Filomat 34:13 (2020), 4367–4374
https://doi.org/10.2298/FIL2013367Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Geometries for a Mutual Connection of Semi-Symmetric Metric Recurrent Connections

Di Zhao[a], Talyun Ho[b], An Jae Hyon[c]

[a]*College of Science, University of Shanghai for Science and Technology, 200093 Shanghai, P. R. China.*
[b]*Faculty of Mathematics, Kim Il Sung University, Pyongyang, D. P. R. K.*
[c]*Department of Mathematics, Kim Hyong Jik Normal University, Pyongyang, D. P. R. K.*

Abstract. A semi-symmetric metric recurrent connection has already been studied. In this paper we newly discovered geometrical properties and conjugate symmetric condition for the mutual connection of a semi-symmetric metric recurrent connection in a Riemannian manifold.

1. Introduction

The study of geometries and physics of a manifolds associated with a semi-symmetric metric(no-metric) connection has been an active fields over the past six decades. There are many celebrated works related to the semi-symmetric metric (no-metric) connections. For instance, Agache and Chafle [1] in 1992 studied the semi-symmetric non-metric connection, and obtained some physical properties. De, Han and Zhao [2] described the geometric properties of a manifold with a semi-symmetric non-metric connection. De and Kamilya [3] studied the hypersurfaces of Kenmotsu manifolds with a quarter-symmetric connection. Peltrovic and Stankovic [10] studied the geometries of F-planar mappings associated with non-symmetric connections. For the further characterization of a manifold associated with semi-symmetric metric connections or Ricci(quarter-)-semi-symmetric metric(non-metric) connections, one can see [4, 5, 9, 11, 12, 16–19, 21, 22] and the references therein.

Recently, many researchers pay attention on the geometric and physics of a manifold with quarter-symmetric connections or quarter-symmetric recurrent connections. Han, Ho and Zhao [7] considered the invariant of a manifold with a quarter-symmetric connection transformation. Tang, Ho, Fu and Zhao [13, 14] investigated the geometric properties of a manifold with a quarter-symmetric projective (conformal) connection. Tang et al [15] studied the geometry of a manifold with a quarter-symmetric recurrent connection.

Surprisingly, Han, Fu and Zhao [6] investigated the similar topics on Sub-Riemannian manifolds, and obtained some interesting results. Furthermore, Zhao, Jen and Ho [20] considered the geometric

2010 *Mathematics Subject Classification*. Primary 53C20; Secondary 53D11
Keywords. Semi-symmetric metric recurrent connection; mutual connection; dual connection; conjugate symmetry.
Received: 10 January 2020; Accepted: 04 March 2020
Communicated by Mića Stanković
Corresponding author: Di Zhao
The authors were supported in part by the NNSF of China(No.11671193), Postgraduate Research & Practice Innovation Program of Jiangsu Province.
Email addresses: jszhaodi@126.com (Di Zhao), jcy19630320@163.com (Talyun Ho), 159043306@163.com (An Jae Hyon)

and physical properties of a special sub-Riemannian manifold with Ricci quarter-symmetric recurrent connection.

Motivated by the previous researches we will introduce the mutual connection and study the geometrical properties of the mutual connection of various forms of a semi-symmetric metric recurrent connection, and investigate the conjugate symmetric conditions in a Riemannian manifold

2. The mutual connection of the first semi-symmetric metric recurrent connection

In a Riemannian manifold (M, g) the first semi-symmetric metric recurrent connection ∇ is a connection satisfying the equation

$$\nabla_k g_{ij} = 2\omega_k g_{ij}, T^k_{ij} = \pi_j \delta^k_i - \pi_i \delta^k_j$$

for a 1-form ω and π, and the connection coefficient Γ^k_{ij} is

$$\Gamma^k_{ij} = \{^k_{ij}\} - \omega_i \delta^k_j - (\omega_j - \pi_j)\delta^k_i + g_{ij}(\omega^k - \pi^k)$$

where $\{^k_{ij}\}$ is the christoffel symbol of Levi-Civita connection $\overset{o}{\nabla}$, and ω_i, π_i are components of the 1-form ω and π. The mutual connection $\overline{\nabla}$ of the connection ∇ is a connection satisfying the equation

