcal{M}f_m$-natural operator A_H (as above) satisfying the Jacobi identity in Leibniz form and some normalization condition is deduced.

1. Introduction

All manifolds considered in this paper are assumed to be finite dimensional second countable Hausdorff without boundary and smooth (of class C^∞). Maps between manifolds are assumed to be smooth (of class C^∞)

In [2, 4], the authors found all $\mathbf{R}$-bilinear operators $A : (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)) \to \mathcal{X}(M) \oplus \Omega^p(M)$ on couples of vector fields and p-forms on a m-dimensional manifold M, which are $\mathcal{M}f_m$-natural, i.e. invariant under the morphisms in the category $\mathcal{M}f_m$ of m-dimensional manifolds and their submersions. The principal result of [3] (or of [2] if $p = 1$) is precisely the full classification of such operators which also, like the Courant bracket, satisfy the Jacobi identity in Leibniz form. The Courant bracket, defined in [1] (if $p = 1$) and in [6] (for any p), is of particular interest, because it involves in the concept of Dirac structures and in the concept of generalized complex structures on M, see [1, 5, 6].

In the present paper, we find all $\mathcal{M}f_m$-natural operators A sending (closed) $(p+2)$-forms $H \in \Omega^{p+2}_{\mathrm{clos}}(M)$ on a m-manifold M into $\mathbf{R}$-bilinear operators

$$A_H : (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)) \to \mathcal{X}(M) \oplus \Omega^p(M)$$

transforming pairs of couples of vector fields and p-forms on M into couples of vector fields and p-forms on M. The principal result of the present paper is precisely the full classification of such $\mathcal{M}f_m$-natural operators A_H which also, like the twisted Courant bracket $[-,-]_H$, satisfy the Jacobi identity in Leibniz form

$$A_H(\rho_1, A_H(\rho_2, \rho_3)) = A_H(A_H(\rho_1, \rho_2), \rho_3) + A_H(\rho_2, A_H(\rho_1, \rho_3))$$

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for all $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^p(M)$ and all $H \in \Omega^{p+2}_{\rm clos}(M)$ and all m-manifolds M. Thus for $p = 1$ we reobtain the results from [9]. It is well-known that the twisted Courant bracket $[-,-]_H$ is of particular interest because its properties (if $p = 1$) were used in [8, 10] to define the concept of exact Courant algebroid.

At the end of the present paper, we observe that the twisted Courant bracket $[-,-]_H$ can be characterized as the unique $\mathcal{M}f_m$-natural operator A_H (as above) satisfying both the Jacobi identity in Leibniz form and the normalization condition

$$A_H(X^1 \oplus 0, X^2 \oplus 0) = [X^1, X^2] \oplus i_{X^1} i_{X^2} H$$

for all vector fields X^1, X^2 and closed $(p+2)$-forms H on m-manifolds.

From now on, (x^i) $(i = 1, ..., m)$ denote the usual coordinates on $\mathbf{R}^m$ and $\partial_i = \frac{\partial}{\partial x^i}$ are the canonical vector fields on $\mathbf{R}^m$.

2. The Courant like brackets

The general concept of natural operators can be found in [7]. In the present paper, we need only some particular cases of natural operators.

Definition 2.1. *An $\mathcal{M}f_m$-natural operator sending pairs $(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M))$ consisting of couples of vector fields and p-forms on m-manifolds M into couples $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \mathcal{X}(M) \oplus \Omega^p(M)$ of vector fields and p-forms on M is an $\mathcal{M}f_m$-invariant family A of operators*

$$A : (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)) \to \mathcal{X}(M) \oplus \Omega^p(M)$$

for m-dimensional manifolds M, where $\mathcal{X}(M)$ is the vector space of vector fields on M and $\Omega^p(M)$ is the vector space of p-forms on M. Such natural operator A is called bilinear if A is a bilinear (over $\mathbf{R}$) function $V \times V \to V$ with $V = \mathcal{X}(M) \oplus \Omega^p(M)$ for any m-manifold M.

Remark 2.2. *The $\mathcal{M}f_m$-invariance of A means that if $(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M))$ and $(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2) \in (\mathcal{X}(\tilde{M}) \oplus \Omega^p(\tilde{M})) \times (\mathcal{X}(\tilde{M}) \oplus \Omega^p(\tilde{M}))$ are φ-related by an $\mathcal{M}f_m$-map $\varphi : M \to \tilde{M}$ (i.e. $\tilde{X}^i \circ \varphi = T\varphi \circ X^i$ and $\tilde{\omega}^i \circ \varphi = \bigwedge^p T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$.*

The most important example of a natural bilinear operator A in the sense of Definition 2.1 is the Courant bracket on couples of vector fields and p-forms.

