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Sup-Hesitant Fuzzy Quasi-Associative Ideals of BCI-Algebras

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Abstract. The notion of Sup-hesitant fuzzy quasi-associative ideal in BCI-algebras is introduced, and related properties are investigated. Characterizations of Sup-hesitant fuzzy quasi-associative ideal are provided. Relations between Sup-hesitant fuzzy ideal and Sup-hesitant fuzzy quasi-associative ideal are displayed. Conditions for a Sup-hesitant fuzzy ideal to be a Sup-hesitant fuzzy quasi-associative ideal are provided. Extension property for Sup-hesitant fuzzy quasi-associative ideal is established.

1. Introduction

Hesitant fuzzy sets are introduced by Torra and Narukawa as another generalization of fuzzy sets, and discussed its properties (see [12] and [13]). After then, several researchers have applied hesitant fuzzy sets to algebraic structure, for example, *BCK*/*BCI*-algebras (see [1, 3–6, 8, 9, 11]). Muhiuddin and Jun [10] introduced the notion of sup-hesitant fuzzy subalgebras and investigate several related properties in BCK/BCI-algebras. They considered characterizations of Sup-hesitant fuzzy subalgebras, and discussed Sup-hesitant fuzzy translation and Sup-hesitant fuzzy extension of Sup-hesitant fuzzy subalgebras. They also investigated relations between Sup-hesitant fuzzy translation and Sup-hesitant fuzzy extension. Muhiuddin, Harizavi and Jun [11] introduced sup-hesitant fuzzy ideals in BCK/BCI-algebras, and investigated several properties. They discussed relations between sup-hesitant fuzzy subalgebras and sup-hesitant fuzzy ideals, and considered characterizations of Sup-hesitant fuzzy ideals.

In this paper, we introduce the Sup-hesitant fuzzy quasi-associative ideal in a BCI-algebra and investigate several properties. We discuss characterizations of Sup-hesitant fuzzy quasi-associative ideal, and consider relations between Sup-hesitant fuzzy ideal and Sup-hesitant fuzzy quasi-associative ideal. We provide conditions for a Sup-hesitant fuzzy ideal to be a Sup-hesitant fuzzy quasi-associative ideal. We establish the extension property for the Sup-hesitant fuzzy quasi-associative ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

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(I) $(\forall v, u, w \in X)\,(((v * u) * (v * w)) * (w * u) = 0)$,

(II) $(\forall v, u \in X)\,((v * (v * u)) * u = 0)$,

(III) $(\forall v \in X)\,(v * v = 0)$,

(IV) $(\forall v, u \in X)\,(v * u = 0, u * v = 0 \Rightarrow v = u)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall v \in X)\,(0 * v = 0)$,

then X is called a *BCK-algebra*.

A BCI-algebra X is said to be *associative* (see [2]) if it satisfies:

$$(\forall v, u, w \in X)\,((v * u) * w = v * (u * w))\,. \tag{1}$$

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall v \in X)\,(v * 0 = v)\,, \tag{2}$$

$$(\forall v, u, w \in X)\,(v \leq u \Rightarrow v * w \leq u * w,\ w * u \leq w * v)\,, \tag{3}$$

$$(\forall v, u, w \in X)\,((v * u) * w = (v * w) * u)\,, \tag{4}$$

$$(\forall v, u, w \in X)\,((v * w) * (u * w) \leq v * u) \tag{5}$$

where $v \leq u$ if and only if $v * u = 0$.

