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Two Iterative Algorithms for Solving the Split Common Fixed Point Problems

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Abstract. The purpose of this paper is to study the split common fixed point problems (SCFP) involved in nonexpansive mappings in real Hilbert space. We introduce two iterative algorithms for finding a solution of the SCFP involved in nonexpansive mappings, where one is a Mann-type iterative algorithm and another is a Halpern-type iterative algorithm.

1. Introduction

In 1994, the split feasibility problem (SFP) was first presented by Censor and Elfving [7] for modeling inverse problems, which arise from phase retrievals and in medical image reconstruction ([1–3]). Recently, the SFP had also been applied to study intensity modulated radiation therapy when one attempted to describe physical dose constraints and equivalent uniform dose constraints within a single model([6]).

To begin with, let us recall that the SFP is to find a point

$$x^* \in C \text{ such that } Ax^* \in Q, \tag{1}$$

where C is a close convex subset of a Hilbert space H_1, Q is a close convex subset of a Hilbert space H_2 and A: $H_1 \to H_2$ is a bounded linear operator. To solve problem (1), Byrne [2] proposed his CQ algorithm, which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(I + \gamma A^*(P_Q - I)A)x_n, \quad n \geq 0, \tag{2}$$

where $\gamma \in (0, 2/\lambda)$ and λ is the spectral radius of the operator A^*A.

Many authors have also made a continuation of the study on the CQ algorithm and its variant form, refer to [11, 14, 17, 18, 22, 27, 30].

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On the other hand, the split common fixed point problem (SCFP) is a generalization of the SFP. The purpose of this paper is to study the SCFP, that is, finding a point x^* with the property

$$x^* \in Fix(T) \text{ and } Ax^* \in Fix(S). \tag{3}$$

This problem was first introduced by Censor and Segal [8]. We use Γ_0 to denote the set of the solutions of (3), that is, $\Gamma_0 = \{x^* : x^* \in Fix(T), Ax^* \in Fix(S)\}$. For solving (3), they suggested an algorithm, which generates a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = T(x_n + \gamma A^*(S - I)Ax_n), \quad n \geq 0. \tag{4}$$

Note that (4) is more general than (1). Some further generation of this algorithm were studied by [4, 5, 12–29, 31–34]and others. One of them, Moudafi extended (4) to the following relaxed algorithm:

$$x_{n+1} = U_{\alpha_n}(x_n + \gamma A^*(T_\beta - I)Ax_n), \quad n \geq 0,$$

where $\beta \in (0, 1)$, $\alpha_n \in (0, 1)$ are relaxation parameters. Using the definition of relaxed operators, observe that this algorithm takes the following form:

$$\begin{cases} y_n = x_n + \gamma\beta A^*(T - I)Ax_n, \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n U y_n, \quad n \geq 0, \end{cases}$$

Recently, Chen et. al. [10] presented the following iterative method to have a common fixed point of three nonexpansive mappings.

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ z_n = \gamma_n x_n + (1 - \gamma_n)Sx_n, \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n)Rz_n, \quad n \geq 0, \end{cases}$$

where $\beta_n \in (0, 1)$, $\gamma_n \in (0, 1)$, $\alpha_n \in (0, 1)$.

Motivated and inspired by the above works and the research going on in these direction, the purpose here is to find a solution of the SCFP involved in nonexpansive mappings by proposed the following algorithm:

$$\begin{cases} y_n = x_n + \gamma\delta A^*(S - I)Ax_n, \\ z_n = (1 - \beta_n)y_n + \beta_n T y_n, \\ w_n = (1 - \gamma_n)y_n + \gamma_n R y_n, \\ x_{n+1} = (1 - \alpha_n)z_n + \alpha_n U w_n, \quad n \geq 0, \end{cases} \tag{5}$$

The weak convergence result of the algorithm will be established. Our results improve and develop previously discussed feasibility problem and related algorithm.

On the other hand, we also suggest the following Halpern-type iterative algorithm to solve the SCFP (3). Define $\{x_n\}$ in the following iteration process:

$$\begin{cases} y_n = Tx_n + (1 - \beta_n)\delta A^*(S - I)ATx_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad n \geq 0, \end{cases} \tag{6}$$

where u is an arbitrary fixed element in H_1. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ defined by (6) converges to a split common fixed point x^* of Γ_0. Strong convergence theorem will be proven.

