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On γ-Preinvex Functions

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Abstract. The aim of the paper is to introduce the notion of γ-preinvex functions.We study this class in perspective of inequalities of Hermite-Hadamard type. We also derive some new estimates of upper bounds involving n-times differentiable γ-preinvex functions. Some special cases are also discussed which shows that the obtained results are quite unifying one.

1. Introduction

The classical concept of convexity is although very simple in nature but has many applications in different fields of pure and applied sciences. During the last century theory of convexity has experienced rapid development and consequently numerous new and significant generalizations of classical convexity have been proposed in the literature, for example, [1–3, 5–8, 16, 17, 19]. Besides its applications another fascinating aspect of theory of convexity is its close relationship with theory of inequalities. Several inequalities known in the literature are direct consequences of the applications of convex functions. For some more information, see [4].
We now discuss some previously known concepts and results. First of all let K be a non empty set in $\mathbb{R}^n$. Let $\Lambda : K \to \mathbb{R}$ and $\xi(.,.) : K \times K \to \mathbb{R}$ be a continuous bifunction.

Definition 1.1 ([9]). *A set K is said to be invex set with respect to bifunction $\xi(.,.)$ if*

$$x + \mu\xi(y,x) \in K \quad \forall\ x, y \in K, \quad \mu \in [0,1].$$

Definition 1.2 ([18]). *A function Λ on the invex set K is said to be preinvex with respect to bifunction $\xi(.,.)$ if*

$$\Lambda(x + \mu\xi(y,x)) \leq (1-\mu)\Lambda(x) + \mu\Lambda(y), \quad \forall x, y \in K,\ \mu \in [0,1].$$

In order to obtain some of the main results of the paper, we need famous condition C, which was introduced and studied by Mohan and Neogy [15]. This condition played a vital role in the development of many results involving preinvex functions.
Condition C. A set $K \subset \mathbb{R}$ is said to be an invex set with respect to bifunction $\xi(.,.)$ if and only if for any $x, y \in K$ and $\mu \in [0,1]$, we have

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1. $\xi(x, x + \mu\xi(y, x)) = -\mu\xi(y, x)$,
2. $\xi(y, x + \mu\xi(y, x)) = (1 - \mu)\xi(y, x)$.

Note that for any $x, y \in K$, $\mu_1, \mu_2 \in [0, 1]$ and from condition C, we can deduce

$$\xi(x + \mu_2\xi(y, x), x + \mu_1\xi(y, x)) = (\mu_2 - \mu_1)\xi(y, x).$$

The following auxiliary result was obtained by Latif [11]. To obtain some new estimates for upper bounds, we use this following result.

Lemma 1.3 ([11]). *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to bifunction $\xi : K\times K \to \mathbb{R}$. Suppose $\Lambda : K \to \mathbb{R}$ is a function such that $\Lambda^{(n)}$ exists on K for $n \in \mathbb{N}, n \geq 1$. If $\Lambda^{(n)}$ is integrable on $[a, a + \xi(b, a)]$, where $a, b \in K$ with $\xi(b, a) > 0$, the following equality holds:*

$$-\frac{\Lambda(a) + \Lambda(a + \xi(b, a))}{2} + \frac{1}{\xi(b, a)}\int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x + \sum_{k=2}^{n-1} \frac{(-1)^k(k-1)\xi^k(b, a)}{2(k+1)!}\Lambda^{(k)}(a + \xi(b, a))$$
$$= \frac{(-1)^{n-1}\xi^n(b, a)}{2n!}\int_0^1 \mu^{n-1}(n - 2\mu)\Lambda^{(n)}(a + \mu\xi(b, a))\mathrm{d}\mu,$$

where the above sum takes 0 when $n = 1$ and $n = 2$.

Some of our calculations involve beta and hypergeometric functions. For the sake of readers convenience, let us recall these classical concepts. The Beta and Hypergeometric functions are defined as

$$B(x, y) = \int_0^1 \nu^{x-1}(1 - \nu)^{y-1}\mathrm{d}\nu$$

also

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

and

$$_2F_1(x, y; c; z) = \frac{1}{B(y, c - y)}\int_0^1 \nu^{y-1}(1 - \nu)^{c-y-1}(1 - z\nu)^{-x}\mathrm{d}\nu, \quad c > y > 0, |z| < 1.$$

Definition 1.4 ([17]). *Functions Λ and g are said to be similarly ordered on K, if*

$$(\Lambda(x) - \Lambda(y))(g(x) - g(y)) \geq 0.$$

2. γ-preinvex functions

In this section, we define the class of γ-preinvex functions. Throughout this section, we suppose that $K \subseteq \mathbb{R}^n$ be invex with respect to the bifunction $\xi(., .) : K \times K \to \mathbb{R}$ unless otherwise specified.

Definition 2.1. *Let $\gamma : (0, 1) \to (0, \infty)$ be a real function. A function Λ on the invex set K is said to be γ-preinvex, if*

$$\Lambda(x + \mu\xi(y, x)) \leq (1 - \mu)\gamma(1 - \mu)\Lambda(x) + \mu\gamma(\mu)\Lambda(y), \quad \forall x, y \in K, \ \mu \in [0, 1].$$

The following classes can be deduced from above definition.
I. If we take $\gamma(\mu) = 1$ in Definition 2.1, then we have the class of classical preinvex function, see [18].
II. If we take $\gamma(\mu) = \mu^{-1}$ in Definition 2.1, then we have the definition of P-preinvex function, see [14].
III. If we take $\gamma(\mu) = \mu^{s-1}$ in Definition 2.1,where $s \in (0, 1)$, then we have the class of s-preinvex functions of Breckner type, see [14].
IV. If we take $\gamma(\mu) = \mu^{-s-1}$, then Definition 2.1 reduces to the definition of s-Godunova-Levin preinvex function, see [13].
V. If we take $\gamma(\mu) = 1 - \mu$ in Definition 2.1, then we have the definition of *tgs*-preinvex function.

Definition 2.2. *Let $\gamma : (0, 1) \to (0, \infty)$ be a real function. A function Λ on the invex set K is said to be tgs-preinvex with respect to bifunction $\xi(.,.)$, if*

$$\Lambda(x + \mu\xi(y, x)) \le \mu(1 - \mu)[\Lambda(x) + \Lambda(y)], \quad \forall x, y \in K,\ \mu \in [0, 1].$$

3. Main Results

In this section, we derive our main results. Throughout this section, we suppose that $K \subseteq \mathbb{R}^n$ be invex with respect to the bifunction $\xi(.,.) : K \times K \to \mathbb{R}$ unless otherwise specified.

Theorem 3.1. *Let Λ and g be two γ-preinvex functions on the invex set K. Then their product fg is also γ-preinvex function provided if Λ and g are similarly ordered functions and $(1 - \mu)\gamma(1 - \mu) + \mu\gamma(\mu) \le 1$.*

Proof. Since Λ and g be two γ-preinvex functions, so we have

$$\begin{aligned}
&\Lambda(a + \xi(b, a))g(a + \xi(b, a)) \\
&\le [(1 - \mu)\gamma(1 - \mu)\Lambda(a) + \mu\gamma(\mu)\Lambda(b)][(1 - \mu)\gamma(1 - \mu)g(a) + \mu\gamma(\mu)g(b)] \\
&= (1 - \mu)^2\gamma^2(1 - \mu)\Lambda(a)g(a) + [\mu(1 - \mu)\gamma(\mu)\gamma(1 - \mu)][\Lambda(a)g(b) + \Lambda(b)g(a)] \\
&\quad + \mu^2\gamma^2(\mu)\Lambda(b)g(b) \\
&\le (1 - \mu)^2\gamma^2(1 - \mu)\Lambda(a)g(a) + [\mu(1 - \mu)\gamma(\mu)\gamma(1 - \mu)][\Lambda(a)g(a) + \Lambda(b)g(b)] \\
&\quad + \mu^2\gamma^2(\mu)\Lambda(b)g(b) \\
&= [(1 - \mu)\gamma(1 - \mu)\Lambda(a)g(a) + \mu\gamma(\mu)\Lambda(b)g(b)][(1 - \mu)\gamma(1 - \mu) + \mu\gamma(\mu)] \\
&\le (1 - \mu)\gamma(1 - \mu)\Lambda(a)g(a) + \mu\gamma(\mu)\Lambda(b)g(b),
\end{aligned}$$

which completes the proof. □

Proposition 3.2. *If Λ is a γ_2-preinvex function on K and $\gamma_2(\mu) \le \gamma_1(\mu)$, $\mu \in (0, 1)$, then Λ is γ_1-preinvex function.*