Example 2.3. *([6]) The Courant bracket (on couples of vector fields and p-forms) is defined by*

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]^C := [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1 + \frac{1}{2} d(i_{X^2}\omega^1 - i_{X^1}\omega^2))$$

for any $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^p(M)$, $i = 1, 2$, where $\mathcal{L}$ denotes the Lie derivative, d the exterior derivative, $[-,-]$ the usual bracket on vector fields and i is the insertion derivative. For $p = 1$ we obtain the usual Courant bracket as in [1].

Remark 2.4. *If $m = p$, we have $\mathcal{L}_X\omega = di_X\omega + i_X d\omega = di_X\omega$ for any vector field X and any m-form ω on a m-manifold M as $d\omega = 0$. Consequently, $[X^1 \oplus \omega^1, X^2 \oplus \omega^2]^C = [X^1, X^2] \oplus \frac{1}{2}(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1)$ for any $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^m(M)$, $i = 1, 2$.*

Theorem 2.5. *([4]) If $m \geq p + 1 \geq 2$ (or $m = p \geq 3$ respectively), any $\mathcal{M}f_m$-natural bilinear operator A in the sense of Definition 2.1 is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + c_1 d(i_{X^2}\omega^1) + c_2 d(i_{X^1}\omega^2))$$

for uniquely determined by A real numbers a, b_1, b_2, c_1, c_2 (or a, b_1, b_2, c_1, c_2 with $c_1 = c_2 = 0$ respectively).

Corollary 2.6. *([4]) If $m \geq p+1 \geq 2$ (or $m = p \geq 3$), any $\mathcal{M}f_m$-natural skew-symmetric bilinear operator A in the sense of Definition 2.1 is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) + cd(i_{X^2}\omega^1 - i_{X^1}\omega^2))$$

for uniquely determined by A real numbers a, b, c (or a, b, c with $c = 0$, respectively).

Roughly speaking, Corollary 2.6 says that if $m \geq p+1 \geq 2$ (or $m = p \geq 3$ respectively), then any skew-symmetric bilinear $\mathcal{M}f_m$-natural operator A in the sense of Definition 2.1 coincides with the generalized Courant bracket up to three (or two respectively) real constants.

Definition 2.7. *A $\mathcal{M}f_m$-natural bilinear operator A in the sense of Definition 2.1 satisfies the Jacobi identity in Leibniz form if*

$$A(\rho_1, A_H(\rho_2, \rho_3)) = A(A(\rho_1, \rho_2), \rho_3) + A(\rho_2, A(\rho_1, \rho_3))$$

for all $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^p(M)$ and all m-manifolds M.

Remark 2.8. *The Courant bracket presented in Example 2.3 is skew-symmetric but not satisfying the Jacobi identity in Leibniz form. The bracket $A^{<4,1,0>}$ from Theorem 2.9 below (also called the Courant bracket) is not skew-symmetric but it satisfies the Jacobi identity in Leibniz form.*

Theorem 2.9. *([3]) If $m \geq p+1 \geq 2$ (or $m = p \geq 3$ respectively), any bilinear $\mathcal{M}f_m$-natural operator A in the sense of Definition 2.1 satisfying the Jacobi identity in Lelbniz form is one of the following operators $A^{<1,a>}, A^{<2,a>}, A^{<3,a>}, A^{<4,a,0>}$ (or $A^{<1,a>}, A^{<2,a>}, A^{<3,a>}$ respectively) given by*

$$\begin{aligned}
&A^{<1,a>}(\rho^1, \rho^2) := a[X^1, X^2] \oplus 0\,, \\
&A^{<2,a>}(\rho^1, \rho^2) := a[X^1, X^2] \oplus (a(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1))\,, \\
&A^{<3,a>}(\rho^1, \rho^2) := a[X^1, X^2] \oplus a\mathcal{L}_{X^1}\omega^2\,, \\
&A^{<4,a,0>}(\rho^1, \rho^2) := a[X^1, X^2] \oplus (a(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1 + di_{X^2}\omega^1))\,,
\end{aligned}$$

where $a \in \mathbf{R}$ is an arbitrary real number, and where $\rho^i = X^i \oplus \omega^i$, $i = 1, 2$. For any $a \in \mathbf{R}$ each of operators $A^{<1,a>}, A^{<2,a>}, A^{<3,a>}, A^{<4,a,0>}$ satisfies the Jacobi identity in Leibniz form.

