Filomat 34:12 (2020), 4161–4169
https://doi.org/10.2298/FIL2012161O

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Oscillatory Behavior of Advanced Difference Equations with General Arguments

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Abstract. In this paper, we introduce some oscillation criteria for the first-order advanced difference equations with general arguments

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = 0, \ n \geq 1, \ n \in \mathbb{N},$$

where $\{p_i(n)\}(i = 1, 2, \ldots, m)$ are sequences of positive real numbers, $\{\tau_i(n)\}(i = 1, 2, \ldots, m)$ are sequences of integers and are not necessarily monotone such that $\tau_i(n) \geq n$ $(i = 1, 2, \ldots, m)$. An example illustrating the results is also given.

1. Introduction

In this paper, we study the oscillatory behavior of all solutions of the first-order advanced difference equations

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = 0, \quad n \in \mathbb{N}, \quad n \geq 1, \tag{1}$$

where $\{p_i(n)\}$ $(i = 1, 2, \cdots, m)$ are sequences of positive real numbers, $\{\tau_i(n)\}$ $(i = 1, 2, \cdots, m)$ are sequences of integers and are not necessarily monotone such that

$$\tau_i(n) \geq n \ \text{ for } n \geq 1. \tag{2}$$

2010 *Mathematics Subject Classification*. 34C10; 39A10; 39A12; 39A21.
Keywords. Advanced; difference equations; nonmonotone; oscillatory solutions.
Received: 25 December 2019; Revised: 17 April 2020; Accepted: 21 June 2020
Communicated by Jelena Manojlović
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Here, ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$. By a solution of (1), we mean a sequence of real numbers $\{x(n)\}$ which is defined for $n \geq 0$ and satisfies (1) for all $n \geq 1$.
Recently, there are too many studies in literature on the oscillation theory of advanced (or delay) type differential or difference equations. See, for example, [1-18] and the references cited therein. As usual, a solution $\{x(n)\}$ of (1) is said to be *oscillatory*, for every positive integer n_0, there exist $n_1, n_2 \geq n_0$ such that $x(n_1)x(n_2) \leq 0$. In other words, a solution $\{x(n)\}$ is *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, the solution is called *nonoscillatory*.

Throughout this paper, we are going to use the notation: $\sum_{i=k}^{k-1} A(i) = 0$.
Now, let's recall some well-known oscillation results on this subject. For $m = 1$, equation (1) reduces to the following equation.

$$\nabla x(n) - p(n)x(\tau(n)) = 0,\ n \in \mathbb{N},\ \ n \geq 1. \tag{3}$$

In 2002, Li and Zhu [15] proved that, when $\tau(n) = n + k$, if there exists an integer $n_1 \geq 0$ and a positive integer l such that

$$\sum_{n=n_1+lk}^{\infty} p(n)\left[\left(\frac{k+1}{k}\right)^l q_l^{1/k+1}(n) - 1\right] = \infty,$$

where

$$\begin{aligned} q_1(n) &= \sum_{i=n-k}^{n-1} p(i),\ \ n \geq k, \\ q_{j+1}(n) &= \sum_{i=n-k}^{n-1} p(i)q_j(n),\ \ j \geq 1,\ \ n \geq (j+1)k, \end{aligned}$$

then all solutions of (3) oscillate.
In 1991, Györi and Ladas [12] studied the following first order linear difference equation with advanced argument $\tau(n) = n + \sigma$.

$$\Delta x(n) - p(n)x(n+\sigma) = 0,\ \ n \geq 0, \tag{4}$$

where Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$, $\sigma \geq 2$ is a positive integer and the authors proved that if

$$\limsup_{n\to\infty} \sum_{i=n}^{n+\sigma-1} p(i) > 1, \tag{5}$$

or

$$\liminf_{n\to\infty} \sum_{i=n+1}^{n+\sigma-1} p(i) > \left(\frac{\sigma-1}{\sigma}\right)^{\sigma}, \tag{6}$$

then all solutions of (4) oscillate.
In 2007, Öcalan and Akın [16] analyzed the following first order linear difference equations

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(n-k_i) = 0,\ \ n \geq 0, \tag{7}$$

where $p_i(n) \leq 0$ and $k_i \leq -1$ for $i = 1, 2, \ldots, m$, and obtained some results for the oscillation of all solutions of (7) (See also [17]). Furthermore, when $p_i(n) = p_i$ $(i = 1, 2, \cdots, m)$ in (7), see [12, Theorems 7.2.1 and 7.3.1].