Proof. Since Λ is a γ_2-preinvex function on K, so we have

$$\begin{aligned}
\Lambda(a + \xi(b, a)) &\le (1 - \mu)\gamma_2(1 - \mu)\Lambda(a) + \mu\gamma_2(\mu)\Lambda(b) \\
&\le (1 - \mu)\gamma_1(1 - \mu)\Lambda(a) + \mu\gamma_1(\mu)\Lambda(b).
\end{aligned}$$

□

Lemma 3.3. *Let Λ be a γ-preinvex function, then*

$$\Lambda(2a + \xi(b, a) - x) \le [(1 - \mu)\gamma(1 - \mu) + \mu\gamma(\mu)][\Lambda(a) + \Lambda(b)] - \Lambda(x).$$

Proof. Given $x = a + \mu\xi(b, a) \in I$, then we have

$$\begin{aligned}
\Lambda(2a + \xi(b, a) - x) &= \Lambda(a + (1 - \mu)\xi(b, a)) \\
&\le \mu\gamma(\mu)\Lambda(a) + (1 - \mu)\gamma(1 - \mu)\Lambda(b).
\end{aligned}$$

Adding and subtracting $(1 - \mu)\gamma(1 - \mu)\Lambda(a) + \mu\gamma(\mu)\Lambda(b)$, we get the required result. □

Theorem 3.4. *Let $\Lambda : K = [a, a + \xi(b; a)] \to \mathbb{R}$ be a γ-preinvex function with $\xi(b, a) > 0$ and $\gamma\left(\frac{1}{2}\right) \neq 0$. If $\xi(.,.)$ satisfies condition C, then*

$$\frac{1}{\gamma(\frac{1}{2})}\Lambda\left(\frac{2a+\xi(b,a)}{2}\right) \leq \frac{1}{\xi(b,a)}\int_{a}^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x \leq [\Lambda(a)+\Lambda(b)]\int_{0}^{1}\mu\gamma(\mu)\mathrm{d}\mu.$$

Proof. The proof is left for interested readers. □

Now we will discuss some special cases.
I. If $\gamma(\mu) = 1$, then Theorem 3.4 reduces to corresponding result in the class of classical preinvex function, Theorem 3.1 from [12].
II. If $\gamma(\mu) = \mu^{-1}$, then Theorem 3.4 reduces to the following result in the class of P-preinvex function.

Corollary 3.5. *Let $\Lambda : K = [a, a + \xi(b; a)] \to \mathbb{R}$ be a P-preinvex function. If $\xi(.,.)$ satisfies condition C, then for $\xi(b, a) > 0$, we have*

$$\frac{1}{2}\Lambda\Big(\frac{2a+\xi(b,a)}{2}\Big) \leq \frac{1}{\xi(b,a)}\int_{a}^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x \leq [\Lambda(a)+\Lambda(b)].$$

III. If $\gamma(\mu) = \mu^{s-1}$, then Theorem 3.4 reduces to the following result in the class of s-preinvex function of Breckner type.

Corollary 3.6. *Let $\Lambda : K = [a, a + \xi(b; a)] \to \mathbb{R}$ be a s-preinvex function of Breckner type. If $\xi(.,.)$ satisfies condition C, then for $\xi(b, a) > 0$, we have*

$$2^{s-1}\Lambda\Big(\frac{2a+\xi(b,a)}{2}\Big) \leq \frac{1}{\xi(b,a)}\int_{a}^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x \leq \frac{\Lambda(a)+\Lambda(b)}{1+s}.$$

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 3.4 reduces to following result in the class of s-Godunova-Levin preinvex functions.

Corollary 3.7. *Let $\Lambda : K = [a, a + \xi(b; a)] \to \mathbb{R}$ be a s-Godunova-Levin preinvex functions. If $\xi(.,.)$ satisfies condition C, then for $\xi(b, a) > 0$, we have*

$$2^{-s-1}\Lambda\Big(\frac{2a+\xi(b,a)}{2}\Big) \leq \frac{1}{\xi(b,a)}\int_{a}^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x \leq \frac{\Lambda(a)+\Lambda(b)}{1-s}.$$

V. If $\gamma(\mu) = 1 - \mu$, then Theorem 3.4 reduces to corresponding result in the class of tgs-preinvex function.

Corollary 3.8. *Let $\Lambda : K = [a, a + \xi(b; a)] \to \mathbb{R}$ be an tgs-preinvex. If $\xi(.,.)$ satisfies condition C, then for $\xi(b, a) > 0$, we have*

$$2\Lambda\Big(\frac{2a+\xi(b,a)}{2}\Big) \leq \frac{1}{\xi(b,a)}\int_{a}^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x \leq \frac{\Lambda(a)+\Lambda(b)}{6}.$$

Theorem 3.9. *Let $\Lambda : K \to (0, \infty)$ be a γ-preinvex function with $\xi(b, a) > 0$, $\gamma\left(\frac{1}{2}\right) \neq 0$ and $w : [a, a + \xi(b, a)] \to \mathbb{R}$ be a non negative, integrable and symmetric function about $\frac{2a+\xi(b,a)}{2}$, then using condition C, we have*

$$\begin{aligned}
&\frac{1}{\gamma(\frac{1}{2})}\Lambda\left(\frac{2a + \xi(b, a)}{2}\right)\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x \\
&\leq \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \\
&\leq \frac{\Lambda(a) + \Lambda(b)}{2}[(1-\mu)\gamma(1-\mu) + \mu\gamma(\mu)]\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x.
\end{aligned}$$

Proof. Since Λ is γ-preinvex function, so we have

$$\begin{aligned}
\Lambda\left(\frac{2a + \xi(b, a)}{2}\right) &= \Lambda\left(\frac{2a + \xi(b, a) - x + x}{2}\right) \\
&\leq \frac{1}{2}\gamma\left(\frac{1}{2}\right)[\Lambda(2a + \xi(b, a) - x) + \Lambda(x)],
\end{aligned}$$

since w is non-negative, so

$$\begin{aligned}
&\frac{2}{\gamma(\frac{1}{2})}\Lambda\left(\frac{2a + \xi(b, a)}{2}\right)\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x \\
&\leq \int_a^{a+\xi(b,a)} \Lambda(2a + \xi(b, a) - x)w(x)\mathrm{d}x + \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \\
&= \int_a^{a+\xi(b,a)} \Lambda(2a + \xi(b, a) - x)w(2a + \xi(b, a) - x)\mathrm{d}x + \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \\
&= 2\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x.
\end{aligned}$$

For right hand side inequality, using Lemma 3.3, we have

$$\begin{aligned}
&\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \\
&= \frac{1}{2}\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x + \frac{1}{2}\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x
\end{aligned}$$

$$= \frac{1}{2}\int_a^{a+\xi(b,a)} \Lambda(2a+\xi(b,a)-x)w(2a+\xi(b,a)-x)\mathrm{d}x + \frac{1}{2}\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x$$

$$= \frac{1}{2}\int_a^{a+\xi(b,a)} \Lambda(2a+\xi(b,a)-x)w(x)\mathrm{d}x + \frac{1}{2}\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x$$

$$\leq [(1-\mu)\gamma(1-\mu)+\mu\gamma(\mu)]\frac{\Lambda(a)+\Lambda(b)}{2}\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x.$$

This completes the proof. □

We now discuss some special cases of Theorem 3.9.

