



Some Steffensen-Type Inequalities Over Time Scale Measure Spaces

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Abstract. In this paper we study some new dynamic Steffensen-type inequalities on a general time scale. More precisely, we deal with time scale spaces with positive σ -finite measures. As an application, our results are compared with some previous results known from the literature. It turns out that our results generalize some previously known Steffensen-type inequalities in a classical setting.

1. Introduction

Let us first recall the integral Steffensen inequality (see [23]):

Theorem 1.1 (Integral Steffensen's inequality). *Let f and g be integrable functions on $[a, b]$ such that f is nonincreasing and $0 \leq g(t) \leq 1$, $t \in [a, b]$. If $\lambda = \int_a^b g(t)dt$, then hold the inequalities*

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt. \quad (1)$$

Clearly, if f is nondecreasing, then the inequalities in (1) are reversed. The discrete version of the Steffensen inequality can be stated as follows (see [12]):

Theorem 1.2 (Discrete Steffensen's inequality). *Let $\{f(k)\}_{k=1}^n$ be nonnegative nonincreasing sequence and let $\{g(k)\}_{k=1}^n$ be such that $0 \leq g(k) \leq 1$, for every k . Furthermore, assume that $\lambda_1, \lambda_2 \in \{1, \dots, n\}$ are such that $\lambda_2 \leq \sum_{k=1}^n g(k) \leq \lambda_1$. Then hold the inequalities*

$$\sum_{k=n-\lambda_2+1}^n f(k) \leq \sum_{k=1}^n f(k)g(k) \leq \sum_{k=1}^{\lambda_1} f(k). \quad (2)$$

Recently, Jakšetić et al. [15], established several Steffensen-type inequalities related to a positive σ -finite measure. In particular, they proved the following two theorems:

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Theorem 1.3. Let $\hat{\mu}$ be a positive finite measure on Borel σ -algebra $\mathfrak{B}([a, b])$, and let $f, g : [a, b] \rightarrow \mathbb{R}$ be measurable functions such that f is nonincreasing and $0 \leq g(t) \leq 1$, for all $t \in [a, b]$. Further, let $\hat{\mu}([c, d]) = \int_{[a,b]} g(t)d\hat{\mu}(t)$, where $[c, d] \subseteq [a, b]$. Then holds the inequality

$$\int_{[a,b]} f(t)g(t)d\hat{\mu}(t) \leq \int_{[c,d]} f(t)g(t)d\hat{\mu}(t) + \int_{[a,c]} (f(t) - f(d))g(t)d\hat{\mu}(t).$$

Theorem 1.4. Let $\hat{\mu}$ be a positive finite measure on $\mathfrak{B}([a, b])$, and let $f, g : [a, b] \rightarrow \mathbb{R}$ be measurable functions such that f is nonincreasing and $0 \leq g(t) \leq 1$, $t \in [a, b]$. If $\hat{\mu}([c, d]) = \int_{[a,b]} g(t)d\hat{\mu}(t)$, where $[c, d] \subseteq [a, b]$, then

$$\int_{[c,d]} f(t)d\hat{\mu}(t) - \int_{[d,b]} (f(c) - f(t))g(t)d\hat{\mu}(t) \leq \int_{[a,b]} f(t)g(t)d\hat{\mu}(t).$$

In order to unify discrete and continuous analysis, Hilger initiated the theory of time scales in his PhD thesis [13] (see also [14]). Since then, this theory has received a lot of attention. The basic idea in time scales theory is to establish a result for a dynamic inequality or a dynamic equation, where the domain of the unknown function is the so-called time scale \mathbb{T} , which is an arbitrary closed subset of \mathbb{R} (see [6, 8]). The most common examples of time scales are continuous calculus, discrete calculus and quantum calculus, that is, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^z : z \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$. For more details about the time scales calculus, the reader is referred to monograph [7] due to Bohner and Peterson.