In 2012, Chatzarakis and Stavroulakis [1] proved that if $\{\tau(n)\}$ is nondecreasing and

$$\limsup_{n\to\infty}\sum_{j=n}^{\tau(n)} p(j) > 1, \tag{8}$$

then all solutions of (3) oscillate.
We note that, in [1], the authors assumed that $\tau(n) \geq n+1,\ n \geq 1$. We would like to state that, in fact, if $\tau(n) \geq n,\ n \geq 1$ is taken, then all results are valid in [1].
Also, in 2012, Chatzarakis and Stavroulakis [1] proved that if $\{\tau(n)\}$ is not necessarily monotone and

$$\limsup_{n\to\infty}\sum_{j=n}^{\sigma(n)} p(j) > 1, \tag{9}$$

where

$$\sigma(n) = \max_{1\leq s\leq n}\{\tau(s)\},\ s \in \mathbb{N}, \tag{10}$$

then all solutions of (3) oscillate. Unfortunately, we consider this result is not applicable. Indeed, if we examine this result, it can not be proved like Theorem 2.1 in [1]. To see this, by using the proof of Theorem 2.1 in [1], since $\sigma(n) \geq \tau(n)$ and $\{x(n)\}$, $\{\sigma(n)\}$ are eventually nondecreasing, from equation (3), we have

$$\nabla x(n) - p(n)x(\sigma(n)) \leq 0,\ n \geq 1. \tag{11}$$

Now, summing up (11) from n to $\sigma(n)$, we obtain

$$x(\sigma(n)) - x(n-1) - \sum_{j=n}^{\sigma(n)} p(j)x(\sigma(j)) \leq 0,$$

and the proof is stopped here (see the proof of Theorem 2.1 in [1]). Hence, Theorem 2.1$''$ and Theorem 2.4$''$ are not applicable in [1].
In 2016, Öcalan and Özkan [18] proved that if $\{\tau(n)\}$ is not necessarily monotone and

$$\limsup_{n\to\infty}\sum_{j=n}^{h(n)} p(j) > 1, \tag{12}$$

where $h(n) = \min_{n\leq s}\{\tau(s)\}$, then all solutions of (3) oscillate.
Also, the authors [18], regarding the lim inf condition, tried to obtain a condition for the oscillatory solution of the equation (3) when $\{\tau(n)\}$ is not necessarily monotone. Unfortunately, the authors have made a mistake in the proof of Theorem 2.4 in [18], caused by induction. That is, the proof of Theorem 2.4 in [18] is invalid. Therefore, one of the aim of this paper is to obtain the lim inf condition for the equation (3) to be oscillatory.

2. Main Results

In this section, we introduce a new sufficient condition, regarding the condition lim inf, for the oscillation of all solutions of (3) when $\{\tau(n)\}$ is not necessarily monotone. Set

$$h(n) := \min_{n\leq s}\{\tau(s)\},\ \ s \in \mathbb{N}. \tag{13}$$

Obviously, $\{h(n)\}$ is nondecreasing and $\tau(n) \geq h(n)$ for all $n \geq 1$. The following lemmas will be needed in the proof of the Theorem 2.3.
The following one was given in [18].

Lemma 2.1. *[18] Assume that* (13) *holds and* $m > 0$. *Then, we have*

$$m = \liminf_{n\to\infty} \sum_{j=n+1}^{h(n)} p(j) = \liminf_{n\to\infty} \sum_{j=n+1}^{\tau(n)} p(j),$$

where $\{h(n)\}$ *is defined by* (13).