Corollary 2.10. *([3]) If $m \geq p+1 \geq 2$ or $m = p \geq 3$, any $\mathcal{M}f_m$-natural Lie algebra brackets on $\mathcal{X}(M) \oplus \Omega^p(M)$ (i.e. $\mathcal{M}f_m$-natural skew-symmetric bilinear operator satisfying the Jacobi identity in Leibniz form) is the constant multiple of the one of the following two Lie algebra brackets:*

$$\begin{aligned}
&[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]^1 := [X^1, X^2] \oplus 0\,, \\
&[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]^2 := [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1)\,.
\end{aligned}$$

3. The twisted Courant like brackets

Definition 3.1. *Let p be a fixed positive integer. A $\mathcal{M}f_m$-natural operator B sending $(p+1)$-forms $F \in \Omega^{p+1}(M)$ on m-manifolds M into $\mathbf{R}$-bilinear operators*

$$B_F : (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)) \to \mathcal{X}(M) \oplus \Omega^p(M)$$

is a $\mathcal{M}f_m$-invariant family of regular operators (functions)

$$B : \Omega^{p+1}(M) \to Lin_2((\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)), \mathcal{X}(M) \oplus \Omega^p(M))$$

for all m-manifolds M, where $Lin_2(U \times V, W)$ denotes the vector space of all bilinear (over $\mathbf{R}$) functions $U \times V \to W$ for any real vector spaces U, V, W.

Such natural operator B is called admissible if $B_F = B_{F+dF'}$ for any $F \in \Omega^{p+1}(M)$ and $F' \in \Omega^p(M)$.

Remark 3.2. *The invariance means that if $F^1 \in \Omega^{p+1}(M)$ and $F^2 \in \Omega^{p+1}(\tilde{M})$ are φ-related and $(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M))$ and $(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2) \in (\mathcal{X}(\tilde{M}) \oplus \Omega^p(\tilde{M})) \times (\mathcal{X}(\tilde{M}) \oplus \Omega^p(\tilde{M}))$ are φ-related by an $\mathcal{M}f_m$-map $\varphi : M \to \tilde{M}$, then so are $B_{F^1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $B_{F^2}(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$.*

Remark 3.3. *The regularity of B means that it transforms smoothly parametrized families $(F_t, X^1_t \oplus \omega^1_t, X^2_t \oplus \omega^2_t)$ into smoothly ones $B_{F_t}(X^1_t \oplus \omega^1_t, X^2_t \oplus \omega^2_t)$.*

Proposition 3.4. *Let B be an admissible $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1. Assume that $m \geq p + 2 \geq 3$. Then there exist uniquely determined real numbers a, b_1, b_2, c_1, c_2, e such that*

$$B_F(\rho^1, \rho^2) = a[X^1, X^2] \oplus$$
$$(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d i_{X^1}\omega^2 + c_2 d i_{X^2}\omega^1 + e i_{X^1} i_{X^2} dF)$$

for any $F \in \Omega^{p+1}(M)$ and any $\rho^1, \rho^2 \in \mathcal{X}(M) \oplus \Omega^p(M)$ and any $\mathcal{M}f_m$-object M, where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$.

Proof. Operator B_0, where 0 is the zero $(p + 1)$-form, can be treated as the natural bilinear operator in the sense of Definition 2.1. Then B_0 is described in Theorem 2.5. So, replacing B by $B - B_0$, we have assumption $B_0 = 0$. Then, by the admissibility of B, $B_{dF'} = 0$ for any p-form F'.

Put $B_F(-,-) = (B^1_F(-,-), B^2_F(-,-))$, where $B^1_F(...) \in \mathcal{X}(M)$ and $B^2_F(...) \in \Omega^{p+1}(M)$. By the $\mathcal{M}f_m$-invariance, B is determined by the values

$$< B^1_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|_0}, \eta >\in \mathbf{R} \text{ and } < B^2_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \mu >\in \mathbf{R}$$

for $F \in \Omega^{p+1}(\mathbf{R}^m)$, $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \mathcal{X}(\mathbf{R}^m) \oplus \Omega^p(\mathbf{R}^m)$, $\eta \in T^*_0\mathbf{R}^m$, $\mu \in \wedge^p T_0\mathbf{R}^m$. We can assume $X^1_{|0} \wedge X^2_{|0} \wedge \mu \neq 0$, and then by the invariance we can assume

$$X^1_{|0} = \partial_{1|0} \ , \ X^2_{|0} = \partial_{2|0} \ , \ \mu = \partial_{3|0} \wedge ... \wedge \partial_{p+2|0} \ .$$

By Corollary 19.9 of the non-linear Petree theorem in [7] there exists a finite number r (possible depending on $(X^1, X^2, \omega^1, \omega^2, F)$) such that from $(j^r_0 F = j^r_0 \overline{F}, j^r_0 X^1 = j^r_0 \overline{X}^1, j^r_0 \omega^1 = j^r_0 \overline{\omega}^1, j^r_0 X^2 = j^r_0 \overline{X}^2, j^r_0 \omega^2 = j^r_0 \overline{\omega}^2)$ it follows $B_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0} = B_{\overline{F}}(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)_{|0}$. So, we may assume $F, X^1, X^2, \omega^1, \omega^2$ are polynomial of degree not more than r.