$$\overline{\nabla}_k g_{ij} = g_{ki}\pi_j + g_{kj}\pi_i + g_{ij}(\omega_k - \pi_k), \overline{T}\,^k_{ij} = \pi_i \delta^k_j - \pi_j \delta^k_i$$

and the connection coefficient $\overline{\Gamma}\,^k_{ij}$ is

$$\overline{\Gamma}\,^k_{ij} = \{^k_{ij}\} - (\omega_i - \pi_i)\delta^k_j - \omega_j \delta^k_i + g_{ij}(\omega^k - \pi^k) \tag{2.1}$$

and the dual connection $\overset{*}{\overline{\nabla}}$ of $\overline{\nabla}$ is a connection satisfying the equation

$$\overset{*}{\overline{\nabla}}_k g_{ij} = -g_{ki}\pi_j - g_{kj}\pi_i - 2g_{ij}(\omega_k - \pi_k), \overset{*}{\overline{T}}\,^k_{ij} = 2[(\omega_i - \pi_i)\delta^k_j - (\omega_j - \pi_j)\delta^k_i]$$

and the connection coefficient $\overset{*}{\overline{\Gamma}}\,^k_{ij}$ is

$$\overset{*}{\overline{\Gamma}}\,^k_{ij} = \{^k_{ij}\} + (\omega_i - \pi_i)\delta^k_j - (\omega_j - \pi_j)\delta^k_i + g_{ij}\pi^k \tag{2.2}$$

From (2.1) and (2.2) we find that the curvature tensor of $\overline{\nabla}$ and the curvature tensor of $\overset{*}{\overline{\nabla}}$ are given respectively by

$$\overline{R}_{ijk}{}^l = K_{ijk}{}^l + \delta^l_i \overline{\omega}_{jk} - \delta^l_j \overline{\omega}_{ik} + g_{jk}\tau^l_i - g_{ik}\tau^l_j - \delta^l_k \beta_{ij} \tag{2.3}$$

$$\overset{*}{\overline{R}}_{ijk}{}^l = K_{ijk}{}^l + \delta^l_i \tau_{jk} - \delta^l_j \tau_{ik} + g_{jk}\overline{\omega}^l_i - g_{ik}\overline{\omega}^l_j + \delta^l_k \beta_{ij} \tag{2.4}$$

and the relation between these curvatures is

$$\overset{*}{\overline{R}}_{ijk}{}^l = \overline{R}_{ijk}{}^l + \delta^l_i \rho_{jk} - \delta^l_j \rho_{ik} + g_{ik}\rho^l_j - g_{jk}\rho^l_i + 2\delta^l_k \beta_{ij} \tag{2.5}$$

where $K_{ijk}{}^l$ is the curvature tensor of $\overset{o}{\nabla}$, and $\overline{\omega}_{jk}, \tau_{jk}, \beta_{jk}, \rho_{jk}$ are denoted, respectively, by

$$\begin{aligned}
\overline{\omega}_{jk} &= \overset{o}{\nabla}_j \omega_k + \omega_j \omega_k \\
\tau_{jk} &= \overset{o}{\nabla}_j(\omega_k - \pi_k) + (\omega_k - \pi_k)(\omega_j - \pi_j) - g_{jk}\omega_l(\omega^l - \pi^l) \\
\beta_{jk} &= \tau_{jk} - \tau_{kj} \\
\rho_{jk} &= \tau_{jk} - \overline{\omega}_{jk}
\end{aligned}$$

Lemma 2.1. *In a Riemannian manifold (M, g), if the Weyl conformal curvature tensor for the connection $\overset{o}{\nabla}, \overline{\nabla}$ and $\overset{*}{\overline{\nabla}}$ are $\overset{o}{C}_{ijk}{}^{l}, \overline{C}_{ijk}{}^{l}$ and $\overset{*}{\overline{C}}_{ijk}{}^{l}$, then*

$$\overline{C}_{ijk}{}^{l} + \overset{*}{\overline{C}}_{ijk}{}^{l} = 2\overset{o}{C}_{ijk}{}^{l} \tag{2.6}$$