Lemma 2.2. *Suppose* $p(n) > 0$ *and* $\{x(n)\}$ *is positive solution of the following inequalities*

$$\nabla x(n) - p(n)x(n) \geq 0, \quad n \geq s. \tag{14}$$

Then

$$x(n) \geq \exp\left\{\sum_{j=s+1}^{n} p(j)\right\} x(s), \quad n \geq s. \tag{15}$$

Proof. Dividing (14) by $x(n)$, we have

$$\frac{\nabla x(n)}{x(n)} - p(n) \geq 0, \quad n \geq s. \tag{16}$$

Summing up (16) from $s + 1$ to n, we obtain

$$\sum_{j=s+1}^{n} \frac{\nabla x(j)}{x(j)} - \sum_{j=s+1}^{n} p(j) \geq 0. \tag{17}$$

Now, we get

$$\begin{aligned}\sum_{j=s+1}^{n} \frac{\nabla x(j)}{x(j)} &= \sum_{j=s+1}^{n} \frac{x(j) - x(j-1)}{x(j)} = (n-s) - \sum_{j=s+1}^{n} \frac{x(j-1)}{x(j)} \\ &= (n-s) - \sum_{j=s+1}^{n} \exp\left\{\ln \frac{x(j-1)}{x(j)}\right\} \\ &\leq (n-s) - \sum_{j=s+1}^{n} \left(1 + \ln \frac{x(j-1)}{x(j)}\right) = \sum_{j=s+1}^{n} \ln \frac{x(j)}{x(j-1)},\end{aligned}$$

where we have used the $e^x \geq 1 + x$ for $x \geq 0$. So, we obtain

$$\begin{aligned}\sum_{j=s+1}^{n} \frac{\nabla x(j)}{x(j)} &\leq \sum_{j=s+1}^{n} \ln \frac{x(j)}{x(j-1)} = \ln x(n) - \ln x(s) \\ &= \ln \frac{x(n)}{x(s)}.\end{aligned}$$

Finally, from (17), we have

$$\ln \frac{x(n)}{x(s)} - \sum_{j=s+1}^{n} p(j) \geq 0,$$

or

$$x(n) \geq \exp\left\{\sum_{j=s+1}^{n} p(j)\right\} x(s),$$

which is desirable. □

Theorem 2.3. *Assume that* (2) *holds. If* $\{\tau(n)\}$ *is not necessarily monotone and*

$$\liminf_{n\to\infty}\sum_{j=n+1}^{\tau(n)} p(j) > \frac{1}{e}, \tag{18}$$

then all solutions of (3) *oscillate.*

Proof. Assume, for the sake of contradiction, that there exists a positive nonoscillatory solution $x(n)$ of (3). Since $-x(n)$ is also a solution of (3), we can confine our discussion only to the case where the solution $x(n)$ is eventually positive. Then, there exists $n_1 > n_0 \geq 1$ such that $x(n),\ x(\tau(n)) > 0$, for all $n \geq n_1$. Thus, from (3) we have

$$\nabla x(n) = p(n)x(\tau(n)) \geq 0, \quad \text{for all } n \geq n_1,$$

which means that $\{x(n)\}$ is an eventually nondecreasing. In view of this and taking into account that $\tau(n) \geq h(n) \geq n$, (3) gives

$$\nabla x(n) - p(n)x(h(n)) \geq 0, \quad n \geq n_1 \tag{19}$$

and

$$\nabla x(n) - p(n)x(n) \geq 0, \quad n \geq n_1. \tag{20}$$

On the other hand, by using Lemma 2.1 and from (18), it follows that there exists a constant $c > 0$ such that

$$\sum_{j=n+1}^{h(n)} p(j) \geq c > \frac{1}{e}, \quad n \geq n_2 > n_1. \tag{21}$$

So, by Lemma 2.2 and (20), we obtain

$$x(h(n)) \geq \exp\left\{\sum_{j=n+1}^{h(n)} p(j)\right\} x(n) \quad \text{for all } h(n) \geq n. \tag{22}$$