2. Preliminaries

In this section, we collect definitions and lemmas which will be used in the sequel.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Assume that C be a nonempty and closed convex subset of H. Denote $F(T)$ the set of fixed points of a mapping $T : C \to C$, that is, $Fix(T) = \{x \in C : x = Tx\}$. We shall use the following notation: 1. $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x; 2. $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x; 3. $\omega_w(x_n) = \{x : \exists x_{n_i} \to x \text{ } weakly\}$ denote the weak w-limit set of $\{x_n\}$.

Definition 2.1. *$T : C \to C$ is said to be nonexpansive mapping if*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is well known that in real Hilbert space H, the following statements hold

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \tag{7}$$

for all $x, y \in H$ and $t \in [0, 1]$, and identical equation

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{8}$$

for all $x, y \in H$. It follows that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{9}$$

for all $x, y \in H$.

Definition 2.2. *A mapping T is said to be demiclosed if for any sequence $\{x_n\}$ which converges weakly to x, and if the sequence $\{Tx_n\}$ converges strongly to z, then $Tx=z$.*

In what follows, only the particular case of demiclosedness at zero will be used, which is the particular case when $z = 0$. It is true that if T be a nonexpansive mapping, then $I - T$ is demiclosed.

Lemma 2.3 ([30]). *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H such that there exists a nonempty set $\mathbb{W} \in H$ satisfying:*
(1) for every $w \in \mathbb{W}$, $\lim_{n\to\infty} \|x_n - w\|$ exists; (2) each weak-cluster point of the sequence $\{x_n\}$ is in $\mathbb{W}$.
Then there exists $w^* \in \mathbb{W}$ such that $\{x_n\}$ weakly converges to w^*.

Lemma 2.4 ([31]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\sigma_n\}$ are such that
(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$; (2) either $\limsup_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$ or $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.
Then $\{a_n\}$ converges to zero.

To attain strong convergence result, we need to use the following lemma.

Lemma 2.5 ([29]). *Let $\{u_n\}$ be a sequence of real numbers. Assume $\{u_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \leq u_{n_k+1}$ for all $k \geq 0$. For every $n \geq N_0$, define an $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \leq n : u_{n_i} < u_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq N_0$,

$$\max\{u_{\tau(n)}, u_n\} \leq u_{\tau(n)+1}.$$

3. Weak convergence to split common fixed point of nonexpansive mappings

In this part, we will focus our attention on the following four-operator SCFP:

$$\text{Find } x^* \in Fix(T) \cap Fix(R) \cap Fix(U) \text{ such that } Ax^* \in Fix(S), \tag{10}$$

to solve (10), we suggested algorithm (5). We use Γ to denote the set of the solutions of (10).

Now, we are in a position to prove our convergence results.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_2 \to H_2$ and $T, R, U: H_1 \to H_1$ are four nonexpansive mappings. Assume that $\Gamma \neq \emptyset$. Given $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset$ (0,1) and constants γ, δ satisfying the following conditions:*
(i) $\liminf_{n\to\infty} \beta_n(1-\alpha_n)(1-\beta_n) > 0$; $\liminf_{n\to\infty} \alpha_n\gamma_n(1-\gamma_n) > 0$; $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $0 < \gamma < 1$ and $0 < \delta < \frac{1}{\gamma\lambda}$, with λ being the spectral radius of the operator AA^*.
Define a sequence $\{x_n\}$ by the algorithm (5), then $\{x_n\}$ converges weakly to a $p \in \Gamma$.

Proof. First, we observe that $\{x_n\}$ is bounded, if we take an arbitrary fixed point x^* of Γ. Then we get $Ax^* \in Fix(S)$ and $x^* \in Fix(T) \cap Fix(R) \cap Fix(U)$, noting that S is nonexpansive mapping and using (8), we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|x_n - x^*\|^2 + 2\gamma\delta\langle Ax_n - Ax^*, (S-I)Ax_n\rangle \\
&\quad + \gamma^2\delta^2\langle AA^*(S-I)Ax_n, (S-I)Ax_n\rangle \\
&\leq \|x_n - x^*\|^2 + 2\gamma\delta\langle Ax_n - Ax^*, (S-I)Ax_n\rangle + \gamma^2\delta^2\lambda\|(S-I)Ax_n\|^2 \\
&= \|x_n - x^*\|^2 + \gamma^2\delta^2\lambda\|SAx_n - Ax_n\|^2 \\
&\quad + \gamma\delta(\|SAx_n - Ax^*\|^2 - \|Ax_n - Ax^*\|^2 - \|SAx_n - Ax_n\|^2) \\
&\leq \|x_n - x^*\|^2 - \gamma\delta(1-\gamma\delta\lambda)\|SAx_n - Ax_n\|^2.
\end{aligned} \tag{11}$$