I. If $\gamma(\mu) = 1$, then Theorem 3.9 reduces to the following result in the class of classical preinvex function.

Corollary 3.10. *Let $\Lambda : K \to (0,\infty)$ be classical preinvex function with $\xi(b,a) > 0$ and $w : [a, a+\xi(b,a)] \to \mathbb{R}$ be a non-negative, integrable and symmetric function about $a + \frac{\xi(b,a)}{2}$, then using condition C, we have*

$$\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x \leq \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x$$
$$\leq \frac{\Lambda(a)+\Lambda(b)}{2}\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x.$$

II. If $\gamma(\mu) = \mu^{-1}$, then Theorem 3.9 reduces to the corresponding result in the class of P-preinvex function.

Corollary 3.11. *Let $\Lambda : K \to (0,\infty)$ be P-preinvex function with $\xi(b,a) > 0$ and $w : [a, a+\xi(b,a)] \to \mathbb{R}$ be a non-negative, integrable and symmetric function about $\frac{2a+\xi(b,a)}{2}$, then using condition C, we have*

$$\frac{1}{2}\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x \leq \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x$$
$$\leq [\Lambda(a)+\Lambda(b)]\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x.$$

III. If $\gamma(\mu) = \mu^{s-1}$, then Theorem 3.9 reduces to the corresponding result in the class of s-preinvex function of Breckner type.

Corollary 3.12. *Let $\Lambda : K \to (0,\infty)$ be s-preinvex function of Breckner type with $\xi(b,a) > 0$ and $w : [a, a+\xi(b,a)] \to \mathbb{R}$ be a non negative, integrable and symmetric function about $\frac{2a+\xi(b,a)}{2}$, then using condition C, we have*

$$2^{s-1}\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x \leq \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x$$
$$\leq \frac{\Lambda(a)+\Lambda(b)}{2}[(1-\mu)^s+\mu^s]\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x.$$

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 3.9 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.13. *Let $\Lambda : K \to (0,\infty)$ be s-Godunova-Levin preinvex function with $\xi(b,a) > 0$ and $w : [a, a+\xi(b,a)] \to \mathbb{R}$ be a non negative, integrable and symmetric function about $\frac{2a+\xi(b,a)}{2}$, then using condition C, we have*

$$2^{-s-1}\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x \leq \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x$$
$$\leq \frac{\Lambda(a)+\Lambda(b)}{2}\left[(1-\mu)^{-s}+\mu^{-s}\right]\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x.$$

V. If $\gamma(\mu) = 1-\mu$, then Theorem 3.9 reduces to the following result in the class of tgs-preinvex function.

Corollary 3.14. *Let $\Lambda : K \to (0,\infty)$ be tgs-preinvex function with $\xi(b,a) > 0$ and $w : [a, a+\xi(b,a)] \to \mathbb{R}$ be a non negative, integrable and symmetric function about $\frac{2a+\xi(b,a)}{2}$, then using condition C, we have*

$$2\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x \leq \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x$$
$$\leq [\Lambda(a)+\Lambda(b)]\,[\mu(1-\mu)]\int_a^{a+\xi(b,a)} w(x)\mathrm{d}x.$$

Theorem 3.15. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be γ_1 and γ_2–preinvex functions respectively with $\xi(b,a) > 0, \gamma_1(\frac{1}{2})\gamma_2(\frac{1}{2}) \neq 0$, then using condition C, we have*

$$\frac{2}{\gamma_1(\frac{1}{2})\gamma_2(\frac{1}{2})}\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)w\left(\frac{2a+\xi(b,a)}{2}\right)$$
$$\leq \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x$$
$$+M(a,b)\int_0^1 \mu(1-\mu)\gamma_1(1-\mu)\gamma_2(\mu)\mathrm{d}\mu$$
$$+N(a,b)\int_0^1 \mu^2\gamma_1(\mu)\gamma_2(\mu)\mathrm{d}\mu,$$

where

$$M(a,b) = \Lambda(a)w(a) + \Lambda(b)w(b)$$

$$N(a,b) = \Lambda(a)w(b) + \Lambda(b)w(a).$$

Proof. Since Λ and w are γ_1 and γ_2-preinvex functions respectively, we have

$$\begin{aligned}
&\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)w\left(\frac{2a+\xi(b,a)}{2}\right)\\
&=\Lambda(a+(1-\mu)\xi(b,a)+\frac{1}{2}\xi(a+\mu\xi(b,a),a+(1-\mu)\xi(b,a)))\\
&\times w(a+(1-\mu)\xi(b,a)+\frac{1}{2}\xi(a+\mu\xi(b,a),a+(1-\mu)\xi(b,a)))\\
&\leq\frac{1}{4}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)[\Lambda(a+\mu\xi(b,a))+\Lambda(a+(1-\mu)\xi(b,a))]\,[w(a+\mu\xi(b,a))+w(a+(1-\mu)\xi(b,a))]\\
&\leq\frac{1}{4}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)[\Lambda(a+\mu\xi(b,a))w(a+\mu\xi(b,a))+\Lambda(a+(1-\mu)\xi(b,a))w(a+(1-\mu)\xi(b,a))]\\
&+\frac{1}{4}\gamma_1\left(\frac{1}{2}\right)\gamma_2\left(\frac{1}{2}\right)\Big[M(a,b)[\mu(1-\mu)\gamma_1(\mu)\gamma_2(1-\mu)+\mu(1-\mu)\gamma_1(1-\mu)\gamma_2(\mu)]\\
&+N(a,b)[(1-\mu)^2\gamma_1(1-\mu)\gamma_2(1-\mu)+\mu^2\gamma_1(\mu)\gamma_2(\mu)]\Big].
\end{aligned}$$

Integrating with respect to μ on $[0,1]$ and using the technique of change of variables, we get the required result. □

We now discuss some special cases of Theorem 3.15.

I. If $\gamma_1(\mu)=\gamma_2(\mu)=1$, then Theorem 3.15 reduces to the following result in the class of classical preinvex function.

Corollary 3.16. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be classical preinvex functions respectively with $\xi(b,a)>0$, then using condition C, we have*

$$2\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)w\left(\frac{2a+\xi(b,a)}{2}\right)\leq\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)w(x)\mathrm{d}x+\frac{M(a,b)}{6}+\frac{N(a,b)}{3}.$$

II. If $\gamma_1(\mu)=\gamma_2(\mu)=\mu^{-1}$, then Theorem 3.15 reduces to the following result in the class of P-preinvex function.

Corollary 3.17. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be P-preinvex functions respectively with $\xi(b,a)>0$, then using condition C, we have*

$$\frac{1}{2}\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)w\left(\frac{2a+\xi(b,a)}{2}\right)\leq\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)w(x)\mathrm{d}x+M(a,b)+N(a,b).$$

III. If $\gamma_1(\mu)=\gamma_2(\mu)=\mu^{s-1}$, then Theorem 3.15 reduces to the following result in the class of s-preinvex function of Breckner type.

Corollary 3.18. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be s-preinvex function of Breckner type respectively with $\xi(b,a)>0$, then using condition C, we have*

$$2^{2s-1}\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)w\left(\frac{2a+\xi(b,a)}{2}\right)\leq\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)w(x)\mathrm{d}x+\frac{\Gamma(1+s)}{1+s}M(a,b)+\frac{N(a,b)}{1+2s}.$$

IV. If $\gamma_1(\mu) = \gamma_2(\mu) = \mu^{-s-1}$, then Theorem 3.15 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.19. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be s-Godunova-Levin preinvex functions respectively with $\xi(b,a) > 0$, then using condition C, we have*

$$2^{-2s-1}\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)w\left(\frac{2a+\xi(b,a)}{2}\right) \le \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x + \frac{\Gamma(1-s)}{1-s}M(a,b) + \frac{N(a,b)}{1-2s}.$$

V. If $\gamma_1(\mu) = \gamma_2(\mu) = 1-\mu$, then Theorem 3.15 reduces to the following result in the class of tgs-preinvex function.