Since $e^x \geq ex$ for $x \in \mathbb{R}$, from (21) and (22), we get

$$x(h(n)) \geq e^c x(n) \geq (ec)\,x(n), \tag{23}$$

where $ec > 1$. Thus, from (19) and (23), we have

$$\nabla x(n) - p(n)(ec)\,x(n) \geq 0, \quad n \geq n_2.$$

Let $p_1(n) := (ec)\,p(n)$. So, we obtain

$$\nabla x(n) - p_1(n)x(n) \geq 0, \quad n \geq n_2. \tag{24}$$

By using Lemma 2.2, we get

$$x(h(n)) \geq \exp\left\{\sum_{j=n+1}^{h(n)} p_1(j)\right\} x(n) \quad \text{for all } h(n) \geq n. \tag{25}$$

Thus, from (21) and (25), we have

$$\begin{aligned} x(h(n)) &\geq \exp\left\{\sum_{j=n+1}^{h(n)} (ec)\, p(j)\right\} x(n) \\ &= \exp\left\{(ec) \sum_{j=n+1}^{h(n)} p(j)\right\} x(n) \geq \exp\left\{ec^2\right\} x(n) \\ &\geq (ec)^2 x(n). \end{aligned}$$

Repeating the above procedures, it follows that by induction for any positive integer k, we obtain

$$\frac{x(h(n))}{x(n)} \geq (ec)^k \quad \text{for sufficiently large } n. \tag{26}$$

On the other hand, from (21), there exists $n^* \in (n, h(n)]$, $n^* \in \mathbb{N}$ such that

$$\sum_{j=n+1}^{n^*} p(j) \geq \frac{c}{2} \quad \text{and} \quad \sum_{j=n^*}^{h(n)} p(j) \geq \frac{c}{2}. \tag{27}$$

Summing up (19) from $n+1$ to n^*, we obtain

$$x(n^*) - x(n) - \sum_{j=n+1}^{n^*} p(j) x(h(j)) \geq 0.$$

Now, using (27) and the fact that the functions $\{x(n)\}$ and $\{h(n)\}$ are nondecreasing, we have

$$x(n^*) \geq x(h(n+1)) \sum_{j=n+1}^{n^*} p(j) \geq x(h(n)) \sum_{j=n+1}^{n^*} p(j),$$

or

$$x(n^*) \geq x(h(n)) \frac{c}{2}. \tag{28}$$

Summing up (19) from n^* to $h(n)$, and using the same arguments we get

$$x(h(n)) - x(n^*-1) - \sum_{j=n^*}^{h(n)} p(j) x(h(j)) \geq 0,$$

or

$$x(h(n)) - x(h(n^*)) \sum_{j=n^*}^{h(n)} p(j) \geq 0,$$

or

$$x(h(n)) \geq x(h(n^*)) \frac{c}{2}. \tag{29}$$

Combining the inequalities (28) and (29), we obtain

$$x(n^*) \geq x(h(n)) \frac{c}{2} \geq x(h(n^*)) \left(\frac{c}{2}\right)^2,$$

or

$$\frac{x(h(n^*))}{x(n^*)} \leq \left(\frac{2}{c}\right)^2 < +\infty,$$

i.e., $\liminf_{n\to\infty} \frac{x(h(n))}{x(n)}$ exists. This contradicts with (26). So, the proof of the theorem is completed. □

A slight modification in the proofs of Theorem 2.3 and [18, Theorem 2.3] leads to the following result.

Theorem 2.4. *Assume that all the conditions of Theorem 2.3 or* (12) *hold. Then*
(*i*) *the difference inequality*

$$\nabla x(n) - p(n)x(\tau(n)) \geq 0, \quad n \in \mathbb{N}, \quad n \geq 1$$

has no eventually positive solutions,
(*ii*) *the difference inequality*

$$\nabla x(n) - p(n)x(\tau(n)) \leq 0, \quad n \in \mathbb{N}, \quad n \geq 1$$

has no eventually negative solutions.