Any BCI-algebra X satisfies the following conditions:

$$(\forall v, u, w \in X)\,(0 * (0 * ((v * w) * (u * w))) = (0 * u) * (0 * v))\,, \tag{6}$$

$$(\forall v, u \in X)\,(0 * (0 * (v * u)) = (0 * u) * (0 * v))\,, \tag{7}$$

$$(\forall v \in X)\,(0 * (0 * (0 * v)) = 0 * v)\,, \tag{8}$$

$$(\forall v, u \in X)\,(0 * (v * u)) = (0 * v) * (0 * u))\,. \tag{9}$$

A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $v * u \in S$ for all $v, u \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in A, \tag{10}$$

$$(\forall v \in X)\,(v * u \in A,\ u \in A \Rightarrow v \in A)\,. \tag{11}$$

A subset Q of a *BCI*-algebra X is called a *quasi-associative ideal* of X (see [14]) if it satisfies (10) and

$$(\forall x, y, z \in X)\,(x * (y * z) \in Q,\ y \in Q \Rightarrow x * z \in Q)\,. \tag{12}$$

Note that an ideal Q of a *BCI*-algebra X is a quasi-associative ideal of X if and only if the following assertion is valid:

$$(\forall x, y \in X)\,(x * (0 * y) \in Q \Rightarrow x * y \in Q)\,. \tag{13}$$

We refer the reader to the books [2, 7] for further information regarding BCK/BCI-algebras.

Torra [12] introduced a new extension for fuzzy sets to manage those situations in which several values are possible for the definition of a membership function of a fuzzy set.

Let X be a reference set. Then we define hesitant fuzzy set on X in terms of a function $\mathcal{H}$ that when applied to X returns a subset of $[0, 1]$ (see [12, 13]).

In what follows, the power set of $[0, 1]$ is denoted by $\mathcal{P}([0, 1])$ and

$$\mathcal{P}^*([0, 1]) = \mathcal{P}([0, 1]) \setminus \{\emptyset\}.$$

For any element $Q \in \mathcal{P}^*([0, 1])$, the supremum of Q is denoted by $\sup Q$. For any hesitant fuzzy set $\mathcal{H}$ on X and $Q \in \mathcal{P}^*([0, 1])$, consider the set

$$\mathrm{Sup}[\mathcal{H}; Q] := \{v \in X \mid \sup \mathcal{H}(v) \geq \sup Q\}\,.$$

Definition 2.1 ([10]). *Let X be a BCK/BCI-algebra. Given an element $Q \in \mathcal{P}^*([0,1])$, a hesitant fuzzy set $\mathcal{H}$ on X is called a Sup-hesitant fuzzy subalgebra of X related to Q (briefly, Q-Sup-hesitant fuzzy subalgebra of X) if the set* $\mathrm{Sup}[\mathcal{H};Q]$ *is a subalgebra of X. If $\mathcal{H}$ is a Q-Sup-hesitant fuzzy subalgebra of X for all $Q \in \mathcal{P}^*([0,1])$, then we say that $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X.*

Lemma 2.2 ([10]). *Every Sup-hesitant fuzzy subalgebra $\mathcal{H}$ of a BCK/BCI-algebra X satisfies:*

$$(\forall v \in X)\,(\sup \mathcal{H}(0) \geq \sup \mathcal{H}(v))\,. \tag{14}$$

Definition 2.3 ([11]). *Let X be a BCK/BCI-algebra. Given an element $Q \in \mathcal{P}^*([0,1])$, a hesitant fuzzy set $\mathcal{H}$ on X is called a Sup-hesitant fuzzy ideal of X related to Q (briefly, Q-Sup-hesitant fuzzy ideal of X) if the set* $\mathrm{Sup}[\mathcal{H};Q]$ *is an ideal of X. If $\mathcal{H}$ is a Q-Sup-hesitant fuzzy ideal of X for all $Q \in \mathcal{P}^*([0,1])$, then we say that $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X.*

Lemma 2.4 ([11]). *A hesitant fuzzy set $\mathcal{H}$ on a BCK/BCI-algebra X is a Sup-hesitant fuzzy ideal of X if and only if it satisfies* (14) *and*

$$(\forall v, u \in X)\,(\sup \mathcal{H}(v) \geq \min\{\sup \mathcal{H}(v * u), \sup \mathcal{H}(u)\})\,. \tag{15}$$

Lemma 2.5 ([11]). *Every Sup-hesitant fuzzy ideal $\mathcal{H}$ of a BCK/BCI-algebra X satisfies:*

$$(\forall v, u \in X)\,(v \leq u \;\Rightarrow\; \sup \mathcal{H}(v) \geq \sup \mathcal{H}(u))\,. \tag{16}$$

3. Sup-hesitant fuzzy quasi-associative ideals

In what follows, let X be a BCI-algebra unless otherwise specified.