Corollary 3.20. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be tgs-preinvex functions respectively with $\xi(b,a) > 0$, then using condition C, we have*

$$8\Lambda\left(\frac{2a+\xi(b,a)}{2}\right)w\left(\frac{2a+\xi(b,a)}{2}\right) \le \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x + \frac{1}{30}[M(a,b)+N(a,b)].$$

Theorem 3.21. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be γ_1 and γ_2-preinvex functions respectively with $\xi(b,a) > 0$, then*

$$\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \le M(a,b)\int_0^1 \mu^2\gamma_1(\mu)\gamma_2(\mu)\mathrm{d}\mu + N(a,b)\int_0^1 \mu(1-\mu)\gamma_1(\mu)\gamma_2(1-\mu)\mathrm{d}\mu.$$

Proof. Since Λ and g are γ_1-preinvex and γ_2-preinvex functions and non-negative, so we have

$$\begin{aligned}&\Lambda(a+\mu\xi(b,a))w(a+\mu\xi(b,a))\\ &\le [(1-\mu)\gamma_1(1-\mu)\Lambda(a)+\mu\gamma_1(\mu)\Lambda(b)][(1-\mu)\gamma_2(1-\mu)w(a)+\mu\gamma_2(\mu)w(b)].\end{aligned}$$

Integrating above inequality with respect to μ on $[0,1]$, we have

$$\begin{aligned}&\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x\\ &\le \Lambda(a)w(a)\int_0^1 (1-\mu)^2\gamma_1(1-\mu)\gamma_2(1-\mu)\mathrm{d}\mu + \Lambda(b)w(b)\int_0^1 \mu^2\gamma_1(\mu)\gamma_2(\mu)\mathrm{d}\mu\\ &+ \Lambda(a)w(b)\int_0^1 \mu(1-\mu)\gamma_1(1-\mu)\gamma_2(\mu)\mathrm{d}\mu + \Lambda(b)w(a)\int_0^1 \mu(1-\mu)\gamma_1(\mu)\gamma_2(1-\mu)\mathrm{d}\mu\\ &= M(a,b)\int_0^1 \mu^2\gamma_1(\mu)\gamma_2(\mu)\mathrm{d}\mu + N(a,b)\int_0^1 \mu(1-\mu)\gamma_1(\mu)\gamma_2(1-\mu)\mathrm{d}\mu.\end{aligned}$$

This completes the proof. □

Now we discuss some special cases of Theorem 3.21.
I. If $\gamma_1(\mu) = \gamma_2(\mu) = 1$, then Theorem 3.21 reduces to classical preinvex function.

Corollary 3.22. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be classical preinvex functions respectively with $\xi(b,a) > 0$, then*

$$\frac{1}{\xi(b,a)} \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \leq \frac{M(a,b)}{3} + \frac{N(a,b)}{6}.$$

II. If $\gamma_1(\mu) = \gamma_2(\mu) = \mu^{-1}$, then Theorem 3.21 reduces to the following result in the class of P-preinvex function.

Corollary 3.23. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be P-preinvex function respectively with $\xi(b,a) > 0$, then using condition C, we have*

$$\frac{1}{\xi(b,a)} \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \leq [M(a,b) + N(a,b)].$$

III. If $\gamma_1(\mu) = \gamma_2(\mu) = \mu^{s-1}$, then Theorem 3.21 reduces to the following result in the class of s-preinvex function of Breckner type.

Corollary 3.24. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be s-preinvex function of Breckner type respectively with $\xi(b,a) > 0$, then*

$$\frac{1}{\xi(b,a)} \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \leq \frac{M(a,b)}{1+2s} + \frac{\Gamma(1+s)}{1+s}N(a,b).$$

IV. If $\gamma_1(\mu) = \gamma_2(\mu) = \mu^{-s-1}$, then Theorem 3.21 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.25. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be s-Godunova-Levin preinvex functions respectively with $\xi(b,a) > 0$, then*

$$\frac{1}{\xi(b,a)} \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \leq \frac{M(a,b)}{1-2s} + \frac{\Gamma(1-s)}{1-s}N(a,b).$$

V. If $\gamma_1(\mu) = \gamma_2(\mu) = 1 - \mu$, then Theorem 3.21 reduces to the following result in the class of tgs-preinvex function.

Corollary 3.26. *Let $\Lambda : K \to (0,\infty)$ and $w : I \to (0,\infty)$ be tgs-preinvex functions respectively with $\xi(b,a) > 0$, then*

$$\frac{1}{\xi(b,a)} \int_a^{a+\xi(b,a)} \Lambda(x)w(x)\mathrm{d}x \leq \frac{1}{30}[M(a,b) + N(a,b)].$$

Theorem 3.27. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\xi : K \times K \to \mathbb{R}$. Suppose $\Lambda : K \to \mathbb{R}$ is a function such that $\Lambda^{(n)}$ exists on K for $n \in \mathbb{N}, n \geq 1$ and $\Lambda^{(n)}$ is integrable on $[a, a + \xi(b,a)]$, where $a, b \in K$ with $\xi(b,a) > 0$. If $|\Lambda^{(n)}|$ is the γ-preinvex function on K, then*

Proof. Suppose $n \geq 2$, using Lemma 1.3, it follows that

$$\begin{aligned}
&\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|\\
&\leq \frac{\xi^n(b,a)}{2n!}\int_0^1\mu^{n-1}(n-2\mu)|\Lambda^{(n)}(a+\mu\xi(b,a))|\mathrm{d}\mu\\
&\leq \frac{\xi^n(b,a)}{2n!}\int_0^1\mu^{n-1}(n-2\mu)[(1-\mu)\gamma(1-\mu)|\Lambda^{(n)}(a)|+\mu\gamma(\mu)|\Lambda^{(n)}(b)|]\mathrm{d}\mu\\
&= \frac{\xi^n(b,a)}{2n!}[|\Lambda^{(n)}(a)|\int_0^1\mu^{n-1}(n-2\mu)(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|\int_0^1\mu^{n-1}(n-2\mu)\mu\gamma(\mu)\mathrm{d}\mu],
\end{aligned}$$

which completes the proof. □

Now we will discuss some special cases.
I. If $\gamma(\mu)=1$, then Theorem 3.27 reduces to corresponding result in the class of classical preinvex function, Theorem 2.3 from [10].
II. If $\gamma(\mu)=\mu^{-1}$, then Theorem 3.27 reduces to the following result in the class of P-preinvex function.

Corollary 3.28. *Under the assumptions of Theorem 3.27 if $|\Lambda^{(n)}|$ is P-preinvex function on K, then*

$$\begin{aligned}
&\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|\\
&\leq \frac{\xi^n(b,a)(n-1)}{2.(n+1)!}[|\Lambda^{(n)}(a)|+|\Lambda^{(n)}(b)|].
\end{aligned}$$

III. If $\gamma(\mu)=\mu^{s-1}$, then Theorem 3.27 reduces to the following result in the class of s-preinvex function.

Corollary 3.29. *Under the assumptions of Theorem 3.27 if $|\Lambda^{(n)}|$ is s-preinvex function on K, then*

$$\begin{aligned}
&\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|\\
&\leq \frac{\xi^n(b,a)}{2n!}[\mu_1|\Lambda^{(n)}(a)|+\mu_2|\Lambda^{(n)}(b)|],
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &= nB(n,1+s)-2B(1+n,1+s)\\
\mu_2 &= \frac{n^2+ns-n-2s}{(n+s)(n+s+1)}.
\end{aligned}$$

IV. If $\gamma(\mu)=\mu^{-s-1}$, then Theorem 3.27 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.30. *Under the assumptions of Theorem 3.27 if $|\Lambda^{(n)}|$ is s-Godunova-Levin preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2n!}[\mu_3|\Lambda^{(n)}(a)|+\mu_4|\Lambda^{(n)}(b)|],$$

where

$$\mu_3 = nB(n,1-s)-2B(1+n,1-s)$$
$$\mu_4 = \frac{n^2-ns-n+2s}{(n-s)(n-s+1)}.$$

V. If $\gamma(\mu)=1-\mu$, then Theorem 3.27 reduces to the following result in the class of *tgs*-preinvex function.

Corollary 3.31. *Under the assumptions of Theorem 3.27 if $|\Lambda^{(n)}|$ is tgs-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)(n-1)}{2.(n+1)!(n+3)}[|\Lambda^{(n)}(a)|+|\Lambda^{(n)}(b)|].$$