In the last two decades, a whole series of dynamic inequalities has been established by numerous researchers. For the reader’s convenience, we refer here to some recent papers due to Bohner, Saker, Srivastava and others: (see [1–4, 7, 10, 17, 18, 20–22, 25, 26]). In particular, Anderson [5], extended the Steffensen inequality to times scales using nabla integrals as follows:

Theorem 1.5. Let \mathbb{T} be a time scale and let $f, g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be nabla integrable functions such that f is of one sign and nonincreasing, and $0 \leq g(t) \leq 1$, for every $t \in [a, b]_{\mathbb{T}}$. If $\lambda = \int_a^b g(t)\nabla t$ with $b - \lambda, a + \lambda \in [a, b]_{\mathbb{T}}$, then

$$\int_{b-\lambda}^b f(t)\nabla t \leq \int_a^b f(t)g(t)\nabla t \leq \int_a^{a+\lambda} f(t)\nabla t.$$

In this paper, we extend some Steffensen-type inequalities established in [15] to a general time scale measure space with positive σ -finite measure. As a special case, we will also obtain some dynamic integral inequalities known from the literature. The paper is arranged as follows. In Section 2, some basic concepts and properties of the time scale calculus are introduced. Further, in Section 3, we state and prove our main results.

2. Basics of Time Scales

In this section, we recall some basic concepts in the theory of time scales. For more details of time scale analysis, we refer the reader to two monographs by Bohner and Peterson [7, 8], which provide a comprehensive insight into the time scale calculus. In addition, we also give several basic results that will be utilized in the proofs of our main results.

A time scale \mathbb{T} is an arbitrary nonempty closed subset of real numbers \mathbb{R} . Throughout the article, we assume that \mathbb{T} has the topology that it inherits from the standard topology on \mathbb{R} . We define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, for any $t \in \mathbb{T}$, by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$, for any $t \in \mathbb{T}$, by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. In the preceding two definitions, we set $\inf \emptyset = \sup \mathbb{T}$ (i.e. if t is the maximum of \mathbb{T} , then $\sigma(t) = t$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. if t is the minimum of \mathbb{T} , then $\rho(t) = t$).

A point $t \in \mathbb{T}$ with $\inf \mathbb{T} < t < \sup \mathbb{T}$ is said to be right-scattered if $\sigma(t) > t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and left-dense if $\rho(t) = t$. Points that are simultaneously right-dense and left-dense are said to be dense points. Furthermore, points that are simultaneously right-scattered and left-scattered

are said to be isolated points. The forward graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined for any $t \in \mathbb{T}$ by $\mu(t) := \sigma(t) - t$.

The interval $[a, b]$ in \mathbb{T} is defined by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. The open intervals and half-closed intervals are defined similarly.

For $f : \mathbb{T} \rightarrow \mathbb{R}$, the function $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$, while $f^\rho : \mathbb{T} \rightarrow \mathbb{R}$ is defined by $f^\rho(t) = f(\rho(t))$. The sets \mathbb{T}^κ , \mathbb{T}_κ and \mathbb{T}_κ^κ are introduced as follows: If \mathbb{T} has a left-scattered maximum t_1 , then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If \mathbb{T} has a right-scattered minimum t_2 , then $\mathbb{T}_\kappa = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally, we have $\mathbb{T}_\kappa^\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be right-dense continuous (rd-continuous) if f is continuous at the right-dense points in \mathbb{T} and its left-sided limits exist at all left-dense points in \mathbb{T} . If $t \in \mathbb{T}^\kappa$, then $f^\Delta(t) \in \mathbb{R}$ is said to be a delta derivative of f at t if for any $\varepsilon > 0$ there exists a neighborhood U of t such that, for every $s \in U$, we have $|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$. Moreover, f is said to be delta differentiable on \mathbb{T}^κ if it is delta differentiable at every $t \in \mathbb{T}^\kappa$.

The delta integration by parts on time scales is given by the following formula:

$$\int_a^b g^\Delta(t)f(t)\Delta t = g(b)f(b) - g(a)f(a) - \int_a^b g^\sigma(t)f^\Delta(t)\Delta t. \tag{3}$$

We will use the following crucial relations between calculus on time scales \mathbb{T} and either differential calculus on \mathbb{R} or difference calculus on \mathbb{Z} . Note that if

- (i) $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, $f^\Delta(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$,
- (ii) $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^\Delta(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, where Δ is the forward difference operator.