From (5), (7) and (11), we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= (1-\alpha_n)\|z_n - x^*\|^2 + \alpha_n\|Uw_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 \\
&\leq (1-\alpha_n)\|(1-\beta_n)(y_n - x^*) + \beta_n(Ty_n - x^*)\|^2 \\
&\quad + \alpha_n\|(1-\gamma_n)(y_n - x^*) + \gamma_n(Ry_n - x^*)\|^2 - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 \\
&\leq \alpha_n(1-\gamma_n)\|y_n - x^*\|^2 + \alpha_n\gamma_n\|y_n - x^*\|^2 - \alpha_n\gamma_n(1-\gamma_n)\|Ry_n - y_n\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)\|y_n - x^*\|^2 + (1-\alpha_n)\beta_n\|y_n - x^*\|^2 \\
&\quad - \beta_n(1-\alpha_n)(1-\beta_n)\|Ty_n - y_n\|^2 - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 \\
&= \|y_n - x^*\|^2 - \beta_n(1-\alpha_n)(1-\beta_n)\|Ty_n - y_n\|^2 - \alpha_n\gamma_n(1-\gamma_n)\|Ry_n - y_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \gamma\delta(1-\gamma\delta\lambda)\|SAx_n - Ax_n\|^2 - \beta_n(1-\alpha_n)(1-\beta_n)\|Ty_n - y_n\|^2 \\
&\quad - \alpha_n\gamma_n(1-\gamma_n)\|Ry_n - y_n\|^2 - \alpha_n(1-\alpha_n)\|Uw_n - z_n\|^2.
\end{aligned} \tag{12}$$

We deduce immediately by conditions (i) and (ii) that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|,$$

for every $x^* \in \Gamma$, and for all $n \geq 0$. Thus, $\{x_n\}$ generated by algorithm (5) is the Féjer-monotone with respect to Γ. So, we obtain $\lim_{n\to\infty} \|x_n - x^*\|$ exists immediately, this implies that $\{x_n\}$ is bounded, the sequence $\{\|x_n - x^*\|\}$ is monotonically decreasing. From (12), we have

$$\lim_{n\to\infty} \|SAx_n - Ax_n\| = 0. \tag{13}$$

Since $\{x_n\}$ is bounded, $\omega_w(x_n) \neq \emptyset$. Denoting by p a weak-cluster point of $\{x_n\}$, we have $w - \lim_{i\to\infty} x_{n_i} = p$. Noting that demiclosedness of $I - S$ at 0, from (13), we obtain

$$SAp = Ap, \tag{14}$$

from which it follows that $Ap \in Fix(S)$. By setting $y_n = x_n + \gamma\delta A^*(S-I)Ax_n$, it follows that $w - \lim_{i\to\infty} y_{n_i} = p$. Again, from (12), we have

$$\lim_{n\to\infty} \|Ty_n - y_n\| = 0 \text{ and } \lim_{n\to\infty} \|Ry_n - y_n\| = 0, \tag{15}$$

by the demiclosedness of $I - T$, $I - R$ at 0 and the weak convergence of $\{y_{n_i}\}$ to p yields

$$Tp = p \text{ and } Rp = p. \tag{16}$$

Next, we prove that $\lim_{n\to\infty} \|Uw_n - w_n\| = 0$.
From (12), we deduce

$$\lim_{n\to\infty} \|Uw_n - z_n\| = 0. \tag{17}$$

By (15), we obtain that $\lim_{n\to\infty} \|z_n - y_n\| = 0$ and $\lim_{n\to\infty} \|w_n - y_n\| = 0$. Thus, we get

$$\lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{18}$$

By (17) and (18), we get $\lim_{n\to\infty} \|Uw_n - w_n\| = 0$. The key limit above combined with $\lim_{n\to\infty} \|w_n - y_n\| = 0$, the demiclosedness of $I - U$ at 0 and the weak convergence of $\{y_{n_i}\}$ to p yield $Up = p$, combined with (14) and (16) we get $p \in \Gamma$. Since there is no more than one weak-cluster point, the weak convergence of the whole sequence $\{x_n\}$ follows by applying Lemma 2.3 with $\mathbb{W} = \Gamma$. This completes the proof. □

Chen et. al. [9], proposed a sequence $\{x_n\}$ generate by the following iterative method:

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ z_n = \gamma_n x_n + (1-\gamma_n)Sx_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)z_n, \quad n \geq 0, \end{cases}$$

where T, S are nonexpansive mappings, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in (0,1).