where

$$\begin{aligned}
\overset{o}{C}_{ijk}{}^{l} &= K_{ijk}{}^{l} + \frac{1}{n-2}\left(\delta^l_j K_{ik} - \delta^l_i K_{jk} + g_{ik}K^l_j - g_{jk}K^l_i\right) + \frac{K}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik}) \\
\overline{C}_{ijk}{}^{l} &= \overline{R}_{ijk}{}^{l} + \frac{1}{n-2}\left(\delta^l_j \overline{R}_{ik} - \delta^l_i \overline{R}_{jk} + g_{ik}\overline{R}^l_j - g_{jk}\overline{R}^l_i\right) + \frac{\overline{R}}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik}) \\
\overset{*}{\overline{C}}_{ijk}{}^{l} &= \overset{*}{\overline{R}}_{ijk}{}^{l} + \frac{1}{n-2}\left(\delta^l_j \overset{*}{\overline{R}}_{ik} - \delta^l_i \overset{*}{\overline{R}}_{jk} + g_{ik}\overset{*}{\overline{R}}_j{}^{l} - g_{jk}\overset{*}{\overline{R}}_i{}^{l}\right) + \frac{\overset{*}{\overline{R}}}{(n-1)(n-2)}(\delta^l_i g_{jk} - \delta^l_j g_{ik})
\end{aligned} \tag{*}$$

Proof. Let $\overline{R}_{ijk}{}^{l} + \overset{*}{\overline{R}}_{ijk}{}^{l} \triangleq S_{ijk}{}^{l}, \overline{\omega}_{jk} + \tau_{jk} \triangleq \gamma_{jk}$, then from (2.3), (2.4), we have

$$S_{ijk}{}^{l} = 2K_{ijk}{}^{l} + \delta^l_i \gamma_{jk} - \delta^l_j \gamma_{ik} + g_{jk}\gamma^l_i - g_{ik}\gamma^l_j \tag{2.7}$$

By using a contraction of the indices i and l of (2.7), we find

$$S_{jk} = 2K_{jk} + (n-2)\gamma_{jk} + g_{jk}\gamma^l_l \tag{2.8}$$

By a further contraction to (2.8) with g^{jk}, we arrive at

$$S = 2K + 2(n-1)\gamma^l_l$$

Thus we get the formula

$$\gamma^l_l = -\frac{2K - S}{2(n-1)}$$

Substituting the formula above into (2.8), we obtain

$$\gamma_{jk} = \frac{1}{n-2}\left(S_{jk} - 2K_{jk} - \frac{2K - S}{2(n-1)} g_{jk}\right)$$

Substituting this formula γ_{jk} into (2.7) and considering the formula (*), it is not hard to see that the formula (2.6) is tenable. □

Theorem 2.2. *If a Riemannian metric admits a mutual connection whose curvature tensor is constant curvature or vanishes, then the Riemannian metric is conformal flat in the Riemannian manifold (M, g).*

Proof. If a Riemannian metric admits a mutual connection $\overline{\nabla}$ whose curvature tensor is constant curvature or vanishes, then the Riemannian metric also admits a dual connection $\overset{*}{\overline{\nabla}}$ of the mutual connection $\overline{\nabla}$ whose curvature tensor is constant curvature or vanishes. Hence $\overline{C}_{ijk}{}^{l} = \overset{*}{\overline{C}}_{ijk}{}^{l} = 0$. Using (2.6), we obtain $\overset{o}{C}_{ijk}{}^{l} = 0$. Hence the Riemannian manifold (M, g) is of conformal flat. □

Lemma 2.3. *The tensor $\overset{*}{\overline{V}}_{ijk}{}^{l}$ is an invariant under the connection transformation $\overline{\nabla} \to \overset{*}{\overline{\nabla}}$, where*