Now, we are ready to state and prove the main results in this paper.

3. Main results

In what follows, $\mathfrak{B}([a, b]_{\mathbb{T}})$ stands for the Borel σ -algebra on $[a, b]_{\mathbb{T}}$.

Theorem 3.1. *Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let $f, g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f is nonincreasing. Further, let $0 \leq g(t) \leq 1$ $\forall t \in [a, b]_{\mathbb{T}}$, and $\hat{\mu}([c, d]_{\mathbb{T}}) = \int_{[a, b]_{\mathbb{T}}} g(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. Then holds the inequality*

$$\int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) \leq \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[a, c]_{\mathbb{T}}} (f(t) - f(d))g(t)\Delta\hat{\mu}(t). \tag{4}$$

If f is nondecreasing, then the sign of inequality (4) is reversed.

Proof. We have

$$\begin{aligned} & \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[a, c]_{\mathbb{T}}} (f(t) - f(d))g(t)\Delta\hat{\mu}(t) - \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) \\ &= \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - f(d) \int_{[a, b]_{\mathbb{T}}} g(t)\Delta\hat{\mu}(t) + f(d) \int_{[c, b]_{\mathbb{T}}} g(t)\Delta\hat{\mu}(t) \\ &= \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[c, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - f(d) \int_{[c, d]_{\mathbb{T}}} \Delta\hat{\mu}(t) + f(d) \int_{[c, b]_{\mathbb{T}}} g(t)\Delta\hat{\mu}(t) \\ &= \int_{[c, d]_{\mathbb{T}}} (f(t) - f(d))(1 - g(t))\Delta\hat{\mu}(t) + \int_{[d, b]_{\mathbb{T}}} (f(d) - f(t))g(t)\Delta\hat{\mu}(t). \end{aligned} \tag{5}$$

Since f is nonincreasing, $0 \leq g(t) \leq 1$, and since $\hat{\mu}$ is positive measure, it follows that the terms under the integral sign of the above relation are nonnegative. Therefore the first sum in the above relation is nonnegative, which yields (4). \square

Theorem 3.2. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let $f, g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f is nonincreasing. Further, let $0 \leq g(t) \leq 1$ $\forall t \in [a, b]_{\mathbb{T}}$, and $\hat{\mu}([c, d]_{\mathbb{T}}) = \int_{[a, b]_{\mathbb{T}}} g(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. Then holds the inequality

$$\int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[d, b]_{\mathbb{T}}} (f(c) - f(t))g(t)k(t)\Delta\hat{\mu}(t) \leq \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t). \quad (6)$$

If f is a nondecreasing function, then the sign of inequality (6) is reversed.

Proof. Similarly to the proof of Theorem 3.1, we have that

$$\begin{aligned} & \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[d, b]_{\mathbb{T}}} (f(c) - f(t))g(t)k(t)\Delta\hat{\mu}(t) \\ &= \int_{[c, d]_{\mathbb{T}}} (f(c) - f(t))(1 - g(t))\Delta\hat{\mu}(t) + \int_{[a, c]_{\mathbb{T}}} (f(t) - f(c))g(t)\Delta\hat{\mu}(t) \geq 0, \end{aligned}$$

i.e. (6) holds due to hypotheses. \square

Remark 3.3. If $\mathbb{T} = \mathbb{R}$, Theorems 3.1 and 3.2 reduce respectively to Theorems 1.3 and 1.4 presented in the Introduction (see also [15]).

Theorem 3.4. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let $f, g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f is nonincreasing. Further, let $0 \leq g(t) \leq 1$ $\forall t \in [a, b]_{\mathbb{T}}$, and

$$\hat{\mu}([c, d]_{\mathbb{T}}) \geq \int_{[a, b]_{\mathbb{T}}} g(t)\Delta\hat{\mu}(t), \quad (7)$$

where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. Then holds the inequality

$$\int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) \leq \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[a, c]_{\mathbb{T}}} (f(t) - f(d))g(t)\Delta\hat{\mu}(t). \quad (8)$$

If f is nondecreasing, then inequality (8) is reversed.