Definition 3.1. *Given an element $Q \in \mathcal{P}^*([0,1])$, a hesitant fuzzy set $\mathcal{H}$ on X is called a Sup-hesitant fuzzy quasi-associative ideal of X related to Q (briefly, Q-Sup-hesitant fuzzy quasi-associative ideal of X) if the set* $\mathrm{Sup}[\mathcal{H};Q]$ *is a quasi-associative ideal of X. If $\mathcal{H}$ is a Q-Sup-hesitant fuzzy quasi-associative ideal of X for all $Q \in \mathcal{P}^*([0,1])$, then we say that $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X.*

Example 3.2. *(1) Let $X = \{0, 1, a\}$ be a BCI-algebra with the following Cayley table.*

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 1.

Table 1: Tabular representation of $\mathcal{H}$

X	0	1	a
$\mathcal{H}(x)$	$(0.8, 0.9]$	$(0.35, 0.9)$	$[0.33, 0.63]$

It is routine to verify that $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X.

(2) Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 2.
It is routine to verify that $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X.

Table 2: Tabular representation of $\mathcal{H}$

X	0	a	b	c
$\mathcal{H}(x)$	$(0.66, 0.88]$	$(0.35, 0.66)$	$[0.44, 0.58]$	$[0.38, 0.58]$

Theorem 3.3. *A hesitant fuzzy set $\mathcal{H}$ on X is a Sup-hesitant fuzzy quasi-associative ideal of X if and only if if it satisfies* (14) *and*

$$(\forall x, y, z \in X)\,(\min\{\sup \mathcal{H}(x * (y * z)), \sup \mathcal{H}(y)\} \leq \sup \mathcal{H}(x * z))\,. \tag{17}$$

Proof. Let $\mathcal{H}$ be a Sup-hesitant fuzzy quasi-associative ideal of X. If (14) is false, then there exists $Q \in \mathcal{P}^*([0,1])$ and $a \in X$ such that $\sup \mathcal{H}(0) < \sup Q \leq \sup \mathcal{H}(a)$. It follows that $a \in \mathrm{Sup}[\mathcal{H}; Q]$ and $0 \notin \mathrm{Sup}[\mathcal{H}; Q]$. This is a contradiction, and so (14) is valid. Now assume that (17) is not valid. Then

$$\min\{\sup \mathcal{H}(a * (b * c)), \sup \mathcal{H}(b)\} > \sup \mathcal{H}(a * c)$$

for some $a, b, c \in X$, and thus there exists $B \in \mathcal{P}^*([0,1])$ such that

$$\min\{\sup \mathcal{H}(a * (b * c)), \sup \mathcal{H}(b)\} \geq \sup B > \sup \mathcal{H}(a * c).$$

which implies that $a * (b * c) \in \mathrm{Sup}[\mathcal{H}; B]$, $b \in \mathrm{Sup}[\mathcal{H}; B]$ but $a * c \notin \mathrm{Sup}[\mathcal{H}; B]$. This is a contradiction, and thus (17) holds.