$$\begin{aligned}
\overset{*}{\overline{V}}_{ijk}{}^{l} &= \overset{*}{\overline{R}}_{ijk}{}^{l} + \frac{1}{n}\Big(\delta^{l}_{j}\overset{*}{\overline{R}}_{ik} - \delta^{l}_{i}\overset{*}{\overline{R}}_{jk} + g_{jk}\overset{*}{\overline{R}}_{i}{}^{l} - g_{ik}\overset{*}{\overline{R}}_{j}{}^{l}\Big) \\
&+ \frac{2}{n(n^2-4)}\Big[\delta^{l}_{j}(\overset{*}{\overline{R}}_{ik} - \overset{*}{\overline{R}}_{ki}) - \delta^{l}_{i}(\overset{*}{\overline{R}}_{jk} - \overset{*}{\overline{R}}_{kj}) + g_{jk}(\overset{*}{\overline{R}}_{i}{}^{l} - \overset{*}{\overline{R}}_{\cdot i}{}^{l}) - g_{ik}(\overset{*}{\overline{R}}_{j}{}^{l} - \overset{*}{\overline{R}}_{\cdot j}{}^{l}) - n\delta^{l}_{k}(\overset{*}{\overline{R}}_{ij}\overset{*}{\overline{R}}_{ji})\Big] \qquad (2.9) \\
&+ \frac{1}{n^2-4}\Big(\delta^{l}_{j}\overset{*}{\overline{P}}_{ik} - \delta^{l}_{i}\overset{*}{\overline{P}}_{jk} + g_{jk}\overset{*}{\overline{P}}{}^{l} - g_{ik}\overset{*}{\overline{P}}_{j}{}^{l} - n\delta^{l}_{k}\overset{*}{\overline{P}}_{ji}\Big)
\end{aligned}$$

where $\overset{*}{\overline{R}}_{ijkl}g^{kj} \hat{=} \overset{*}{\overline{P}}_{ij}$ is called the quasi-Ricci (volume) curvature w.r.t. $\overset{*}{\overline{\nabla}}$, and $\overset{*}{\overline{R}}_{j}{}^{l} \hat{=} \overset{*}{\overline{R}}_{js}g^{sl}, \overset{*}{\overline{R}}_{\cdot j}{}^{l} \hat{=} \overset{*}{\overline{R}}_{sj}g^{sl}$.

Proof. By using a contraction of indices i and l of (2.5), we obtain

$$\overset{*}{\overline{R}}_{jk} = \overline{R}_{jk} + n\rho_{jk} - g_{jk}\rho^{l}_{l} + 2\beta_{kj} \qquad (2.10)$$

and by using the contraction of indices k and l of (2.5), we then get

$$\overset{*}{\overline{P}}_{ij} = \overline{P}_{ij} + 2(\rho_{ji} - \rho_{ij}) + 2n\beta_{ji} \qquad (2.11)$$

From (2.10) and (2.11), we find

$$\begin{aligned}
\beta_{jk} &= \frac{1}{2(n^2-4)}\Big\{[2(\overset{*}{\overline{R}}_{jk} - \overset{*}{\overline{R}}_{kj}) + n\overset{*}{\overline{P}}_{jk}] - [2(\overline{R}_{jk} - \overline{R}_{kj}) + n\overline{P}_{jk}]\Big\} \\
\rho_{jk} &= \frac{1}{n}\Big\{\overset{*}{\overline{R}}_{jk} - \overline{R}_{jk} + g_{jk}\rho^{l}_{l} + 2\beta_{jk}\Big\}
\end{aligned}$$

Substituting this formula into (2.5) and considering the formula (2.9), we obtain

$$\overset{*}{\overline{V}}_{ijk}{}^{l} = \overline{V}_{ijk}{}^{l}. \qquad (2.12)$$

This ends the proof of Lemma 2.3. □

Using Lemma 2.3, we have the following

Theorem 2.4. *In the Riemannian manifold (M, g), in order that the mutual connection $\overline{\nabla}$ of the first semi-symmetric metric recurrent connection ∇ is the conjugate symmetry, it is necessary and sufficient that the connection $\overline{\nabla}$ should be the conjugate Ricci symmetry and the conjugate volume symmetry. If the 1-form ω and π are of closed 1-forms, in order that the mutual connection $\overline{\nabla}$ should be the conjugate symmetry it is necessary and sufficient that the connection $\overline{\nabla}$ should be the conjugate Ricci symmetry.*

3. The mutual connection of the second semi-symmetric metric recurrent connection

In a Riemannian manifold (M, g), the second semi-symmetric metric recurrent connection ∇ is a connection satisfying the equation

$$\nabla_k g_{ij} = -\omega_i g_{jk} - \omega_j g_{ik}, T^{k}_{ij} = \pi_j \delta^{k}_{i} - \pi_i \delta^{k}_{j}$$

for a 1-form ω and π, and the connection coefficient Γ^{k}_{ij} is

$$\Gamma^{k}_{ij} = \{^{k}_{ij}\} + \pi_j \delta^{k}_{i} + g_{ij}(\omega^{k} - \pi^{k})$$