Using the invariance of B with respect to the homotheties and the bi-linearity of B_F we obtain the homogeneity condition

$$< B^1_{(\frac{1}{t}\mathrm{id})_* F}(t(\frac{1}{t}\mathrm{id})_* X^1 \oplus \mathrm{t}(\frac{1}{\mathrm{t}}\mathrm{id})_*\omega^1, \mathrm{t}(\frac{1}{\mathrm{t}}\mathrm{id})_* X^2 \oplus \mathrm{t}(\frac{1}{\mathrm{t}}\mathrm{id})_*\omega^2)_{|_0}, \eta >$$
$$= t < B^1_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|_0}, \eta > \ .$$

Then, by the homogeneous function theorem, since $B_0 = 0$ and $p+1 \geq 2$, we have $< B^1_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|_0}, \eta >= 0$.

Using the same arguments we get homogeneity condition

$$< B^2_{(\frac{1}{t}\mathrm{id})_* F}(t(\frac{1}{t}\mathrm{id})_* X^1 \oplus \mathrm{t}(\frac{1}{\mathrm{t}}\mathrm{id})_*\omega^1, \mathrm{t}(\frac{1}{\mathrm{t}}\mathrm{id})_* X^2 \oplus \mathrm{t}(\frac{1}{\mathrm{t}}\mathrm{id})_*\omega^2)_{|_0}, \mu >$$
$$= t^{p+2} < B^2_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|_0}, \mu > \ .$$

Then, by the homogeneous function theorem and the bi-linearity of B_F and the assumption $B_{dF'} = 0$, $< B^2_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \mu >$ is linear in F and it is determined by the values

$$< B^2_{x^{i_1} dx^{i_2} \wedge ... \wedge dx^{i_{p+2}}}(\partial_1 \oplus 0, \partial_2 \oplus 0), \partial_{3|0} \wedge ... \wedge \partial_{p+2|0} >$$

for all $i_1 = 1, ..., m$ and $i_2, ..., i_{p+2}$ with $1 \leq i_2 < ... < i_{p+2} \leq m$. Then using the invariance of B with respect to $(\tau^1 x^1, ... \tau^m x^m)$ for $\tau^i > 0$ we deduce that only

$$< B^2_{x^i dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^{p+2}}(\partial_1 \oplus 0, \partial_2 \oplus 0), \partial_{3|0} \wedge ... \wedge \partial_{p+2|0} >$$

for $i = 1, ..., p+2$ may be non-zero. But if $p+2 \geq i \geq 2$, then

$$(-1)^i x^1 dx^2 \wedge ... \wedge dx^{p+2} = -x^i dx^1 \wedge ... \wedge \widehat{dx^i} \wedge .... \wedge dx^{p+2} + d(x^1 x^i dx^2 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^{p+2}) .$$

Then $< B^2_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|0}, \mu >$ is determined by

$$< B^2_{x^1 dx^2 \wedge ... \wedge dx^{p+2}}(\partial_1 \oplus 0, \partial_2 \oplus 0), \partial_{3|0} \wedge ... \wedge \partial_{p+2|0} >$$

because of the assumption $B_F = B_{F+dF'}$.

Then the vector space of all such B (with $B_0 = 0$) is at most 1-dimensional. On the other hand, $B_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = 0 \oplus i_{X^1} i_{X^2} dF$ is an example of such B. □

Definition 3.5. *Let p be a fixed positive integer. A $\mathcal{M}f_m$-natural operator A sending closed (p+2)-forms $H \in \Omega^{p+2}_{\rm clos}(M)$ on m-manifolds M into bilinear operators*

$$A_H : (\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)) \to \mathcal{X}(M) \oplus \Omega^p(M)$$

is a $\mathcal{M}f_m$-invariant family of regular operators (functions)

$$A : \Omega^{p+2}_{\rm clos}(M) \to Lin_2((\mathcal{X}(M) \oplus \Omega^p(M)) \times (\mathcal{X}(M) \oplus \Omega^p(M)), \mathcal{X}(M) \oplus \Omega^p(M))$$

for all m-manifolds M.

Example 3.6. *The most important example of such A_H is the twisted Courant bracket*

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_H := [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - i_{X^2} d\omega^1 + i_{X^1} i_{X^2} H)$$

for all closed $(p+2)$-forms $H \in \Omega^{p+2}_{\rm clos}(M)$ and all m-manifolds M.

Lemma 3.7. *Any natural operator A in the sense of Definition 3.5 defines an admissible natural operator $B^{<A>}$ in the sense of Definition 3.1 by $B^{<A>}_F := A_{dF}$ for any $F \in \Omega^{p+1}(M)$. If A^1 is another natural operator in the sense of Definition 3.5 such that $B^{<A>} = B^{<A^1>}$ then $A = A^1$.*

Proof. The first sentence is clear. To prove the second one, we observe that $B^{<A>} = B^{<A^1>}$ means that $A_H = A^1_H$ for exact $(p+2)$-forms H. Since the $\mathcal{M}f_m$-invariance of A and A^1 implies that A and A^1 are local operators, we can replace "for exact" by "for closed" because of the Poincare lemma. □

Combining Lemma 3.7 and Proposition 3.4 we immediately get the following complete description of natural operators in the sense of Definitin 3.5.