Theorem 3.32. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\xi: K\times K\to\mathbb{R}$. Suppose $\Lambda: K\to\mathbb{R}$ is a function such that $\Lambda^{(n)}$ exists on K for $n\in\mathbb{N}, n\geq 1$ and $\Lambda^{(n)}$ is integrable on $[a,a+\xi(b,a)]$, where $a,b\in K$ with $\xi(b,a)>0$. If $|\Lambda^{(n)}|^r$ is the γ-preinvex function on K for $r>1$ and $p^{-1}+r^{-1}=1$, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2n!}\lambda^{\frac{1}{p}}\left(|\Lambda^{(n)}(a)|^r\int_0^1\mu^{n-1}(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|^r\int_0^1\mu\gamma(\mu)\mu^{n-1}\mathrm{d}\mu\right)^{\frac{1}{r}},$$

where

$$\lambda = n^{p-1}\,{}_2F_1\left(-p,n;n+1;\frac{2}{n}\right).$$

Proof. Suppose $n\geq 2$, using Lemma 1.3 and Holder's inequality, it follows that

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2n!}\int_0^1\mu^{n-1}(n-2\mu)|\Lambda^{(n)}(a+\mu\xi(b,a))|\mathrm{d}\mu$$
$$\leq \frac{\xi^n(b,a)}{2n!}\left(\int_0^1\mu^{n-1}(n-2\mu)^p\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1\mu^{n-1}|\Lambda^{(n)}(a+\mu\xi(b,a))|^r\mathrm{d}\mu\right)^{\frac{1}{r}}$$
$$\leq \frac{\xi^n(b,a)}{2n!}\lambda^{\frac{1}{p}}\left(|\Lambda^{(n)}(a)|^r\int_0^1\mu^{n-1}(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|^r\int_0^1\mu\gamma(\mu)\mu^{n-1}\mathrm{d}\mu\right)^{\frac{1}{r}}.$$

This completes the proof. □

Now we discuss some special cases.
I. If $\gamma(\mu) = 1$, then Theorem 3.32 reduces to the following result in the class of classical preinvex function.

Corollary 3.33. *Under the assumptions of Theorem 3.32 if $|\Lambda^{(n)}|^r$ is classical preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2} - \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\le \frac{\xi^n(b,a)}{2n!(n+1)^{\frac{1}{r}}}\lambda^{\frac{1}{p}}\left(\frac{1}{n}|\Lambda^{(n)}(a)|^r + |\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

II. If $\gamma(\mu) = \mu^{-1}$, then Theorem 3.32 reduces to the following result in the class of P-preinvex function.

Corollary 3.34. *Under the assumptions of Theorem 3.32 if $|\Lambda^{(n)}|^r$ is P-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2} - \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\le \frac{\xi^n(b,a)}{2n!n^{\frac{1}{r}}}\lambda^{\frac{1}{p}}\left(|\Lambda^{(n)}(a)|^r + |\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

III. If $\gamma(\mu) = \mu^{s-1}$, then Theorem 3.32 reduces to the following result in the class of s-preinvex function.

Corollary 3.35. *Under the assumptions of Theorem 3.32 if $|\Lambda^{(n)}|^r$ is s-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2} - \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\le \frac{\xi^n(b,a)}{2n!}\lambda^{\frac{1}{p}}\left(B(n,s+1)|\Lambda^{(n)}(a)|^r + \frac{1}{n+s}|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 3.32 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.36. *Under the assumptions of Theorem 3.32 if $|\Lambda^{(n)}|^r$ is s-Godunova-Levin preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2} - \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\le \frac{\xi^n(b,a)}{2n!}\lambda^{\frac{1}{p}}\left(B(n,1-s)|\Lambda^{(n)}(a)|^r + \frac{1}{n-s}|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

V. If $\gamma(\mu) = 1-\mu$, then Theorem 3.32 reduces to the following result in the class of tgs-preinvex function.

Corollary 3.37. *Under the assumptions of Theorem 3.32 if $|\Lambda^{(n)}|^r$ is tgs-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2} - \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\le \frac{\xi^n(b,a)}{2n!}\lambda^{\frac{1}{p}}\left(\frac{|\Lambda^{(n)}(a)|^r + |\Lambda^{(n)}(b)|^r}{(n+1)(n+2)}\right)^{\frac{1}{r}}.$$

Theorem 3.38. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\xi : K \times K \to \mathbb{R}$. Suppose $\Lambda : K \to \mathbb{R}$ is a function such that $\Lambda^{(n)}$ exists on K for $n \in \mathbb{N}, n \geq 1$ and $\Lambda^{(n)}$ is integrable on $[a, a + \xi(b, a)]$, where $a, b \in K$ with $\xi(b, a) > 0$. If $|\Lambda^{(n)}|^r$ is the γ-preinvex function on K for $r > 1$ and $p^{-1} + r^{-1} = 1$, then*

$$\left|\frac{\Lambda(a) + \Lambda(a + \xi(b, a))}{2} - \frac{1}{\xi(b, a)} \int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(-1)^k(k-1)\xi^k(b, a)}{2(k+1)!}\Lambda^{(k)}(a + \xi(b, a))\right|$$
$$\leq \frac{\xi^n(b, a)}{2n!}\vartheta^{\frac{1}{p}}\left(|\Lambda^{(n)}(a)|^r \int_0^1 (1-\mu)\gamma(1-\mu)\mathrm{d}\mu + |\Lambda^{(n)}(b)|^r \int_0^1 \mu\gamma(\mu)\mathrm{d}\mu\right)^{\frac{1}{r}},$$

where

$$\vartheta = \frac{n^p}{p(n-1)+1}\, {}_2F_1\left(-p, p(n-1)+1; p(n-1)+2; \frac{2}{n}\right).$$

Proof. Suppose $n \geq 2$, using Lemma 1.3 and Holder's inequality, it follows that

$$\left|\frac{\Lambda(a) + \Lambda(a + \xi(b, a))}{2} - \frac{1}{\xi(b, a)} \int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(-1)^k(k-1)\xi^k(b, a)}{2(k+1)!}\Lambda^{(k)}(a + \xi(b, a))\right|$$
$$\leq \frac{\xi^n(b, a)}{2n!} \int_0^1 \mu^{n-1}(n - 2\mu)|\Lambda^{(n)}(a + \mu\xi(b, a))|\mathrm{d}\mu$$
$$\leq \frac{\xi^n(b, a)}{2n!}\left(\int_0^1 \mu^{p(n-1)}(n-2\mu)^p\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1 |\Lambda^{(n)}(a + \mu\xi(b, a))|^r\mathrm{d}\mu\right)^{\frac{1}{r}}$$
$$\leq \frac{n\xi^n(b, a)}{2n!}\vartheta^{\frac{1}{p}}\left(|\Lambda^{(n)}(a)|^r \int_0^1 (1-\mu)\gamma(1-\mu)\mathrm{d}\mu + |\Lambda^{(n)}(b)|^r \int_0^1 \mu\gamma(\mu)\mathrm{d}\mu\right)^{\frac{1}{r}}.$$

This completes the proof. □

Now we will discuss some special cases.
I. If $\gamma(\mu) = 1$, then Theorem 3.38 reduces to the following result in the class of classical preinvex function.

Corollary 3.39. *Under the assumptions of Theorem 3.38 if $|\Lambda^{(n)}|^r$ is classical preinvex function on K, then*

$$\left|\frac{\Lambda(a) + \Lambda(a + \xi(b, a))}{2} - \frac{1}{\xi(b, a)} \int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(-1)^k(k-1)\xi^k(b, a)}{2(k+1)!}\Lambda^{(k)}(a + \xi(b, a))\right|$$
$$\leq \frac{n\xi^n(b, a)}{2n!}\vartheta^{\frac{1}{p}}\left(\frac{|\Lambda^{(n)}(a)|^r + |\Lambda^{(n)}(b)|^r}{2}\right)^{\frac{1}{r}}.$$

II. If $\gamma(\mu) = \mu^{-1}$, then Theorem 3.38 reduces to the following result in the class of P-preinvex function.

Corollary 3.40. *Under the assumptions of Theorem 3.38 if $|\Lambda^{(n)}|^r$ is P-preinvex function on K, then*

$$\left|\frac{\Lambda(a) + \Lambda(a + \xi(b, a))}{2} - \frac{1}{\xi(b, a)} \int_a^{a+\xi(b,a)} \Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(-1)^k(k-1)\xi^k(b, a)}{2(k+1)!}\Lambda^{(k)}(a + \xi(b, a))\right|$$
$$\leq \frac{n\xi^n(b, a)}{2n!}\vartheta^{\frac{1}{p}}\left(|\Lambda^{(n)}(a)|^r + |\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

III. If $\gamma(\mu) = \mu^{s-1}$, then Theorem 3.38 reduces to the following result in the class of s-preinvex function.