The mutual connection $\overline{\nabla}$ of the connection ∇ is a connection satisfying the equation

$$\overline{\nabla}_k g_{ij} = -(\omega_i - \pi_i)\delta^{l}_{j} - (\omega_j - \pi_j)\delta^{l}_{i} - 2\pi_k g_{ij}, \overline{T}{}^{k}_{ij} = \pi_i \delta^{k}_{j} - \pi_j \delta^{k}_{i}$$

and the connection coefficient Γ_{ij}^{k} is

$$\overline{\Gamma}\,^{k}_{ij} = \{^{k}_{ij}\} + \pi_j\delta^k_i + g_{ij}(\omega^k - \pi^k) \tag{3.1}$$

and the dual connection $\overset{*}{\overline{\nabla}}$ of the connection $\overline{\nabla}$ is a connection satisfying the equation

$$\overset{*}{\overline{\nabla}}_k g_{ij} = (\omega_i - \pi_i)\delta^k_j + (\omega_j - \pi_j)\delta^k_i + 2\pi_k g_{ij}, \overset{*}{\overline{T}}\,^{k}_{ij} = (\omega_i - 2\pi_i)\delta^k_j - (\omega_j - 2\pi_j)\delta^k_i$$

and the connection coefficient $\overset{*}{\overline{\Gamma}}_{ij}{}^{k}$ is Γ^k_{ij} is

$$\overset{*}{\overline{\Gamma}}\,^{k}_{ij} = \{^{k}_{ij}\} - \delta^k_i(\omega_j - \pi_j) - \delta^k_j\pi_i \tag{3.2}$$

From (3.1) and (3.2) we find that the curvature tensor of the connection $\overline{\nabla}$ and the curvature tensor of the connection $\overset{*}{\overline{\nabla}}$ are given respectively by

$$\overline{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + g_{jk}\overline{\tau}^l_i - g_{ik}\overline{\tau}^l_j + \delta^l_k\pi_{ij} \tag{3.3}$$

$$\overset{*}{\overline{R}}_{ijk}{}^{l} = K_{ijk}{}^{l} + \delta^l_i\overline{\tau}_{jk} - \delta^l_j\overline{\tau}_{ik} - \delta^l_k\pi_{ij} \tag{3.4}$$

and the relation between the curvatures is

$$\overset{*}{\overline{R}}_{ijk}{}^{l} = \overline{R}_{ijk}{}^{l} + \delta^l_i\overline{\tau}_{jk} - \delta^l_j\overline{\tau}_{ik} + g_{ik}\overline{\tau}^l_j - g_{jk}\overline{\tau}^l_i - 2\delta^l_k\pi_{ij} \tag{3.5}$$

where $\overline{\tau}_{ik} = \overset{o}{\nabla}_j(\omega_k - \pi_k) + (\omega_i - \pi_i)(\omega_k - \pi_k)$, and $\pi_{ij} = \overset{o}{\nabla}_i\pi_j - \overset{o}{\nabla}_j\pi_i$.

Lemma 3.1. *The tensor $\overline{U}_{ijk}{}^{l}$ is an invariant under the connection transformation $\overset{o}{\nabla} \to \overline{\nabla}$, where the tensor $\overline{U}_{ijk}{}^{l}$ is denoted by*

$$\begin{aligned}\overline{U}_{ijk}{}^{l} &= R_{ijk}{}^{l} + g_{jk}\overline{R}^l_i - g_{ik}\overline{R}^l_j + \frac{\overline{R}}{n-1}(\delta^l_j g_{ik} - \delta^l_i g_{jk}) + \frac{1}{n-2}(g_{jk}\overline{P}^l_i - g_{ik}\overline{P}^l_j - \delta^l_k\overline{P}_{ij}) \\ &+ \frac{1}{n-2}\Big[g_{jk}(\overline{R}^l_i - \overline{R}^l_{.i}) - g_{ik}(\overline{R}^l_j - \overline{R}^l_{.j}) - \delta^l_k(\overline{R}_{ij} - \overline{R}_{ji})\Big]\end{aligned} \tag{3.6}$$

(20) If 1-form ω and 1-form π are of closed 1-forms, then formula (3.6) is

$$\overline{U}_{ijk}{}^{l} = R_{ijk}{}^{l} + g_{jk}\overline{R}^l_i - g_{ik}\overline{R}^l_j + \frac{\overline{R}}{n-1}(\delta^l_j g_{ik} - \delta^l_i g_{jk}) \tag{3.7}$$