Theorem 3.8. *Let p be a fixed positive integer. Let A be a $\mathcal{M}f_m$-natural operator in the sense of Definition 3.5. Assume that $m \geq p+2 \geq 3$. Then there exist uniquely determined real numbers a, b_1, b_2, c_1, c_2, e such that*

$$A_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus$$
$$(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d i_{X^1}\omega^2 + c_2 d i_{X^2}\omega^1 + e i_{X^1} i_{X^2} H)$$

for any $H \in \Omega^{p+2}_{\rm clos}(M)$ and any $\rho^1, \rho^2 \in \mathcal{X}(M) \oplus \Omega^p(M)$ and any m-manifold M, where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$.

4. The twisted Courant like brackets satisfying the Jacobi identity in Leibniz form

Definition 4.1. *Let p be a fixed positive integer. A $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.5 satisfies the Jacobi identity in Leibniz form if*

$$A_H(\rho_1, A_H(\rho_2, \rho_3)) = A_H(A_H(\rho_1, \rho_2), \rho_3) + A_H(\rho_2, A_H(\rho_1, \rho_3))$$

for all $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^p(M)$ and all $H \in \Omega^{p+2}_{\rm clos}(M)$ and all m-manifolds M.

Example 4.2. *The twisted Courant bracket $[-,-]_H$ is an example of natural operator in question satisfying the Jacobi identity in Leibniz form. (Namely, it is $A^{<4,1,1>}$ from Theorem 4.3, below.)*

Theorem 4.3. *Let $m \geq p+2 \geq 3$. Any $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.5 satisfying the Jacobi identity in Leibniz form is one of the following operators*

$$\begin{aligned}
&A_H^{<1,a>}(\rho_1,\rho_2) := a[X^1,X^2]\oplus 0\,,\\
&A_H^{<2,a>}(\rho^1,\rho^2) := a[X^1,X^2]\oplus (a(\mathcal{L}_{X^1}\omega^2-\mathcal{L}_{X^2}\omega^1))\,,\\
&A_H^{<3,a>}(\rho^1,\rho^2) := a[X^1,X^2]\oplus (a\mathcal{L}_{X^1}\omega^2)\,,\\
&A_H^{<4,a,e>}(\rho^1,\rho^2) := a[X^1,X^2]\oplus (a(\mathcal{L}_{X^1}\omega^2-\mathcal{L}_{X^2}\omega^1+di_{X^2}\omega^1)+ei_{X^1}i_{X^2}H)\,,
\end{aligned}$$

where $\rho^1 = X^1\oplus\omega^1$ and $\rho^2 = X^2\oplus\omega^2$, and a and e are arbitrary real numbers. For any $a,e\in\mathbf{R}$ each of operators $A^{<1,a>}, A^{<2,a>}, A^{<3,a>}, A^{<4,a,e>}$ satisfies the Jacobi identity in Leibniz form.

Proof. Let A be a $\mathcal{M}f_m$-natural operator in the sense of Definition 3.5 satisfying the Jacobi identity in Leibniz form. By Theorem 3.8, A is of the form

$$\begin{aligned}
&A_H(X^1\oplus\omega^1,X^2\oplus\omega^2) = a[X^1,X^2]\oplus\\
&(b_1\mathcal{L}_{X^2}\omega^1+b_2\mathcal{L}_{X^1}\omega^2+c_1di_{X^2}\omega^1+c_2di_{X^1}\omega^2+ei_{X^1}i_{X^2}H)
\end{aligned}$$

for (uniquely determined by A) real numbers a,b_1,b_2,c_1,c_2,e. Then for any $X^1,X^2,X^3\in\mathcal{X}(M)$ and $\omega^1,\omega^2,\omega^3\in\Omega^p(M)$ and $H\in\Omega^{p+2}_{\mathrm{clos}}(M)$ we have

$$A_H(X^1\oplus\omega^1,A_H(X^2\oplus\omega^2,X^3\oplus\omega^3)) = a^2[X^1,[X^2,X^3]]\oplus\Omega\,,$$

$$A_H(A_H(X^1\oplus\omega^1,X^2\oplus\omega^2),X^3\oplus\omega^3) = a^2[[X^1,X^2],X^3]\oplus\Theta\,,$$

$$A_H(X^2\oplus\omega^2,A_H(X^1\oplus\omega^1,X^3\oplus\omega^3)) = a^2[X^2,[X^1,X^3]]\oplus\mathcal{T}\,,$$