Corollary 3.41. *Under the assumptions of Theorem 3.38 if $|\Lambda^{(n)}|^r$ is s-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{n\xi^n(b,a)}{2n!}\vartheta^{\frac{1}{p}}\left(\frac{|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r}{1+s}\right)^{\frac{1}{r}}.$$

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 3.38 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.42. *Under the assumptions of Theorem 3.38 if $|\Lambda^{(n)}|^r$ is s-Godunova-Levin preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{n\xi^n(b,a)}{2n!}\vartheta^{\frac{1}{p}}\left(\frac{|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r}{1-s}\right)^{\frac{1}{r}}.$$

V. If $\gamma(\mu) = 1-\mu$, then Theorem 3.38 reduces to the following result in the class of tgs-preinvex function.

Corollary 3.43. *Under the assumptions of Theorem 3.38 if $|\Lambda^{(n)}|^r$ is tgs-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{n\xi^n(b,a)}{2n!}\vartheta^{\frac{1}{p}}\left(\frac{|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r}{6}\right)^{\frac{1}{r}}.$$

Theorem 3.44. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\xi : K \times K \to \mathbb{R}$. Suppose $\Lambda : K \to \mathbb{R}$ is a function such that $\Lambda^{(n)}$ exists on K for $n \in \mathbb{N}, n \geq 1$ and $\Lambda^{(n)}$ is integrable on $[a, a+\xi(b,a)]$, where $a, b \in K$ with $\xi(b,a) > 0$. If $|\Lambda^{(n)}|^r$ is the γ-preinvex function on K for $r \geq 1$, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2n!n^{1-\frac{1}{r}}}\left(|\Lambda^{(n)}(a)|^r\int_0^1\mu^{n-1}(n-2\mu)^r(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|^r\int_0^1\mu^{n-1}(n-2\mu)^r\mu\gamma(\mu)\mathrm{d}\mu\right)^{\frac{1}{r}}.$$

Proof. Suppose $n \geq 2$, using Lemma 1.3 and Holder's inequality, it follows that

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$

$$\leq \frac{\xi^n(b,a)}{2n!}\int_0^1 \mu^{n-1}(n-2\mu)|\Lambda^{(n)}(a+\mu\xi(b,a))|\mathrm{d}\mu$$

$$\leq \frac{\xi^n(b,a)}{2n!}\left(\int_0^1 \mu^{n-1}\mathrm{d}\mu\right)^{1-\frac{1}{r}}\left(\int_0^1 \mu^{n-1}(n-2\mu)^r|\Lambda^{(n)}(a+\mu\xi(b,a))|^r\mathrm{d}\mu\right)^{\frac{1}{r}}$$

$$\leq \frac{\xi^n(b,a)}{2n!n^{1-\frac{1}{r}}}\left(|\Lambda^{(n)}(a)|^r\int_0^1 \mu^{n-1}(n-2\mu)^r(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|^r\int_0^1 \mu^{n-1}(n-2\mu)^r\mu\gamma(\mu)\mathrm{d}\mu\right)^{\frac{1}{r}}.$$

This completes the proof. □

Now we will discuss some special cases.

I. If $\gamma(\mu)=1$, then Theorem 3.44 reduces to the following result in the class of classical preinvex function.

Corollary 3.45. *Under the assumptions of Theorem 3.44 if $|\Lambda^{(n)}|^r$ is classical preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$

$$\leq \frac{\xi^n(b,a)}{2n!}\left(\frac{n}{n+1}\right)^{\frac{1}{r}}\left(\frac{1}{n}|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

II. If $\gamma(\mu)=\mu^{-1}$, then Theorem 3.44 reduces to the following result in the class of P-preinvex function.

Corollary 3.46. *Under the assumptions of Theorem 3.44 if $|\Lambda^{(n)}|^r$ is P-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$

$$\leq \frac{\xi^n(b,a)}{2n!}\,{}_2F_1^{\frac{1}{r}}\left(-r,n;n+1;\frac{2}{n}\right)\left(|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

III. If $\gamma(\mu)=\mu^{s-1}$, then Theorem 3.44 reduces to the following result in the class of s-preinvex function.

Corollary 3.47. *Under the assumptions of Theorem 3.44 if $|\Lambda^{(n)}|^r$ is s-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$

$$\leq \frac{\xi^n(b,a)n^{\frac{1}{r}}}{2n!}\left(m_1|\Lambda^{(n)}(a)|^r+m_2|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}},$$

where

$$m_1=\frac{\Gamma(n)\Gamma(s+1)}{\Gamma(n+s+1)}\,{}_2F_1\left(-r,n;n+s+1;\frac{2}{n}\right)$$

$$m_2=\frac{1}{n+s}\,{}_2F_1\left(-r,n;n+s+1;\frac{2}{n}\right).$$

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 3.44 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.48. *Under the assumptions of Theorem 3.44 if $|\Lambda^{(n)}|^r$ is s-Godunova-Levin preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)n^{\frac{1}{r}}}{2n!}\left(m_3|\Lambda^{(n)}(a)|^r+m_4|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}},$$

where

$$m_3=\frac{\Gamma(n)\Gamma(1-s)}{\Gamma(n-s+1)}\,{}_2F_1\left(-r,n;n-s+1;\frac{2}{n}\right)$$
$$m_4=\frac{1}{n-s}\,{}_2F_1\left(-r,n;n-s+1;\frac{2}{n}\right).$$

V. If $\gamma(\mu)=1-\mu$, then Theorem 3.44 reduces to the following result in the class of tgs-preinvex function.

Corollary 3.49. *Under the assumptions of Theorem 3.44 if $|\Lambda^{(n)}|^r$ is tgs-preinvex function on, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)n^{\frac{1}{r}}}{2n!}\,{}_2F_1^{\frac{1}{r}}\left(-r,n+1;n+3;\frac{2}{n}\right)\left(\frac{|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r}{(n+1)(n+2)}\right)^{\frac{1}{r}}.$$

Theorem 3.50. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\xi:K\times K\to\mathbb{R}$. Suppose $\Lambda:K\to\mathbb{R}$ is a function such that $\Lambda^{(n)}$ exists on K for $n\in\mathbb{N}, n\geq 1$ and $\Lambda^{(n)}$ is integrable on $[a,a+\xi(b,a)]$, where $a,b\in K$ with $\xi(b,a)>0$. If $|\Lambda^{(n)}|^r$ is the γ-preinvex function on K for $r\geq 1$, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)}{2n!}\left(|\Lambda^{(n)}(a)|^r\int_0^1\mu^{r(n-1)}(n-2\mu)^r(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|^r\int_0^1\mu^{r(n-1)}(n-2\mu)^r\mu\gamma(\mu)\mathrm{d}\mu\right)^{\frac{1}{r}}.$$

Proof. Suppose $n \geq 2$, using Lemma 1.3 and Holder's inequality, it follows that

$$\begin{aligned}
&\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|\\
&\leq \frac{\xi^n(b,a)}{2n!}\int_0^1 \mu^{n-1}(n-2\mu)|\Lambda^{(n)}(a+\mu\xi(b,a))|\mathrm{d}\mu\\
&\leq \frac{\xi^n(b,a)}{2n!}\left(\int_0^1 1\mathrm{d}\mu\right)^{1-\frac{1}{r}}\left(\int_0^1 \mu^{r(n-1)}(n-2\mu)^r|\Lambda^{(n)}(a+\mu\xi(b,a))|^r\mathrm{d}\mu\right)^{\frac{1}{r}}\\
&\leq \frac{\xi^n(b,a)}{2n!}\left(|\Lambda^{(n)}(a)|^r\int_0^1 \mu^{r(n-1)}(n-2\mu)^r(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|^r\int_0^1 \mu^{r(n-1)}(n-2\mu)^r\mu\gamma(\mu)\mathrm{d}\mu\right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Now we will discuss some special cases.