Conversely, suppose that $\mathcal{H}$ satisfies two conditions (14) and (17). Let $Q \in \mathcal{P}^*([0,1])$ be such that $\mathrm{Sup}[\mathcal{H}; Q] \neq \emptyset$. Obviously, $0 \in \mathrm{Sup}[\mathcal{H}; Q]$. Let $x, y, z \in X$ be such that $x * (y * z) \in \mathrm{Sup}[\mathcal{H}; Q]$ and $y \in \mathrm{Sup}[\mathcal{H}; Q]$. Then $\sup \mathcal{H}(x * (y * z)) \geq \sup Q$ and $\sup \mathcal{H}(y) \geq \sup Q$. It follows from (17) that

$$\sup \mathcal{H}(x * z) \geq \min\{\sup \mathcal{H}(x * (y * z)), \sup \mathcal{H}(y)\} \geq \sup Q$$

and that $x * z \in \mathrm{Sup}[\mathcal{H}; Q]$. Hence $\mathrm{Sup}[\mathcal{H}; Q]$ is a quasi-associative ideal of X for all $Q \in \mathcal{P}^*([0,1])$, and therefore $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X. □

Proposition 3.4. *If $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X, then $\mathcal{H}(x * z) \geq \mathcal{H}(x * (0 * z))$ for all $x, z \in X$. In particular, $\mathcal{H}(0 * z) \geq \mathcal{H}(0 * (0 * z))$ for all $z \in X$.*

Proof. Straightforward. □

We consider relations between a Sup-hesitant fuzzy ideal and a Sup-hesitant fuzzy quasi-associative ideal.

Theorem 3.5. *Every Sup-hesitant fuzzy quasi-associative ideal is a Sup-hesitant fuzzy ideal.*

Proof. Let $\mathcal{H}$ be a Sup-hesitant fuzzy quasi-associative ideal of X. Since $x * 0 = x$ for all $x \in X$, it follows from (17) that

$$\begin{aligned}\sup \mathcal{H}(x) = \sup \mathcal{H}(x * 0) &\geq \min\{\sup \mathcal{H}(x * (y * 0)), \sup \mathcal{H}(y)\} \\ &= \min\{\sup \mathcal{H}(x * y), \sup \mathcal{H}(y)\}\end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X. □

The following example shows that the converse of Theorem 3.5 is not true in general.

Example 3.6. *Consider a BCI-algebra $X = \{0, a, 1, 2, \}$ with the following Cayley table.*

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by

$$\mathcal{H} : X \to \mathcal{P}([0,1]), \quad x \mapsto \begin{cases} (0.55, 0.89] & \text{if } x = 0, \\ (0.37, 0.77) & \text{if } x = a, \\ [0.25, 0.65] & \text{if } x = 1, \\ [0.35, 0.56] \cup \{0.65\} & \text{if } x = 2, \\ \{0.54\} \cup [0.60, 0.65] & \text{if } x = 3 \end{cases}$$

It is routine to verify that $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X, but it is not a Sup-hesitant fuzzy quasi-associative ideal of X since

$$\sup \mathcal{H}(3 * 1) < \min\{\sup \mathcal{H}(3 * (0 * 1)), \sup \mathcal{H}(0)\}.$$

Proposition 3.7. *Every Sup-hesitant fuzzy quasi-associative ideal $\mathcal{H}$ of X satisfies the following assertions.*

(1) $(\forall x, y \in X)(x \leq y \;\Rightarrow\; \sup \mathcal{H}(x) \geq \sup \mathcal{H}(y))$.
(2) $(\forall x, y \in X)(\sup \mathcal{H}(x * y) = \sup \mathcal{H}(0) \;\Rightarrow\; \sup \mathcal{H}(x) \geq \sup \mathcal{H}(y))$.
(3) $(\forall x, y \in X)(\sup \mathcal{H}(x * y) \geq \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\})$.
(4) $(\forall x, y, z \in X)(\sup \mathcal{H}(x * y) \geq \min\{\sup \mathcal{H}(x * z), \sup \mathcal{H}(z * y)\})$.
(5) $(\forall x \in X)(\sup \mathcal{H}((0 * x) * x) = \sup \mathcal{H}(0)\})$.