where

$$\begin{aligned}
\Omega &= b_1\mathcal{L}_{a[X^2,X^3]}\omega^1+c_1di_{a[X^2,X^3]}\omega^1+ei_{X^1}i_{a[X^2,X^3]}H\\
&+b_2\mathcal{L}_{X^1}(b_1\mathcal{L}_{X^3}\omega^2+b_2\mathcal{L}_{X^2}\omega^3+c_1di_{X^3}\omega^2+c_2di_{X^2}\omega^3+ei_{X^2}i_{X^3}H)\\
&+c_2di_{X^1}(b_1\mathcal{L}_{X^3}\omega^2+b_2\mathcal{L}_{X^2}\omega^3+c_1di_{X^3}\omega^2+c_2di_{X^2}\omega^3+ei_{X^2}i_{X^3}H)\,,
\end{aligned}$$

$$\begin{aligned}
\Theta &= b_2\mathcal{L}_{a[X^1,X^2]}\omega^3+c_2di_{a[X^1,X^2]}\omega^3+ei_{a[X^1,X^2]}i_{X^3}H\\
&+b_1\mathcal{L}_{X^3}(b_1\mathcal{L}_{X^2}\omega^1+b_2\mathcal{L}_{X^1}\omega^2+c_1di_{X^2}\omega^1+c_2di_{X^1}\omega^2+e_{X^1}i_{X^2}H)\\
&+c_1di_{X^3}(b_1\mathcal{L}_{X^2}\omega^1+b_2\mathcal{L}_{X^1}\omega^2+c_1di_{X^2}\omega^1+c_2di_{X^1}\omega^2+ei_{X^1}i_{X^2}H)\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{T} &= b_1\mathcal{L}_{a[X^1,X^3]}\omega^2+c_1di_{a[X^1,X^3]}\omega^2+ei_{X^2}i_{a[X^1,X^3]}H\\
&+b_2\mathcal{L}_{X^2}(b_1\mathcal{L}_{X^3}\omega^1+b_2\mathcal{L}_{X^1}\omega^3+c_1di_{X^3}\omega^1+c_2di_{X^1}\omega^3+ei_{X^1}i_{X^3}H)\\
&+c_2di_{X^2}(b_1\mathcal{L}_{X^3}\omega^1+b_2\mathcal{L}_{X^1}\omega^3+c_1di_{X^3}\omega^1+c_2di_{X^1}\omega^3+ei_{X^1}i_{X^3}H)\,.
\end{aligned}$$

The Jacobi identity in Leibniz form of A_H is equivalent to $\Omega = \Theta+\mathcal{T}$.

Putting $H=0$, we are in the situation of Theorem 2.9. Then by Theorem 2.9 we get $(b_1,b_2,c_1,c_2) = (0,0,0,0)$ or $(b_1,b_2,c_1,c_2) = (0,a,0,0)$ or $(b_1,b_2,c_1,c_2) = (-a,a,0,0)$ or $(b_1,b_2,c_1,c_2) = (-a,a,a,0)$. More, A_0 for such (b_1,b_2,c_1,c_2) satisfies the Jacobi identity in Leibniz form.

Therefore (as $c_2 = 0$) the Jacobi identity in Leibniz form of A_H is equivalent to the equality

$$\begin{aligned}&eai_{X^1}i_{[X^2,X^3]}H + b_2e\mathcal{L}_{X^1}i_{X^2}i_{X^3}H\\&= eai_{[X^1,X^2]}i_{X^3}H + b_1e\mathcal{L}_{X^3}i_{X^1}i_{X^2}H + c_1edi_{X^3}i_{X^1}i_{X^2}H\\&+eai_{X^2}i_{[X^1,X^3]}H + b_2e\mathcal{L}_{X^2}i_{X^1}i_{X^3}H\ .\end{aligned}$$

Put $\omega^o := dx^4 \wedge ... \wedge dx^{p+2}$ if $p + 2 \geq 4$ and $\omega^o := 1$ if $p + 2 = 3$.

If $(b_1, b_2, c_1, c_2) = (0, 0, 0, 0)$, the above equality is equivalent to

$$eai_{X^1}i_{[X^2,X^3]}H = eai_{[X^1,X^2]}i_{X^3}H + eai_{X^2}i_{[X^1,X^3]}H\ .$$

Putting $X^1 = \partial_1$, $X^2 = \partial_1 + x^1\partial_3$ and $X^3 = \partial_2$ we have $[X^2, X^3] = 0$, $[X^1, X^3] = 0$ and $[X^1, X^2] = \partial_3$, and then $0 = eai_{\partial_3}i_{\partial_2}H$ for any closed H (for example for $H = dx^1 \wedge dx^2 \wedge dx^3 \wedge \omega^o$). Consequently $e = 0$ or $a = 0$.