Proof. By using a contraction of the indices i and l of (3.3), we get

$$\overline{U}_{jk} = R_{jk} + g_{jk}\overline{\tau}^l_l - \overline{\tau}_{jk} + \pi_{jk} \tag{3.8}$$

And by contracting (3.8) with g^{jk}, we obtain

$$\overline{R} = K + (n-1)\overline{\tau}^l_l.$$

Thus we have

$$\overline{\tau}^l_l = \frac{1}{n-1}(\overline{R} - K).$$

On the other hand, by using contraction of the indices k and l of (3.3), there holds

$$\overline{P}_{ij} = \overset{o}{P}_{ij} + \overline{\tau}_{ij} - \overline{\tau}_{ij} + n\pi_{ij} \tag{3.9}$$

From (3.8) and (3.9) we find

$$\begin{aligned} \pi_{ij} &= \frac{1}{n-2}\Big[(\overline{R}_{ij} - \overline{R}_{ji} + \overline{P}_{ij}) - (K_{ij} - K_{ji} + \overset{o}{P}_{ij})\Big] \\ \overline{\tau}_{jk} &= K_{jk} - \overline{R}_{jk} + \frac{1}{n-1}(\overline{R} - K)g_{jk} + \pi_{jk} \end{aligned}$$

Substituting these formulas above into (3.3) and considering

$$\begin{aligned} \overset{o}{U}_{ijk}{}^{l} &= K_{ijk}{}^{l} + g_{jk}K_i^l - g_{ik}K_j^l + \frac{K}{n-1}(\delta_j^l g_{ik} - \delta_i^l g_{jk}) + \frac{1}{n-2}(g_{jk}\overset{o}{P}{}_i^l - g_{ik}\overset{o}{P}{}_j^l - \delta_k^l \overset{o}{P}_{ij}) \\ &+ \frac{1}{n-2}\Big[g_{jk}(K_i^l - K_{.i}^l) - g_{ik}(K_j^l - K_{.j}^l) - \delta_k^l(K_{ij} - K_{ji})\Big] \\ &= K_{ijk}{}^{l} + g_{jk}K_i^l - g_{ik}K_j^l + \frac{K}{n-1}(\delta_j^l g_{ik} - \delta_i^l g_{jk}) \end{aligned} \tag{3.10}$$

we obtain

$$\overline{U}_{ijk}{}^{l} = \overset{o}{U}_{ijk}{}^{l} \tag{3.11}$$

if 1-form π is a closed 1-form, then $\pi_{ji} = 0$. Hence the formula (3.6) becomes (3.7). This completes the proof of Lemma 3.1. □

Using lemma 3.1, we have the following

Theorem 3.2. *If a Riemannian metric admits a mutual connection $\overline{\nabla}$ of the second semi-symmetric metric recurrent connection ∇ whose curvature tensor is constant curvature or vanishes, then the Riemannian metric is of constant curvature(or projective flat).*

Proof. If $\overline{R}_{ijk}{}^{l} = k(\delta_i^l g_{jk} - \delta_j^l g_{ik})$ or $\overline{R}_{ijk}{}^{l} = 0$, then $\overline{U}_{ijk}{}^{l} = 0$. Hence from (3.11), there holds $\overset{o}{U}_{ijk}{}^{l} = 0$. By (3.10) we obtain

$$K_{ijk}{}^{l} = g_{jk}K_i^l - g_{ik}K_j^l + \frac{k}{n-1}(\delta_i^l g_{jk} - \delta_j^l g_{ik}) \tag{3.12}$$

By using $K_{ijkl} = -K_{ijlk}$, we arrive at

$$g_{jk}K_{il} - g_{ik}K_{jl} + g_{jl}K_{ik} - g_{il}K_{jk} = 0.$$

By contracting the expression with g^{il}, we find

$$K_{jl} = \frac{k}{n}g_{jl}.$$

Substituting these expressions into (3.12), then we have

$$K_{ijk}{}^{l} = \frac{(2n-1)k}{n(n-1)}(\delta_i^l g_{jk} - \delta_j^l g_{ik}).$$

Hence the Riemannian manifold $(M, g, \overset{o}{\nabla})$ is a manifold with constant curvature. □

4. 3. The mutual connection of the third semi-symmetric metric recurrent connection