Proof. (1) If $x \leq y$, then $x * y = 0$ and thus

$$\begin{aligned} \sup \mathcal{H}(x) = \sup \mathcal{H}(x * 0) &\geq \min\{\sup \mathcal{H}(x * (y * 0)), \sup \mathcal{H}(y)\} \\ &= \min\{\sup \mathcal{H}(x * y), \sup \mathcal{H}(y)\} \\ &= \min\{\sup \mathcal{H}(0), \sup \mathcal{H}(y)\} = \sup \mathcal{H}(y). \end{aligned}$$

(2) It is similar to the proof of (1).
(3) For any $x, y \in X$, we have

$$\begin{aligned} \sup \mathcal{H}(x * y) &\geq \min\{\sup \mathcal{H}(x * (y * y)), \sup \mathcal{H}(y)\} \\ &= \min\{\sup \mathcal{H}(x * 0), vy)\} \\ &= \min\{\sup \mathcal{H}(x), \sup \mathcal{H}(y)\}. \end{aligned}$$

(4) Using (I) and (1), we get $\sup \mathcal{H}((x*y)*(x*z)) \geq \sup \mathcal{H}(z*y)$ for all $x, y, z \in X$. Since $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X (see Theorem 3.5), it follows that

$$\begin{aligned} \sup \mathcal{H}(x * y) &\geq \min\{\sup \mathcal{H}((x * y) * (x * z)), \sup \mathcal{H}(x * z)\} \\ &\geq \min\{\sup \mathcal{H}(z * y), \sup \mathcal{H}(x * z)\} \end{aligned}$$

for all $x, y, z \in X$.
(5) If we put $x := 0 * x$, $y := 0$ and $z := x$ in (17), then

$$\sup \mathcal{H}(0) = \min\{\sup \mathcal{H}((0 * x) * (0 * x)), \sup \mathcal{H}(0)\} \leq \sup \mathcal{H}((0 * x) * x)$$

for all $x \in X$. Combining this and (14) induces $\sup \mathcal{H}((0 * x) * x) = \sup \mathcal{H}(0)$ for all $x \in X$. □

By combining Proposition 3.7(3) and Theorem 3.5, we know that every Sup-hesitant fuzzy quasi-associative ideal is a Sup-hesitant fuzzy closed ideal.

We provide a condition for a Sup-hesitant fuzzy ideal to be a Sup-hesitant fuzzy quasi-associative ideal.

Theorem 3.8. *In an associative BCI-algebra, every Sup-hesitant fuzzy ideal is a Sup-hesitant fuzzy quasi-associative ideal.*

Proof. Let $\mathcal{H}$ be a Sup-hesitant fuzzy ideal of an associative BCI-algebra X. Then

$$\begin{aligned}\sup\mathcal{H}(x*z) &\geq \min\{\sup\mathcal{H}((x*z)*y), \sup\mathcal{H}(y)\}\\ &= \min\{\sup\mathcal{H}((x*y)*z), \sup\mathcal{H}(y)\}\\ &= \min\{\sup\mathcal{H}(x*(y*z)), \sup\mathcal{H}(y)\}\end{aligned}$$

for all $x, y, z \in X$. Hence $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X. □

Theorem 3.9. *If $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X, then the following are equivalent.*

(1) *$\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X.*
(2) *$\mathcal{H}$ satisfies:*

$$(\forall x, y \in X)(\sup\mathcal{H}(x*y) \geq \sup\mathcal{H}(x*(0*y))). \tag{18}$$

(3) *$\mathcal{H}$ satisfies:*

$$(\forall x, y, z \in X)(\sup\mathcal{H}((x*y)*z) \geq \sup\mathcal{H}(x*(y*z))). \tag{19}$$

Proof. (1) ⇒ (2). Assume that $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X. Then

$$\sup\mathcal{H}(x*y) \geq \min\{\sup\mathcal{H}(x*(0*y)), \sup\mathcal{H}(0)\} = \sup\mathcal{H}(x*(0*y)).$$