If $(b_1, b_2, c_1, c_2) = (0, a, 0, 0)$, the above equality is equivalent to

$$eai_{X^1}i_{[X^2,X^3]}H + ea\mathcal{L}_{X^1}i_{X^2}i_{X^3}H = eai_{[X^1,X^2]}i_{X^3}H + eai_{X^2}i_{[X^1,X^3]}H + ea\mathcal{L}_{X^2}i_{X^1}i_{X^3}H\ .$$

Putting $X^1 = \partial_1$, $X^2 = \partial_2$ and $X^3 = \partial_3$ and $H = x^2dx^1 \wedge dx^2 \wedge dx^3 \wedge \omega^o$ (it is closed) we have $[X^2, X^3] = 0$, $[X^1, X^2] = 0$, $[X^1, X^3] = 0$, $\mathcal{L}_{X^2}i_{X^1}i_{X^3}H = \mathcal{L}_{\partial_2}x^2dx^2 \wedge \omega^o = dx^2 \wedge \omega^o$ and $\mathcal{L}_{X^1}i_{X^2}i_{X^3}H = \mathcal{L}_{\partial_1}(-x^2dx^1 \wedge \omega^o) = 0$. Then $eadx^2 \wedge \omega^o = 0$. So, $a = 0$ or $e = 0$.

If $(b_1, b_2, c_1, c_2) = (-a, a, 0, 0)$, the above equality is equivalent to

$$\begin{aligned}&eai_{X^1}i_{[X^2,X^3]}H + ea\mathcal{L}_{X^1}i_{X^2}i_{X^3}H\\&= eai_{[X^1,X^2]}i_{X^3}H - ea\mathcal{L}_{X^3}i_{X^1}i_{X^2}H + eai_{X^2}i_{[X^1,X^3]}H + ea\mathcal{L}_{X^2}i_{X^1}i_{X^3}H\ .\end{aligned}$$

Putting $X^1 = \partial_1$, $X^2 = \partial_2$ and $X^3 = \partial_3$ and $H = x^2dx^1 \wedge dx^2 \wedge dx^3 \wedge \omega^o$ we have (see above) $[X^2, X^3] = 0$, $[X^1, X^2] = 0$, $[X^1, X^3] = 0$, $\mathcal{L}_{X^2}i_{X^1}i_{X^3}H = dx^2 \wedge \omega^o$, $\mathcal{L}_{X^1}i_{X^2}i_{X^3}H = 0$ and $\mathcal{L}_{X^3}i_{X^1}i_{X^2}H = \mathcal{L}_{\partial_3}(-x^2dx^3 \wedge \omega^o) = 0$. Then $eadx^2 \wedge \omega^o = 0$. So, $a = 0$ or $e = 0$.

If $(b_1, b_2, c_1, c_2) = (-a, a, a, 0)$, the above equality is equivalent to

$$ea\sum\{i_{X^1}i_{[X^2,X^3]}H + \mathcal{L}_{X^1}i_{X^2}i_{X^3}H\} = eadi_{X^1}i_{X^2}i_{X^3}H\ ,$$

where $\sum$ is the cyclic sum $\sum_{cycl(X^1,X^2,X^3)}$. Then e is arbitrary real number because of from $dH = 0$ it follows

$$\sum\{i_{X^1}i_{[X^2,X^3]}H + \mathcal{L}_{X^1}i_{X^2}i_{X^3}H\} = di_{X^1}i_{X^2}i_{X^3}H\ ,$$

see Lemma 4.4, below.

Summing up, given a real number $a \neq 0$ we have $(b_1, b_2, c_1, c_2, e) = (0, 0, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2, e) = (0, a, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2, e) = (-a, a, 0, 0, 0,)$ or $(b_1, b_2, c_1, c_2, e) = (-a, a, a, 0, e)$. If $a = 0$ we have $(b_1, b_2, c_1, c_2, e) = (0, 0, 0, 0, e)$. Theorem 4.3 is complete. □

Lemma 4.4. *Let $H \in \Omega^{p+2}_{\text{clos}}(M)$ and $X^1, X^2, X^3 \in \mathcal{X}(M)$. Then the equality*

$$\sum\{i_{X^1}i_{[X^2,X^3]}H + \mathcal{L}_{X^1}i_{X^2}i_{X^3}H\} = di_{X^1}i_{X^2}i_{X^3}H\ ,$$

holds.

Proof. We have an $\mathcal{M}f_m$-natural 4-linear operator $C : \mathcal{X}(M) \oplus \mathcal{X}(M) \oplus \mathcal{X}(M) \oplus \Omega^{p+1}(M) \to \Omega^p(M)$ given by

$$C(X^1, X^2, X^3, F) := \sum_{cycl(X^1,X^2,X^3)}\{i_{X^1}i_{[X^2,X^3]}dF + \mathcal{L}_{X^1}i_{X^2}i_{X^3}dF\} - di_{X^1}i_{X^2}i_{X^3}dF$$

for any $X^1, X^2, X^3 \in \mathcal{X}(M)$ and $F \in \Omega^{p+1}(M)$.

By the Poincare lemma, it is sufficient to show $C = 0$, i.e. that $C(X^1, X^2, X^3, F)_x = 0$ for any X^1, X^2, X^3, F as above and $x \in M$.