Proof. Since $f(d) \geq 0$, combining (5) and (7), we obtain the claim of this theorem. \square

Theorem 3.5. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let $f, g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f is nonincreasing. Further, let $0 \leq g(t) \leq 1$ $\forall t \in [a, b]_{\mathbb{T}}$, and $\hat{\mu}([c, d]_{\mathbb{T}}) \leq \int_{[a, b]_{\mathbb{T}}} g(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. Then holds the inequality

$$\int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[d, b]_{\mathbb{T}}} (f(c) - f(t))g(t)k(t)\Delta\hat{\mu}(t) \leq \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t). \quad (9)$$

If f is a nondecreasing function, then inequality (9) is reversed.

Proof. Similar to the proof of Theorem 3.4. \square

Remark 3.6. If $\mathbb{T} = \mathbb{R}$, then Theorems 3.4 and 3.5 reduce to the corresponding results obtained in [15].

Remark 3.7. If $\mathbb{T} = \mathbb{R}$, then considering Theorems 3.1 and 3.4 with $c = a$ and $d = a + \lambda$, or considering Theorems 3.2 and 3.5 with $c = b - \lambda$ and $d = b$, we obtain the corresponding results from [16]. Moreover, by putting $c = a$, $d = a + \lambda$ in Theorem 3.1 or by putting $c = b - \lambda$, $d = b$ in Theorem 3.2, we obtain the corresponding results from [9].

In the sequel we need the following two lemmas to generalize the above results for the function of the form f/k .

Lemma 3.8. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$. Further, let $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$ with $\int_{[c, d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a, b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, and let $z \in [a, b]_{\mathbb{T}}$. Then holds the following identity:

$$\begin{aligned} \int_{[c, d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) - \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) &= \int_{[a, c]_{\mathbb{T}}} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ &+ \int_{[c, d]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \\ &+ \int_{[d, b]_{\mathbb{T}}} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t). \end{aligned} \tag{10}$$

Proof. By a straightforward calculation, we have that

$$\begin{aligned} &\int_{[c, d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) - \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) \\ &= \int_{[c, d]_{\mathbb{T}}} k(t)[h(t) - g(t)]\frac{f(t)}{k(t)}\Delta\hat{\mu}(t) - \left[\int_{[a, c]_{\mathbb{T}}} \frac{f(t)}{k(t)}g(t)k(t)\Delta\hat{\mu}(t) + \int_{[d, b]_{\mathbb{T}}} \frac{f(t)}{k(t)}g(t)k(t)\Delta\hat{\mu}(t) \right] \\ &= \int_{[a, c]_{\mathbb{T}}} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) + \int_{[c, d]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \\ &+ \int_{[d, b]_{\mathbb{T}}} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) + \frac{f(d)}{k(d)} \left[\int_{[c, d]_{\mathbb{T}}} k(t)h(t)\Delta\hat{\mu}(t) \right. \\ &\left. - \int_{[a, c]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t) - \int_{[c, d]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t) - \int_{[d, b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t) \right]. \end{aligned} \tag{11}$$

Now, since $\int_{[c, d]_{\mathbb{T}}} k(t)h(t)\Delta\hat{\mu}(t) = \int_{[a, b]_{\mathbb{T}}} k(t)g(t)\Delta\hat{\mu}(t)$, it follows that

$$\int_{[c, d]_{\mathbb{T}}} k(t)h(t)\Delta\hat{\mu}(t) - \int_{[a, c]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t) - \int_{[c, d]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t) - \int_{[d, b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t) = 0. \tag{12}$$

Now, the desired result follows by virtue of (11) and (12). \square

Now, following the lines of the proof of the previous lemma, we also obtain the following result:

Lemma 3.9. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$. Further, let $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$ with $\int_{[c, d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a, b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, and let $z \in [a, b]_{\mathbb{T}}$. Then,

$$\begin{aligned} \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[c, d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) &= \int_{[a, c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ &+ \int_{[c, d]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \\ &+ \int_{[d, b]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right)g(t)k(t)\Delta\hat{\mu}(t). \end{aligned} \tag{13}$$

Theorem 3.10. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Further, let $0 \leq g(t) \leq h(t) \forall t \in [a, b]_{\mathbb{T}}$, and

$$\int_{[c, d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a, b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t), \tag{14}$$

where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. Then holds the inequality

$$\int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) \leq \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[a, c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)g(t)k(t)\Delta\hat{\mu}(t). \tag{15}$$

If f/k is nondecreasing, then the sign of inequality (15) is reversed.

Proof. Since f/k is nonincreasing, k is positive and $0 \leq g \leq h$, we have

$$\int_{[c, d]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \geq 0 \quad \text{and} \quad \int_{[d, b]_{\mathbb{T}}} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \geq 0.$$

Now, taking into account identity (10) as well as the above two relations, we have

$$\begin{aligned} & \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[a, c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ &= \int_{[c, d]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) + \int_{[d, b]_{\mathbb{T}}} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \geq 0, \end{aligned}$$

which proves our assertion. \square

Remark 3.11. If $\mathbb{T} = \mathbb{R}$, Theorem 3.10 reduces to the corresponding result from [15]. Moreover, in the case of the Lebesgue scale measure Δt , we arrive at the result established in [11].

Theorem 3.12. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Further, let $0 \leq g(t) \leq h(t) \forall t \in [a, b]_{\mathbb{T}}$, and

$$\int_{[c, d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a, b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t), \tag{16}$$

where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. Then holds the inequality

$$\int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[d, b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \leq \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t). \tag{17}$$

If f/k is nondecreasing function, then inequality (17) is reversed.

Proof. Since f/k is nonincreasing, k is positive and $0 \leq g \leq h$, we have

$$\int_{[c, d]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \geq 0 \quad \text{and} \quad \int_{[a, c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right)g(t)k(t)\Delta\hat{\mu}(t) \geq 0.$$

Now, identity (13) and the previous two inequalities yield

$$\begin{aligned} & \int_{[a, b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[c, d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[d, b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ &= \int_{[c, d]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) + \int_{[a, c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right)g(t)k(t)\Delta\hat{\mu}(t) \geq 0, \end{aligned}$$

which completes the proof. \square

Obviously, if $h \equiv 1$ and $k \equiv 1$ Theorems 3.10 and 3.12 reduce to Theorems 3.1 and 3.2.

Remark 3.13. It should be noted here that Theorem 3.12 is an extension of the corresponding results derived in [11] and [15].

Remark 3.14. If we additionally assume that the function f is nonnegative, conditions (14) and (16) can be relaxed by weaker conditions

$$\int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) \geq \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t) \quad \text{and} \quad \int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) \leq \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t).$$

Remark 3.15. If $\mathbb{T} = \mathbb{R}$, then Theorem 3.10 with $c = a$, $d = a + \lambda$, and Theorem 3.12 with $c = b - \lambda$, $d = b$ reduce to Steffensen-type inequalities established in [19].

Theorem 3.16. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Further, let $0 \leq g(t) \leq h(t) \forall t \in [a, b]_{\mathbb{T}}$, and $\int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. Then hold the inequalities

$$\begin{aligned} \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) &\leq \int_{[c,d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[c,d]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \\ &\quad + \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ &\leq \int_{[c,d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)g(t)k(t)\Delta\hat{\mu}(t). \end{aligned} \quad (18)$$

If f/k is nondecreasing, then the inequalities in (18) are reversed.

Proof. We follow the lines as in the proof of Theorem 3.10. \square

Theorem 3.17. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Further, let $0 \leq g(t) \leq h(t) \forall t \in [a, b]_{\mathbb{T}}$, and $\int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. Then hold the inequalities

$$\begin{aligned} &\int_{[c,d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ &\leq \int_{[c,d]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[c,d]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) - \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ &\leq \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t). \end{aligned} \quad (19)$$

If f/k is nondecreasing, then inequality (19) is reversed.