In a Riemannian manifold (M, g), the third semi-symmetric metric recurrent connection ∇ is a connection satisfying the equation

$$\nabla_k g_{ij} = -2\omega_k g_{ij} - 2\omega_j g_{ik} - 2\omega_i g_{jk}, T^k_{ij} = \pi_j \delta^k_i - \pi_i \delta^k_j$$

For a 1-form ω and π, and the connection coefficient Γ^k_{ij} is

$$\Gamma^{\,k}_{\,ij} = \{^k_{ij}\} + \pi_j \delta^k_i - g_{ij}\pi^k + \omega_i \delta^k_j + \omega_j \delta^k_i$$

The mutual connection $\overline{\nabla}$ of the third semi-symmetric metric recurrent connection ∇ is a connection satisfying the equation

$$\overline{\nabla}_k g_{ij} = -2(\omega_k + \pi_k)g_{ij} - (2\omega_j - \pi_j)g_{ik} - (2\omega_i - \pi_i)g_{jk}, \overline{T}^{\,k}_{\,ij} = \pi_i \delta^k_i - \pi_j \delta^k_i$$

and the connection coefficient $\overline{\Gamma}^k_{ij}$ is

$$\overline{\Gamma}^{\,k}_{\,ij} = \{^k_{ij}\} + (\omega_i + \pi_i)\delta^k_j + \omega_j \delta^k_i + g_{ij}(\omega^k - \pi^k) \tag{4.1}$$

and the dual connection $\overset{*}{\overline{\nabla}}$ of the $\overline{\nabla}$ is a connection satisfying the equation

$$\overset{*}{\overline{\nabla}}_k g_{ij} = 2(\omega_k + \pi_k)g_{ij} + (2\omega_j - \pi_j)g_{ik} + (2\omega_i - \pi_i)g_{jk}, \overset{*}{\overline{T}}{}^{\,k}_{\,ij} = 2(\pi_j \delta^k_i - \omega_i \delta^k_j)$$

and $\overset{*}{\overline{\Gamma}}{}^k_{ij}$ is

$$\overset{*}{\overline{\Gamma}}{}^{\,k}_{\,ij} = \{^k_{ij}\} - (\omega_i + \pi_i)\delta^k_j - (\omega_j - \pi_j)\delta^k_i - g_{ij}\omega^k \tag{4.2}$$

From (4.1) and (4.2), we find the curvature tensor of $\overline{\nabla}$ and the curvature tensor of $\overset{*}{\overline{\nabla}}$ are given respectively by

$$\overline{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + \delta^l_j \omega_{ik} - \delta^l_i \omega_{jk} + g_{jk}\Phi^l_i - g_{ik}\Phi^l_j + \delta^l_k \overline{\beta}_{ij} \tag{4.3}$$

$$\overset{*}{\overline{R}}_{ijk}{}^{l} = K_{ijk}{}^{l} + \delta^l_i \omega_{jk} - \delta^l_j \omega_{ik} + g_{ik}\Phi^l_j - g_{jk}\Phi^l_i - \delta^l_k \overline{\beta}_{ij} \tag{4.4}$$

and the relation between these curvatures is

$$\overset{*}{\overline{R}}_{ijk}{}^{l} = \overline{R}_{ijk}{}^{l} + \delta^l_i \rho_{jk} - \delta^l_j \rho_{ik} + g_{ik}\rho^l_j - g_{jk}\rho^l_i - 2\delta^l_k \overline{\beta}_{ij} \tag{4.5}$$

where $\omega_{ik} = \overset{o}{\nabla}_i \omega_k - \frac{1}{2}\omega_i \omega_k$, $\Phi_{ik} = \overset{o}{\nabla}_i(\omega_k - \pi_k) + (\omega_i - \pi_i)(\omega_k - \pi_k) + g_{ik}\omega_l(\omega^l - \pi^l)$, $\overline{\beta}_{ik} = \overset{o}{\nabla}_i(\omega_k - \pi_k) - \overset{o}{\nabla}_k(\omega_i + \pi_i)$, $\rho_{ik} = \omega_{ik} + \Phi_{ik}$

The expressions (4.3), (4.4) and (4.5) are nor different in the forms from the expressions (2.3), (2.4) and (2.5). Hence the geometrical properties and conjugate symmetry condition of the mutual connection of the third semi-symmetric metric recurrent connection coincide with those of the first semi-symmetric metric recurrent connection.