Because of the invariance of C we may assume that $M = \mathbf{R}^m$ and $x = 0$. Since C is 4-linear, we may assume that $F = f dg^1 \wedge ... \wedge dg^{p+1}$, where $f, g^1, ..., g^{p+1} : \mathbf{R}^m \to \mathbf{R}$ are maps. More, we may assume that $d_0(f, g_1, ..., g_{p+1})$ is of rank $p + 2$. Then (by the $\mathcal{M}f_m$-invariance of C) we may assume $F = (x^1 + \lambda)dx^2 \wedge ... \wedge dx^{p+2}$. We else may assume $X^1 = h^1\partial_j$ and $X^2 = h^2\partial_k$ and $X^3 = h^3\partial_l$ for some $h^1, h^2, h^3 : \mathbf{R}^m \to \mathbf{R}$.

Now, to complete the lemma, it is sufficient to verify the following two facts:

(1) We have $C(\partial_j, \partial_k, \partial_l, (x^1 + \lambda)dx^2 \wedge ... \wedge dx^{p+2})_0 = 0$;

(2) We have implication: If $C(X^1, X^2, X^3, F)_0 = 0$, then $C(hX^1, X^2, X^3, F)_0 = 0$ and $C(X^1, hX^2, X^3, F)_0 = 0$ and $C(X^1, X^2, hX^3, F)_0 = 0$ for any $h : \mathbf{R}^m \to \mathbf{R}$.

ad(1) We can easily see that $\sum_{cycl(\partial_j,\partial_k,\partial_l)} i_{\partial_j} i_{[\partial_k,\partial_l]} dx^1 \wedge ... \wedge dx^{p+2} = 0$ and $\sum_{cycl(\partial_j,\partial_k,\partial_l)} \mathcal{L}_{\partial_j} i_{\partial_k} i_{\partial_l} dx^1 \wedge ... \wedge dx^{p+2} = 0$ and $d i_{\partial_j} i_{\partial_k} i_{\partial_l} dx^1 \wedge ... \wedge dx^{p+2} = 0$. That is why $C(\partial_j, \partial_k, \partial_l, (x^1 + \lambda)dx^2 \wedge ... \wedge dx^{p+2})_0 = 0$.

ad(2) We have

$$
\begin{aligned}
&C(hX^1, X^2, X^3, F) = i_{hX^1} i_{[X^2,X^3]} dF + i_{X^3} i_{[hX^1,X^2]} dF + i_{X^2} i_{[X^3,hX^1]} dF + \\
&+\mathcal{L}_{hX^1} i_{X^2} i_{X^3} dF + \mathcal{L}_{X^3} i_{hX^1} i_{X^2} dF + \mathcal{L}_{X^2} i_{X^3} i_{hX^1} dF - d i_{hX^1} i_{X^2} i_{X^3} dF \\
&= h i_{X^1} i_{[X^2,X^3]} dF + h i_{X^3} i_{[X^1,X^2]} dF - X^2(h) i_{X^3} i_{X^1} dF + \\
&+h i_{X^2} i_{[X^3,X^1]} dF + X^3(h) i_{X^2} i_{X^1} dF + h\mathcal{L}_{X^1} i_{X^2} i_{X^3} dF + dh \wedge i_{X^1} i_{X^2} i_{X^3} dF + \\
&+h\mathcal{L}_{X^3} i_{X^1} i_{X^2} dF + X^3(h) i_{X^1} i_{X^2} dF + h\mathcal{L}_{X^2} i_{X^3} i_{X^1} dF + X^2(h) i_{X^3} i_{X^1} dF + \\
&-h d i_{X^1} i_{X^2} i_{X^3} dF - dh \wedge i_{X^1} i_{X^2} i_{X^3} dF = hC(X^1, X^2, X^3, F) + 0\ .
\end{aligned}
$$

So, $C(hX^1, X^2, X^3, F)_0 = 0$ if $C(X^1, X^2, X^3, F)_0 = 0$. Similarly, we get that $C(X^1, hX^2, X^3, F)_0 = 0$ and $C(X^1, X^2, hX^3, F)_0 = 0$ if $C(X^1, X^2, X^3, F)_0 = 0$. $\square$

From Theorem 4.3 it follows the following interesting characterization of the twisted Courant bracket $[-,-]_H$ (from Example 3.6).

Corollary 4.5. *Let $m \geq p + 2 \geq 3$. The twisted Courant bracket is the unique $\mathcal{M}f_m$-natural operator A in the sense of Definition 3.5 satisfying the Jacobi identity in Leibniz form and the normalization condition $A_H(X^1 \oplus 0, X^2 \oplus 0) = [X^1, X^2] \oplus i_{X^1} i_{X^2} H$ for all vector fields X^1, X^2 and closed $(p + 2)$-forms H on m-manifolds.*

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