Proof. The proof is similar to the proof of Theorem 3.12 and is omitted. \square

Remark 3.18. It should be noted here that Theorems 3.16 and 3.17 represent generalized Steffensen-type inequalities established in [11] and [15].

Remark 3.19. If $\mathbb{T} = \mathbb{R}$, then considering Theorem 3.16 with $c = a$, $d = a + \lambda$, or considering Theorem 3.17 with $c = b - \lambda$, $d = b$, we obtain generalizations of Wu and Srivastava refinement of Steffensen's inequality established in [19]. In addition, if $k \equiv 1$ we obtain more accurate Steffensen-type inequality derived by Wu and Srivastava (see [24]).

Remark 3.20. It should be noted here that if $h \equiv 1$ and $k \equiv 1$, then Theorems 3.16 and 3.17 represent more accurate forms of Theorems 3.1 and 3.2.

Theorem 3.21. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Moreover, let $\int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. If

$$\int_{[c,x]} k(t)g(t)\Delta\hat{\mu}(t) \leq \int_{[c,x]} k(t)h(t)\Delta\hat{\mu}(t), \quad c \leq x \leq d, \quad \text{and} \quad \int_{[x,b]} k(t)g(t)\Delta\hat{\mu}(t) \geq 0, \quad d \leq x \leq b,$$

then holds the inequality

$$\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) \leq \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) + \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)\Delta\hat{\mu}(t). \tag{20}$$

Proof. Utilizing (13) and integration by parts formula (3), we have that

$$\begin{aligned} & \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) + \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)\Delta\hat{\mu}(t) \\ &= - \int_{[c,d]_{\mathbb{T}}} \left(\int_{[c,x]} k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \right) \left(\frac{f(x)}{k(x)} \right)^{\Delta} \Delta\hat{\mu}(x) \\ & \quad - \int_{[d,b]_{\mathbb{T}}} \left(\int_{[x,b]} g(t)k(t)\Delta\hat{\mu}(t) \right) \left(\frac{f(x)}{k(x)} \right)^{\Delta} \Delta\hat{\mu}(x) \geq 0. \end{aligned}$$

Consequently, it follows that

$$\int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) + \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)} \right) g(t)k(t)\Delta\hat{\mu}(t) \geq 0,$$

which proves our assertion. \square

Remark 3.22. If $\mathbb{T} = \mathbb{R}$, then Theorem 3.21 reduces to the corresponding results from [11] and [15].

By putting $c = a$ and $d = a + \lambda$ in Theorem 3.21, we obtain the following consequence.

Corollary 3.23. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Moreover, let λ be defined by $\int_{[a,a+\lambda]} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$. If

$$\int_{[a,x]} k(t)g(t)\Delta\hat{\mu}(t) \leq \int_{[a,x]} k(t)h(t)\Delta\hat{\mu}(t), \quad a \leq x \leq a + \lambda, \quad \text{and} \quad \int_{[x,b]} k(t)g(t)\Delta\hat{\mu}(t) \geq 0, \quad a + \lambda \leq x \leq b,$$

then holds the inequality

$$\int_{[a,b]} f(t)g(t)\Delta\hat{\mu}(t) \leq \int_{[a,a+\lambda]} f(t)h(t)\Delta\hat{\mu}(t).$$

Remark 3.24. It should be noted here that Corollary 3.23 represents extension of Steffensen-type inequality established in [11] to the time scales setting.

The following theorem also represents generalization of Steffensen-type inequalities form [11] and [15] to the time scales setting.