I. If $\gamma(\mu)=1$, then Theorem 3.50 reduces to the following result in the class of classical preinvex function.

Corollary 3.51. *Under the assumptions of Theorem 3.50 if $|\Lambda^{(n)}|^r$ is classical preinvex function on K, then*

$$\begin{aligned}
&\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|\\
&\leq \frac{\xi^n(b,a)}{2.(2n-1)!}\left(k_1|\Lambda^{(n)}(a)|^r+k_2|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}},
\end{aligned}$$

where

$$\begin{aligned}
k_1&=\frac{1}{(r+1)}\,{}_2F_1\left(-r,r(n-1)+1;r(n-1)+2;\frac{2}{n}\right)\\
k_2&=\frac{1}{(r+2)}\,{}_2F_1\left(-r,r(n-1)+2;r(n-1)+3;\frac{2}{n}\right).
\end{aligned}$$

II. If $\gamma(\mu)=\mu^{-1}$, then Theorem 3.50 reduces to the following result in the class of P-preinvex function.

Corollary 3.52. *Under the assumptions of Theorem 3.50 if $|\Lambda^{(n)}|^r$ is P-preinvex function on K with respect to $\xi(.,.)$, then*

$$\begin{aligned}
&\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|\\
&\leq \frac{\xi^n(b,a)}{2.(2n-1)!}\left(\frac{|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r}{r(n-1)+1}\right)^{\frac{1}{r}}.
\end{aligned}$$

III. If $\gamma(\mu)=\mu^{s-1}$, then Theorem 3.50 reduces to the following result in the class of s-preinvex function.

Corollary 3.53. *Under the assumptions of Theorem 3.50 if $|\Lambda^{(n)}|^r$ is s-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)}{2.(2n-1)!}\left(k_3|\Lambda^{(n)}(a)|^r+k_4|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}},$$

where

$$k_3=\frac{\Gamma(r(n-1)+1)\Gamma(s+1)}{\Gamma(r(n-1)+s+2)}\,{}_2F_1\left(-r,r(n-1)+1;r(n-1)+s+2;\frac{2}{n}\right)$$
$$k_4=\frac{1}{r(n-1)+s+1}\,{}_2F_1\left(-r,r(n-1)+s+1;r(n-1)+s+2;\frac{2}{n}\right).$$

IV. If $\gamma(\mu)=\mu^{-s-1}$, then Theorem 3.50 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.54. *Under the assumptions of Theorem 3.50 if $|\Lambda^{(n)}|^r$ is s-Godunova-Levin preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)}{2n!}\left(k_5|\Lambda^{(n)}(a)|^r+k_6|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}},$$

where

$$k_5=\frac{\Gamma(r(n-1)+1)\Gamma(s+1)}{\Gamma(r(n-1)-s+2)}\,{}_2F_1\left(-r,r(n-1)+1;r(n-1)-s+2;\frac{2}{n}\right)$$
$$k_6=\frac{1}{r(n-1)-s+1}\,{}_2F_1\left(-r,r(n-1)-s+1;r(n-1)-s+2;\frac{2}{n}\right).$$

V. If $\gamma(\mu)=1-\mu$, then Theorem 3.50 reduces to the following result in the class of tgs-preinvex function.

Corollary 3.55. *Under the assumptions of Theorem 3.50 if $|\Lambda^{(n)}|^r$ is tgs-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)}{2.(2n-1)!}\,{}_2F_1^{\frac{1}{r}}\left(-r,r(n-1)+2;r(n-1)+4;\frac{2}{n}\right)\left(\frac{|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r}{(r(n-1)+2)(r(n-1)+31)}\right)^{\frac{1}{r}}.$$

Theorem 3.56. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\xi:K\times K\to\mathbb{R}$. Suppose $\Lambda:K\to\mathbb{R}$ is a function such that $\Lambda^{(n)}$ exists on K for $n\in\mathbb{N},n\geq1$ and $\Lambda^{(n)}$ is integrable on $[a,a+\xi(b,a)]$, where $a,b\in K$ with $\xi(b,a)>0$. If $|\Lambda^{(n)}|^r$ is the γ-preinvex function on K for $r>1$ and $p^{-1}+r^{-1}=1$, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)}{2.(2n-1)!}\omega^{\frac{1}{p}}\left(|\Lambda^{(n)}(a)|^r\int_0^1\mu^{r(n-1)}(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|^r\int_0^1\mu^{r(n-1)}\mu\gamma(\mu)\mathrm{d}\mu\right)^{\frac{1}{r}},$$

where

$$\omega = \frac{n[(1-\frac{2}{n})^{p+1}-1]}{2(p+1)}.$$

Proof. Suppose $n \geq 2$, using Lemma 1.3 and Holder's inequality, it follows that

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2} - \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2n!}\int_0^1 \mu^{n-1}(n-2\mu)|\Lambda^{(n)}(a+\mu\xi(b,a))|\mathrm{d}\mu$$
$$\leq \frac{\xi^n(b,a)}{2n!}\left(\int_0^1 (n-2\mu)^p\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1 \mu^{r(n-1)}|\Lambda^{(n)}(a+\mu\xi(b,a))|^r\mathrm{d}\mu\right)^{\frac{1}{r}}$$
$$\leq \frac{\xi^n(b,a)}{2.(2n-1)!}\omega^{\frac{1}{p}}\left(|\Lambda^{(n)}(a)|^r\int_0^1 \mu^{r(n-1)}(1-\mu)\gamma(1-\mu)\mathrm{d}\mu + |\Lambda^{(n)}(b)|^r\int_0^1 \mu^{r(n-1)}\mu\gamma(\mu)\mu^{n-1}\mathrm{d}\mu\right)^{\frac{1}{r}}.$$

This completes the proof. □

Now we will discuss some special cases.
I. If $\gamma(\mu) = 1$, then Theorem 3.56 reduces to the following result in the class of classical preinvex function.

Corollary 3.57. *Under the assumptions of Theorem 3.56 if* $|\Lambda^{(n)}|^r$ *is classical preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2} - \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2,(2n-1)!(r(n-1)+2)^{\frac{1}{r}}}\omega^{\frac{1}{p}}\left(\frac{1}{r(n-1)+1}|\Lambda^{(n)}(a)|^r + |\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

II. If $\gamma(\mu) = \mu^{-1}$, then Theorem 3.56 reduces to the following result in the class of P-preinvex function.

Corollary 3.58. *Under the assumptions of Theorem 3.56 if* $|\Lambda^{(n)}|^r$ *is P-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2} - \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2.(2n-1)!(r(n-1)+1)^{\frac{1}{r}}}\omega^{\frac{1}{p}}\left(|\Lambda^{(n)}(a)|^r + |\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

III. If $\gamma(\mu) = \mu^{s-1}$, then Theorem 3.56 reduces to the following result in the class of s-preinvex function.