Example 2.5. *Consider*

$$\nabla x(n) - p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}, \quad n \geq 1. \tag{30}$$

We take $p(n) = 0.19$ *and* $\tau(n) = n + 2$. *We observe that*

$$\limsup_{n\to\infty} \sum_{j=n}^{n+2} p(j) = 0.57 \ngtr 1.$$

shows that condition (12) *fails. However, since*

$$\liminf_{n\to\infty} \sum_{j=n+1}^{n+2} p(j) = 0.38 > \frac{1}{e},$$

every solution of (30) *is oscillatory.*

3. Equations with several arguments

Now, we consider the first-order advanced difference equations with several arguments and coefficients

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \quad n \in \mathbb{N}, \quad n \geq 1 \tag{31}$$

where $\{p_i(n)\}$ $(i = 1, 2, \cdots, m)$ are positive sequences, $\{\tau_i(n)\}$ $(i = 1, 2, \ldots, m)$ are sequences of integers and are not necessarily monotone such that

$$\tau_i(n) \geq n \quad \text{for all } n \in \mathbb{N}, \quad n \geq 1. \tag{32}$$

In this section, we present some new sufficient conditions for the oscillation of all solutions of (31).
In 2014, Chatzarakis et al. [2] proved that if $\{\tau_i(n)\}$ $(i = 1, 2, \cdots, m)$ are nondecreasing and

$$\limsup_{n\to\infty} \sum_{j=n}^{\tau(n)} \sum_{i=1}^{m} p_i(j) > 1, \tag{33}$$

where $\tau(n) = \min_{1\le i\le m}\{\tau_i(n)\}$, then all solutions of (31) oscillate.
Set

$$h_i(n) := \inf_{n\le s} \tau_i(s) \ \text{ and } \ h(n) = \min_{1\le i\le m} h_i(n), \ \ n \ge n_0. \tag{34}$$

Clearly, $\{h_i(n)\}$ $(i = 1, 2, \cdots, m)$ are nondecreasing and $\tau_i(n) \ge h_i(n) \ge h(n)$ for all $n \ge n_0$. Now, we have the following result.

Theorem 3.1. *Assume that* (32) *holds. If* $\{\tau_i(n)\}$ $(i = 1, 2, \cdots, m)$ *are not necessarily monotone and*

$$\limsup_{n\to\infty} \sum_{j=n}^{h(n)} \sum_{i=1}^{m} p_i(j) > 1, \tag{35}$$

or

$$\liminf_{n\to\infty} \sum_{j=n+1}^{\tau(n)} \sum_{i=1}^{m} p_i(j) > \frac{1}{e}, \tag{36}$$

where $\tau(n) = \min_{1\le i\le m}\{\tau_i(n)\}$ *and* $h(n)$ *is defined by* (34)*, then all solutions of* (31) *oscillate.*

Proof. Assume, for the sake of contradiction, that there exists a positive nonoscillatory solution $x(n)$ of (31). Then there exists $n_1 > n_0$ such that $x(n), \ x(\tau_i(n)) > 0$ for all $n \ge n_1$. Thus, from (31) we have

$$\nabla x(n) - \left(\sum_{i=1}^{m} p_i(n)\right) x(\tau(n)) \ge 0.$$

Comparing (35) and (36), we obtain a contradiction to Theorem 2.4. Here, we have used the following equality

$$\liminf_{n\to\infty} \sum_{j=n+1}^{\tau(n)} \sum_{i=1}^{m} p_i(j) = \liminf_{t\to\infty} \sum_{j=n+1}^{h(n)} \sum_{i=1}^{m} p_i(j),$$

which is easily obtained as similar to the proof of Lemma 2.1. □

A slight modification in the proof of Theorem 3.1 leads to the following result.

Theorem 3.2. *Assume that all the conditions of Theorem 3.1 hold. Then*
(*i*) *the difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \ge 0, \ \ n \in \mathbb{N}, \ \ n \ge 1$$

has no eventually positive solutions,
(*ii*) *the difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \le 0, \ \ n \in \mathbb{N}, \ \ n \ge 1$$

has no eventually negative solutions.

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