(2) ⇒ (3). Note that

$$\begin{aligned}&((x*y)*(0*z))*(x*(y*z)) = ((x*y)*(x*(y*z)))*(0*z)\\ &\leq ((y*z)*y)*(0*z) = (0*z)*(0*z) = 0\end{aligned}$$

for all $x, y, z \in X$. Since $\mathcal{H}$ is a Sup-hesitant fuzzy ideal, it follows from (14), (15), (16) and (18) that

$$\begin{aligned}&\sup\mathcal{H}((x*y)*z) \geq \sup\mathcal{H}((x*y)*(0*z))\\ &\geq \min\{\sup\mathcal{H}(((x*y)*(0*z))*(x*(y*z))), \sup\mathcal{H}(x*(y*z))\}\\ &\geq \min\{\sup\mathcal{H}(0), \sup\mathcal{H}(x*(y*z))\} = \sup\mathcal{H}(x*(y*z))\end{aligned}$$

for all $x, y, z \in X$.

(3) ⇒ (1). For any $x, y, z \in X$, we have

$$\begin{aligned}\sup\mathcal{H}(x*z) &\geq \min\{\sup\mathcal{H}((x*z)*y), \sup\mathcal{H}(y)\}\\ &= \min\{\sup\mathcal{H}((x*y)*z), \sup\mathcal{H}(y)\}\\ &\geq \min\{\sup\mathcal{H}(x*(y*z), \sup\mathcal{H}(y)\}\end{aligned}$$

by (4), (15) and (19). Therefore $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X. □

Theorem 3.10. *If a Sup-hesitant fuzzy ideal $\mathcal{H}$ of X satisfies the following assertion*

$$(\forall x, y \in X)(\sup\mathcal{H}(x*y) \geq \sup\mathcal{H}(x)), \tag{20}$$

then $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X.

Proof. Let $\mathcal{H}$ be a Sup-hesitant fuzzy ideal of X that satisfies the condition (20). Then

$$\sup \mathcal{H}((x*z)*(y*z)) = \sup \mathcal{H}((x*(y*z))*z) \geq \sup \mathcal{H}(x*(y*z))$$

for all $x, y, z \in X$ by (4) and (20). It follows from (15) that

$$\begin{aligned}\sup \mathcal{H}(x*z) &\geq \min\{\sup \mathcal{H}((x*z)*(y*z)), \sup \mathcal{H}(y*z)\} \\ &\geq \min\{\sup \mathcal{H}(x*(y*z)), \sup \mathcal{H}(y)\}\end{aligned}$$

for all $x, y, z \in X$. Therefore $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X. □

For any hesitant fuzzy set $\mathcal{H}$ on X and an element a of X, consider a set

$$\mathcal{H}_a := \{x \in X \mid \sup \mathcal{H}(a) \leq \sup \mathcal{H}(x)\}.$$

Lemma 3.11 ([11]). *If $\mathcal{H}$ is a hesitant fuzzy ideal of X, then the set $\mathcal{H}_a$ is an ideal of X for all $a \in X$.*

Theorem 3.12. *If $\mathcal{H}$ is a hesitant fuzzy quasi-associative ideal of X, then the set $\mathcal{H}_a$ is a quasi-associative ideal of X for all $a \in X$.*

Proof. Let $x, y \in X$ be such that $x*(0*y) \in \mathcal{H}_a$. Then

$$\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x*(0*y)) \leq \sup \mathcal{H}(x*y)$$

by Theorem 3.9, and so $x*y \in \mathcal{H}_a$. Hence $\mathcal{H}_a$ is a quasi-associative ideal of X for all $a \in X$. □

Proposition 3.13. *Given an element $a \in X$, if $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X, then the following conditions are valid:*

$$(\forall x, y \in X)\left(\begin{array}{c}\sup \mathcal{H}(a) \leq \min\{\sup \mathcal{H}(x*y), \sup \mathcal{H}(y)\} \\ \Rightarrow\ \sup \mathcal{H}(a) \leq \sup \mathcal{H}(x)\end{array}\right). \tag{21}$$

$$(\forall x, y, z \in X)\left(\begin{array}{c}\sup \mathcal{H}(a) \leq \min\{\sup \mathcal{H}(x*(y*z)), \sup \mathcal{H}(y)\} \\ \Rightarrow\ \sup \mathcal{H}(a) \leq \sup \mathcal{H}(x*z)\end{array}\right). \tag{22}$$

$$(\forall x, y \in X)\left(\begin{array}{c}\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x*(0*y)) \\ \Rightarrow\ \sup \mathcal{H}(a) \leq \sup \mathcal{H}(x*y)\end{array}\right). \tag{23}$$