Theorem 3.25. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Moreover, let $\int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. If

$$\int_{[x,d]} k(t)g(t)\Delta\hat{\mu}(t) \leq \int_{[x,d]} k(t)h(t)\Delta\hat{\mu}(t), \quad c \leq x \leq d, \quad \text{and} \quad \int_{[a,x]} k(t)g(t)\Delta\hat{\mu}(t) \geq 0, \quad a \leq x \leq c,$$

then holds the inequality

$$\int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) - \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \leq \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t). \tag{21}$$

Proof. Utilizing (10) and integration by parts formula (3), it follows that

$$\begin{aligned} & \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) + \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ &= - \int_{[a,c]_{\mathbb{T}}} \left(\int_{[a,x]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t) \right) \left(\frac{f(x)}{k(x)}\right)^{\Delta} \Delta\hat{\mu}(x) \\ & \quad - \int_{[c,d]_{\mathbb{T}}} \left(\int_{[x,d]_{\mathbb{T}}} k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \right) \left(\frac{f(x)}{k(x)}\right)^{\Delta} \Delta\hat{\mu}(x) \geq 0. \end{aligned}$$

Therefore we have

$$\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) + \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \geq 0$$

and the proof is completed. \square

By putting $c = b - \lambda$ and $d = b$ in Theorem 3.25, we obtain the following consequence which represents generalization of the corresponding Steffensen-type inequality established in [11].

Corollary 3.26. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$, let k be a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and let $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Moreover, let λ be defined by $\int_{[b-\lambda,b]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$. If

$$\int_{[x,b]_{\mathbb{T}}} k(t)g(t)\Delta\hat{\mu}(t) \leq \int_{[x,b]_{\mathbb{T}}} k(t)h(t)\Delta\hat{\mu}(t), \quad b - \lambda \leq x \leq b, \quad \text{and} \quad \int_{[a,x]_{\mathbb{T}}} k(t)g(t)\Delta\hat{\mu}(t) \geq 0, \quad a \leq x \leq b - \lambda,$$

then holds the inequality

$$\int_{[b-\lambda,b]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) \leq \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t).$$

By putting $k \equiv 1$ and $h \equiv 1$ in Theorem 3.25, we obtain weaker conditions for function g in Theorem 3.1. In the same way, the following theorem provides relaxed conditions for g in Theorem 3.2.

Theorem 3.27. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$. Assume that k is a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and suppose $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ are $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Further, assume $\int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. If

$$\int_{[t,b]_{\mathbb{T}}} k(s)g(s)\Delta\hat{\mu}(s) \geq 0, \quad d \leq t < b, \quad \text{and} \quad \int_{[c,t]_{\mathbb{T}}} k(s)g(s)\Delta\hat{\mu}(s) \leq \int_{[c,t]_{\mathbb{T}}} k(s)h(s)\Delta\hat{\mu}(s), \quad c \leq t < d,$$

then hold the inequalities

$$\begin{aligned} & \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) \\ & \leq \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) + \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)g(t)k(t)\Delta\hat{\mu}(t) - \int_{[c,d]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \\ & \leq \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) + \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)g(t)k(t)\Delta\hat{\mu}(t). \end{aligned} \tag{22}$$

Proof. Applying identity (10) and integration by parts formula (3), we get

$$\begin{aligned} & \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) - \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) + \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)\Delta\hat{\mu}(t) \\ & - \int_{[c,d]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \\ & = \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) = - \int_{[d,b]_{\mathbb{T}}} \left(\int_{[t,b]} g(s)k(s)\Delta\hat{\mu}(s)\right)\left(\frac{f(t)}{k(t)}\right)^{\Delta} \Delta\hat{\mu}(t) \geq 0. \end{aligned}$$

Moreover, it follows that

$$\int_{[c,d]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) = - \int_{[c,d]_{\mathbb{T}}} \left(\int_{[c,t]} k(s)[h(s) - g(s)]\Delta\hat{\mu}(s)\right)\left(\frac{f(t)}{k(t)}\right)^{\Delta} \Delta\hat{\mu}(t) \geq 0,$$

and the proof is completed. \square

Remark 3.28. If $\mathbb{T} = \mathbb{R}$, then (22) reduces to Steffensen-type inequality established in [15].