5. Ackonowedement

The third author would like to thank Professors P. Zhao, U. C. De and H. Li for their encouragement and help!

References

[1] N. S. Agache and M. R. Chafle: *A semi-symmetric non-metric connection on a Riemannian manifold,* Indian J. Pure Appl. Math., 1992, **23**, 399-409

[2] U. C. De, Y. L. Han and P. B. Zhao: *A special type of semi- symmetric non-metric connection on a Riemannian manifold,* Facta Universitatis, Series, Math and Informatics, 2016, **31**(2): 529-541

[3] U. C. De and D. Kamilya: *Hypersurfaces of Kenmotsu manifolds Endowed with a quariter-symmetric non-metric connection,* Kuwait Journal of Sci. Eng, 2012, **39**(1), A, 43-56

[4] F. Fu, X. Yang and P. Zhao: *Geometrical and physical characteristics of a class conformal mapping*, J. Geom. Phys. **62**(2012), 1467-1479.

[5] S. Golab: *On semi-symmetric and quarter-symmetric linear connections*, Tensor. N. S. 1975, **29**, 249-254

[6] Y. Han, F. Fu and P. Zhao: *On a semi-symmetric metric connection in sub-Riemannian manifolds.* Tamkang Journal of Mathmatics, 2016, **47**(4): 373-384

[7] Y. Han, T. Ho and P. Zhao: *Some invariants of quarter-symmetric metric connections under the projective transformation*, Filomat, 2013, 27(4): 679-691

[8] H. A. Hayden: *Subspace of a space with torsion,* Proc, London Math. Soc. 1932, **34**, 27-50

[9] T. Ho: *On a semi-symmetric non-metric connection satisfying the Schur's theorem on a Riemannian manifold*, arXiv : 1212 4748 V1

[10] M. Z. Petrovic and M. S. Stankovic: *A note on F-planar mappings of manifolds with non-symmetric linear connection,* International Journal of Geometric Methods in Modern Physics, 2019, **16**(5), id. 1950078-205

[11] E. S. Stepanova: *Dual symmetric statistical manifold.* J. of Mathematical Sciences, 2007, **147**(1): 6507-6509

[12] I. Suhendro: *A New Semi-symmetric unified field theory of the classical fields of gravity and electromagnetism*, Progress in physics, 2007, **4**:, 47-62

[13] W. Tang, T. Ho, F. Fu and P. Zhao: *On a Quarter-symmetric Projective conformal connection*, IEJG, 2017, **10**, 2, 37-45

[14] W. Tang, T. Ho, K. Ri, F. Fu and P. Zhao: *On a Generalized quarter-symmetric metric recurrent connection,* Filomat, 2018, **32**:1, 207-215

[15] W. Tang, J. Yong, T. Ho, G. He and P. Zhao: *Geometries of manifolds equipped with a Ricci (projection-Ricci) quarter-symmetric connection*, Filomat, 2019, **33**(16): 5237-5248.

[16] M. M. Tripath: *A new connection in a Riemannian manifold*, IEJG, 2008, **1**, 1, 15-24

[17] N. O. Vesic and M. S. Stankovic: *Invariants of Special Second-Type Almost Geodesic Mappings of Generalized Riemannian Space,* Mediterranean Journal of Mathematics, 2018, **15**:60

[18] K. Yano: *On semi-symmetric metric connection*, Rev, Roum, Math. Pures. et Appl, 1971, **15**, 1679-1586

[19] K. Yano, J. Imai: *Quarter-symmetric connection and their curvature tensors*, Tensor. N. S, 1982, **38**, 13-18

[20] D. Zhao, C. Jen and T. Ho: *On a Ricci quarter-symmetric metric recurrent connection and a projective Ricci quarter-symmetric metric recurrent connection in a Riemannian manifold,* Filomat, 2020, **34**(3):795-806

[21] P. Zhao : *The invariant of projective transformation of semi-symmetric metric recurrent connections and curvature tensor expression,* J of Engineering Math, 2000, **17**(2): 105-108

[22] P. Zhao, H. Song and X. Yang: *Some invariant properties of semi-symmetric recurrent connections and curvature tensor expression,* Chin. Quart. J. of Math, 2004, **14**(4): 355-361