Proof. Straightforward by definition of Sup-hesitant fuzzy (quasi-assoiative) ideal. □

Given a hesitant fuzzy set $\mathcal{H}$ on X, we provide conditions for the set $\mathcal{H}_a$ to be a quasi-associative ideal.

Theorem 3.14. *If a hesitant fuzzy set $\mathcal{H}$ on X satisfies* (14) *and* (22)*, then the set $\mathcal{H}_a$ is a quasi-associative ideal of X for any $a \in X$.*

Proof. Let $a \in X$. The condition (14) implies that $0 \in \mathcal{H}_a$. Let $x, y, z \in X$ be such that $x*(y*z) \in \mathcal{H}_a$ and $y \in \mathcal{H}_a$. Then $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x*(y*z))$ and $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(y)$, which imply that

$$\sup \mathcal{H}(a) \leq \min\{\sup \mathcal{H}(x*(y*z)), \sup \mathcal{H}(y)\}.$$

It follows from (22) that $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x*z)$. Hence $x*z \in \mathcal{H}_a$, and therefore $\mathcal{H}_a$ is a quasi-associative ideal of X. □

Theorem 3.15. *If a hesitant fuzzy set $\mathcal{H}$ on X satisfies* (14), (21) *and* (23)*, then the set $\mathcal{H}_a$ is a quasi-associative ideal of X for any $a \in X$.*

Proof. Let $a \in X$. The condition (14) implies that $0 \in \mathcal{H}_a$. Let $x, y \in X$ be such that $x * y \in \mathcal{H}_a$ and $y \in \mathcal{H}_a$. Then $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x * y)$ and $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(y)$, which imply that

$$\sup \mathcal{H}(a) \leq \min\{\sup \mathcal{H}(x * y), \sup \mathcal{H}(y)\}.$$

It follows from (21) that $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x)$. Thus $x \in \mathcal{H}_a$, and so $\mathcal{H}_a$ is an ideal of X. Let $x, y \in X$ be such that $x * (0 * y) \in \mathcal{H}_a$. Then $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x * (0 * y))$, and so $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x * y)$ by (23), that is, $x * y \in \mathcal{H}_a$. Therefore $\mathcal{H}_a$ is a quasi-associative ideal of X. □

In the following theorem, we establish the extension property for a Sup-hesitant fuzzy quasi-associative ideal.

Theorem 3.16. *Let $\mathcal{H}$ and $\mathcal{G}$ be Sup-hesitant fuzzy ideals of X such that $\mathcal{H}(0) = \mathcal{G}(0)$ and $\mathcal{H}(x) \leq \mathcal{G}(x)$ for all $x(\neq 0) \in X$. If $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X, then so is $\mathcal{G}$.*

Proof. Assume that $\mathcal{H}$ is a Sup-hesitant fuzzy quasi-associative ideal of X. Using (III), (4), (14), (15), (18) and given conditions, we have