Theorem 3.29. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$. Assume that k is a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and suppose $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ are $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Further, assume $\int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. If $\int_{[t,b]} k(s)g(s)\Delta\hat{\mu}(s) \geq 0$, for $d \leq t < b$, then

$$\begin{aligned} \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) & \leq \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) + \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ & - \int_{[c,d]_{\mathbb{T}}} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t). \end{aligned} \tag{23}$$

In addition, if $\int_{[c,t]} k(s)g(s)\Delta\hat{\mu}(s) \leq \int_{[c,t]} k(s)h(s)\Delta\hat{\mu}(s)$, for $c \leq t < d$, then holds inequality (22).

Proof. The proof is similar to the proof of Theorem 3.27, and hence is omitted. \square

Theorem 3.30. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$. Assume that k is a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and suppose $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ are $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Further, assume $\int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. If

$$\int_{[a,t]} k(s)g(s)\Delta\hat{\mu}(s) \geq 0, \quad a \leq t < c, \quad \text{and} \quad \int_{[t,d]} k(s)g(s)\Delta\hat{\mu}(s) \leq \int_{[t,d]} k(s)h(s)\Delta\hat{\mu}(s), \quad c \leq t < d,$$

then hold the inequalities

$$\begin{aligned} & \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) - \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ & \leq \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) - \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) + \int_{[c,d]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \\ & \leq \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t). \end{aligned} \tag{24}$$

Proof. Utilizing identity (13) and integration by parts formula (3), we have that

$$\begin{aligned} & \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t) - \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) + \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) \\ & - \int_{[c,d]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \\ & = - \int_{[a,c]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) = - \int_{[a,c]_{\mathbb{T}}} \left(\int_{[a,t]} g(s)k(s)\Delta\hat{\mu}(s)\right)\left(\frac{f(t)}{k(t)}\right)^{\Delta} \Delta\hat{\mu}(t) \geq 0. \end{aligned}$$

Furthermore, we have

$$\int_{[c,d]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) = - \int_{[c,d]_{\mathbb{T}}} \left(\int_{[t,d]} k(s)[h(s) - g(s)]\Delta\hat{\mu}(s)\right)\left(\frac{f(t)}{k(t)}\right)^{\Delta} \Delta\hat{\mu}(t) \geq 0,$$

and the proof is completed. \square

Remark 3.31. It should be noted here that Theorem 3.30 generalizes the corresponding Steffensen-type inequality established in [15].

Theorem 3.32. Let $([a, b]_{\mathbb{T}}, \mathfrak{B}([a, b]_{\mathbb{T}}), \hat{\mu})$ be time scale measure space with positive σ -finite measure on $\mathfrak{B}([a, b]_{\mathbb{T}})$. Assume that k is a positive $\Delta\hat{\mu}$ -integrable function on $[a, b]_{\mathbb{T}}$, and suppose $f, g, h : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ are $\Delta\hat{\mu}$ -integrable functions on $[a, b]_{\mathbb{T}}$ such that f/k is nonincreasing. Further, assume $\int_{[c,d]_{\mathbb{T}}} h(t)k(t)\Delta\hat{\mu}(t) = \int_{[a,b]_{\mathbb{T}}} g(t)k(t)\Delta\hat{\mu}(t)$, where $[c, d]_{\mathbb{T}} \subseteq [a, b]_{\mathbb{T}}$. If $\int_{[a,t]} k(s)g(s)\Delta\hat{\mu}(s) \geq 0$, for $a \leq t < c$, then holds the inequality

$$\begin{aligned} & \int_{[c,d]_{\mathbb{T}}} f(t)h(t)\Delta\hat{\mu}(t) - \int_{[d,b]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\Delta\hat{\mu}(t) + \int_{[c,d]_{\mathbb{T}}} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right)k(t)[h(t) - g(t)]\Delta\hat{\mu}(t) \\ & \leq \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\hat{\mu}(t). \end{aligned} \tag{25}$$

In addition, if $\int_{[t,d]} k(s)g(s)\Delta\hat{\mu}(s) \leq \int_{[t,d]} k(s)h(s)\Delta\hat{\mu}(s)$, $c \leq t < d$, then holds inequality (24).

Proof. We follow the lines as in the proof of Theorem 3.30. \square

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