Corollary 3.59. *Under the assumptions of Theorem 3.56 if* $|\Lambda^{(n)}|^r$ *is s-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2} - \frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2.(2n-1)!}\omega^{\frac{1}{p}}\left(B(r(n-1)+1,s+1)|\Lambda^{(n)}(a)|^r + \frac{1}{r(n-1)+s+1}|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 3.56 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.60. *Under the assumptions of Theorem 3.56 if $|\Lambda^{(n)}|^r$ is s-Godunova-Levin preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2.(2n-1)!}\omega^{\frac{1}{p}}\left(B(r(n-1)+1,1-s)|\Lambda^{(n)}(a)|^r+\frac{1}{r(n-1)-s+1}|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}}.$$

V. If $\gamma(\mu) = 1-\mu$, then Theorem 3.56 reduces to the following result in the class of tgs-preinvex function.

Corollary 3.61. *Under the assumptions of Theorem 3.56 if $|\Lambda^{(n)}|^r$ is tgs-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2.(2n-1)!}\omega^{\frac{1}{p}}\left(\frac{|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r}{(r(n-1)+2)(r(n-1)+3)}\right)^{\frac{1}{r}}.$$

Theorem 3.62. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\xi: K\times K\to\mathbb{R}$. Suppose $\Lambda: K\to\mathbb{R}$ is a function such that $\Lambda^{(n)}$ exists on K for $n\in\mathbb{N}, n\geq 1$ and $\Lambda^{(n)}$ is integrable on $[a, a+\xi(b,a)]$, where $a,b\in K$ with $\xi(b,a)>0$. If $|\Lambda^{(n)}|^r$ is the γ-preinvex function on K for $r\geq 1$, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{r}}\left(|\Lambda^{(n)}(a)|^r\int_0^1\mu^{n-1}(n-2\mu)(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|^r\int_0^1\mu^{n-1}(n-2\mu)\mu\gamma(\mu)\mathrm{d}\mu\right)^{\frac{1}{r}}.$$

Proof. Suppose $n\geq 2$, using Lemma 1.3 and Holder's inequality, it follows that

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq \frac{\xi^n(b,a)}{2n!}\int_0^1\mu^{n-1}(n-2\mu)|\Lambda^{(n)}(a+\mu\xi(b,a))|\mathrm{d}\mu$$
$$\leq \frac{\xi^n(b,a)}{2n!}\left(\int_0^1\mu^{n-1}(n-2\mu)\mathrm{d}\mu\right)^{1-\frac{1}{r}}\left(\int_0^1\mu^{n-1}(n-2\mu)|\Lambda^{(n)}(a+\mu\xi(b,a))|^r\mathrm{d}\mu\right)^{\frac{1}{r}}$$
$$\leq \frac{\xi^n(b,a)}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{r}}\left(|\Lambda^{(n)}(a)|^r\int_0^1\mu^{n-1}(n-2\mu)(1-\mu)\gamma(1-\mu)\mathrm{d}\mu+|\Lambda^{(n)}(b)|^r\int_0^1\mu^{n-1}(n-2\mu)\mu\gamma(\mu)\mathrm{d}\mu\right)^{\frac{1}{r}}.$$

This completes the proof. □

Now we will discuss some special cases.
I. If $\gamma(\mu) = 1$, then Theorem 3.62 reduces to the corresponding result in the class of classical preinvex function, Theorem 2.4 from [10].
II. If $\gamma(\mu) = \mu^{-1}$, then Theorem 3.62 reduces to the following result in the class of P-preinvex function.

Corollary 3.63. *Under the assumptions of Theorem 3.62, if $|\Lambda^{(n)}|^r$ is P-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)(n-1)}{2.(n+1)!}\left(\frac{|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r}{r(n-1)+1}\right)^{\frac{1}{r}}.$$

III. If $\gamma(\mu) = \mu^{s-1}$, then Theorem 3.62 reduces to the following result in the class of s-preinvex function.

Corollary 3.64. *Under the assumptions of Theorem 3.62 if $|\Lambda^{(n)}|^r$ is s-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{r}}\left(\mu_1|\Lambda^{(n)}(a)|^r+\mu_2|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}},$$

where μ_1 and μ_2 are given in Theorem 3.27.

IV. If $\gamma(\mu) = \mu^{-s-1}$, then Theorem 3.62 reduces to the following result in the class of s-Godunova-Levin preinvex function.

Corollary 3.65. *Under the assumptions of Theorem 3.62 if $|\Lambda^{(n)}|^r$ is s-Godunova-Levin preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{r}}\left(\mu_3|\Lambda^{(n)}(a)|^r+\mu_4|\Lambda^{(n)}(b)|^r\right)^{\frac{1}{r}},$$

where μ_3 and μ_4 are given in Theorem 3.27.

V. If $\gamma(\mu) = 1-\mu$, then Theorem 3.62 reduces to the following result in the class of tgs-preinvex function.

Corollary 3.66. *Under the assumptions of Theorem 3.62 if $|\Lambda^{(n)}|^r$ is tgs-preinvex function on K, then*

$$\left|\frac{\Lambda(a)+\Lambda(a+\xi(b,a))}{2}-\frac{1}{\xi(b,a)}\int_a^{a+\xi(b,a)}\Lambda(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)\xi^k(b,a)}{2(k+1)!}\Lambda^{(k)}(a+\xi(b,a))\right|$$
$$\leq\frac{\xi^n(b,a)(n-1)}{2.(n+1)!}\left(\frac{|\Lambda^{(n)}(a)|^r+|\Lambda^{(n)}(b)|^r}{n+3}\right)^{\frac{1}{r}}.$$

4. Conclusion

We have introduced the notion of γ-preinvex functions. We have shown that the class of γ-preinvex functions unifies several other new and known classes of preinvexity. Several new integral inequalities of Hermite-Hadamard's type are obtained. New and known special cases are also discussed in detail. These results may be useful where bounds for natural phenomena described by integrals such as mechanical work are frequently required and are also helpful in the feld of numerical analysis where error analysis is required. We hope that the ideas of this paper will inspire interested readers. One can also obtain fractional and quantum analogues of the obtained main results. This can be an interesting problem for future research work.

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References

[1] W. W. Breckner, Stetigkeitsaussagen fureine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen, Publ. Inst. Math., 23, (1978), 13-20.
[2] G. Cristescu, M. A. Noor, M. U. Awan, Bounds of the second degree cumulative frontier gaps of functions with generalized convexity, Carpath. J. Math., 31, 173-180, (2015).
[3] S. S. Dragomir, Inequalities of Hermite-Hadamard type for φ-convex functions, RGMIA, Vol 16, Article No. 87, (2013).
[4] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequality and applications. Victoria University, Melbourne, (2000).
[5] S. S. Dragomir, J.E. Pecaric, L.E. Persson, Some inequalities of Hadamard type, Soochow J. Math., 21, 335-341, (1995).
[6] T. S. Du, J. G. Liao, Y. J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions, J. Nonlinear Sci. Appl., 9, (2016), 3112-3126.
[7] E. K. Godunova, V. I. Levin, Neravenstva dlja funkcii sirokogo klassa soderzascego vypuklye, monotonnye I nekotorye drugie vidy funkii, Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, 138-142, (1985).
[8] P. Gua, Z. Huang, T. S. Du, Riemann-Liouville fractional trapezium-like inequalities via generalized (m, h_1, h_2)-preinvexity, Proyecciones J. Math., 37(2), (2018), 345-378.
[9] A. B.-Israel, B. Mond, What is invexity? J. Austral. Math. Soc. Ser. B, 28(1), 1-9 (1986).
[10] M. A. Latif, On Hermite-Hadamard type integral inequalities for n-times differentiable preinvex functions with applications, Stud. Univ. Babes-Bolyai Math, 58, 325-343, (2013).
[11] M. A. Latif, S. S. Dragomir, On Hermite-Hadamard type integral inequalities for n-times differentiable log-preinvex functions. Filomat, 29(7), 1651-1661, (2015).
[12] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory, 2, 126-131, (2007).
[13] M. A. Noor, K. I. Noor, M. U. Awan, S. Khan, Hermite-Hadamard inequalities for s-Godunova-Levin preinvex functions, J. Adv. Math. Stud. 7(2), 12-19, (2014).
[14] M. A. Noor, K. I. Noor, M. U. Awan, J. Li, On Hermite-Hadamard inequalities for h-preinvex functions, Filomat, 28(7), (2014), 1463-1474.
[15] S. R. Mohan, S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl., 189(3), 901-908, (1995).
[16] M. Tunc, E. Gov, U. Sanal, On tgs-convex function and their inequalities, FACTA Uni. (NIS) Ser. Math. Inform., 30(5), 679-691, (2015).
[17] S. Varosanec, On h-convexity. J. Math. Anal. Appl., 326(1), 303-311, (2007).
[18] T. Weir, B. Mond, Preinvex functions in multiple objective optimization, J. Math. Anal. Appl., 136, 29-38, (1988).
[19] Y. Zhang, T. S. Du, H. Wang, Some new k-fractional integral inequalities containing multiple parameters via generalized (s, m)-preinvexity, Italian J. Pure and Appl. Math., 40, 510-527, (2018).