$$\begin{aligned}
\sup \mathcal{G}(x * y) &\geq \min\{\sup \mathcal{G}((x * y) * (x * (0 * y))), \sup \mathcal{G}(x * (0 * y))\} \\
&\geq \min\{\sup \mathcal{H}((x * y) * (x * (0 * y))), \sup \mathcal{G}(x * (0 * y))\} \\
&= \min\{\sup \mathcal{H}((x * (x * (0 * y))) * y), \sup \mathcal{G}(x * (0 * y))\} \\
&\geq \min\{\sup \mathcal{H}((x * (x * (0 * y))) * (0 * y)), \sup \mathcal{G}(x * (0 * y))\} \\
&= \min\{\sup \mathcal{H}((x * (0 * y)) * (x * (0 * y))), \sup \mathcal{G}(x * (0 * y))\} \\
&= \min\{\sup \mathcal{H}(0), \sup \mathcal{G}(x * (0 * y))\} \\
&= \min\{\sup \mathcal{G}(0), \sup \mathcal{G}(x * (0 * y))\} \\
&= \sup \mathcal{G}(x * (0 * y))
\end{aligned}$$

for all $x, y \in X$. It follows from Theorem 3.9 that $\mathcal{G}$ is a Sup-hesitant fuzzy quasi-associative ideal of X. □

Conclusion

The concept of a hesitant fuzzy set has many applications in the domain of mathematics and elsewhere; among them are many logical algebras. Based on this, we applied this concept to introduce the Sup-hesitant fuzzy quasi-associative ideal in BCI-algebras. The researchers can apply this concept for more subjects of BCK/BCI-algebra. In this paper, we discussed characterizations of Sup-hesitant fuzzy quasi-associative ideal. Also, we considered relations between Sup-hesitant fuzzy ideal and Sup-hesitant fuzzy quasi-associative ideal. We provided conditions for a Sup-hesitant fuzzy ideal to be a Sup-hesitant fuzzy quasi-associative ideal. Finally, we established the extension property for Sup-hesitant fuzzy quasi-associative ideal.

References

[1] S. Aldhafeeri and G. Muhiuddin, Commutative ideals of BCK-algebras based on uni-hesitant fuzzy set theory. Missouri J. Math. Sci. 31 (2019), no. 1, 56–65.
[2] Y. Huang, BCI-algebra, Science Press, Beijing 2006.
[3] Y.B. Jun, A new type of p-ideals in BCI-algebras based on hesitant fuzzy sets. Azerb. J. Math. 8 (2018), no. 1, 104–115.
[4] Y. B. Jun and S. S. Ahn, Hesitant fuzzy set theory applied to BCK/BCI-algerbas, J. Comput. Anal. Appl. 20 (2016), no. 4, 635–646.
[5] Y.B. Jun, H.S. Kim and S.Z. Song, A new type of subsemigroups and ideals of semigroups in the framework of hesitant fuzzy set theory. J. Mult.-Valued Logic Soft Comput. 29 (2017), no. 1-2, 67–81.
[6] Y. B. Jun and S. Z. Song, Hesitant fuzzy set theory applied to filters in *MTL*-algebras, Honam Mathematical J. 36 (2014), No. 4, 813–830. http://dx.doi.org/10.5831/HMJ.2014.36.4.813.
[7] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa Co. Seoul 1994.
[8] G. Muhiuddin, Hesitant fuzzy filters and hesitant fuzzy *G*-filters in residuated lattices, J. Comput. Anal. Appl. 20 (2016), no. 2, 394–404.

[9] G. Muhiuddin and S. Aldhafeeri, Subalgebras and ideals in BCK/BCI-algebras based on uni-hesitant fuzzy set theory. Eur. J. Pure Appl. Math. 11 (2018), no. 2, 417–430.
[10] G. Muhiuddin and Y.B. Jun, Sup-hesitant fuzzy subalgebras and its translations and extensions. Annals of Commu- nications in Mathematics, Volume 2, Number 1 (2019), 48-56.
[11] G. Muhiuddin, H. Harizavi and Y.B. Jun, Ideal theory in BCK/BCI-algebras in the frame of hesitant fuzzy set theory, Applications and Applied Mathematics (in press).
[12] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (2010), 529–539.
[13] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, in: The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, 2009, pp. 1378-1382.
[14] Z. Yue and X. H. Zhang, Quasi-associative ideals of *BCI*-algebras, Selected papers on BCK and BCI-algebras (in P. R. China) 1 (1992) 85–86.