Inspired by above algorithm, we suggest and analyze the corollary 3.2 as below. We denote $\Gamma_1 = \{x^* : x^* \in Fix(T) \cap Fix(R), Ax^* \in Fix(S)\}$.

By the careful analysis of the proof of Theorem 3.1, we can obtain the following result. Because its proof is much simpler than that of Theorem 3.1, we omit its proof.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_2 \to H_2$ and T, R: $H_1 \to H_1$ are three nonexpansive mappings. Assume that $\Gamma_1 \neq \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\} \subset$ (0,1) and constants γ, δ satisfying the following conditions:*

(1) $\liminf_{n\to\infty} \beta_n(1-\alpha_n)(1-\beta_n) > 0$; $\liminf_{n\to\infty} \alpha_n\gamma_n(1-\gamma_n) > 0$;

(2) $0 < \gamma < 1$ *and* $0 < \delta < \frac{1}{\gamma\lambda}$, *with* λ *being the spectral radius of the operator* AA^*.

Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} y_n = x_n + \gamma\delta A^*(S-I)Ax_n, \\ z_n = (1-\beta_n)y_n + \beta_n Ty_n, \\ w_n = (1-\gamma_n)y_n + \gamma_n Ry_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n w_n, \quad n \geq 0, \end{cases}$$

then $\{x_n\}$ converges weakly to a split common fixed point $p \in \Gamma_1$.

Next, applying Theorem 3.1, we get the result as below.

Corollary 3.3. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_2 \to H_2$ and T, R: $H_1 \to H_1$ are three nonexpansive mappings. Assume that $\Gamma_1 \neq \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and constants γ, δ satisfying the following conditions:*
(1) $\liminf_{n\to\infty} \alpha_n\beta_n(1-\beta_n) > 0$; $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(2) $0 < \gamma < 1$ *and* $0 < \delta < \frac{1}{\gamma\lambda}$, *with λ being the spectral radius of the operator AA^*.*
Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} y_n = x_n + \gamma\delta A^*(S-I)Ax_n, \\ z_n = (1-\beta_n)y_n + \beta_n T y_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n R z_n, \quad n \geq 0, \end{cases}$$

then $\{x_n\}$ converges weakly to a split common fixed point $p \in \Gamma_1$.

Now, we suggest a new split common fixed point problem, that is, finding a point x^* with the property

$$x^* \in Fix(R) \text{ and } Ax^* \in Fix(T) \cap Fix(S). \tag{19}$$

We use Γ_2 to denote the set of the solutions of (19).

Theorem 3.4. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let S, $T : H_2 \to H_2$ and R: $H_1 \to H_1$ are three nonexpansive mappings. Assume that $\Gamma_2 \neq \emptyset$. The following conditions hold for the sequence $\{\alpha_n\} \subset (0,1)$ and constants γ, δ:*
(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $0 < \gamma, \delta < \frac{1}{\lambda}$, *with λ being the spectral radius of the operator AA^*.*
Then a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} y_n = x_n + \gamma A^*(T-I)Ax_n, \\ z_n = x_n + \delta A^*(S-I)Ax_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n R z_n, \quad n \geq 0, \end{cases} \tag{20}$$

converges weakly to a split common fixed point $p \in \Gamma_2$.

Proof. As proved in Theorem 3.1, only a sketch of the proof is given here.

First, we observe that $\{x_n\}$ is bounded, taking x^* of Γ_2, i.e., $Ax^* \in Fix(T) \cap Fix(S)$ and $x^* \in Fix(R)$, noting that T is nonexpansive mapping and (8), using the same argument in Theorem 3.1, we have

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \gamma(1-\gamma\lambda)\|TAx_n - Ax_n\|^2. \tag{21}$$

Similarly, we get

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \delta(1-\delta\lambda)\|SAx_n - Ax_n\|^2. \tag{22}$$

From (20), (7), (21) and (22), we obtain that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1-\alpha_n)\|y_n - x^*\|^2 + \alpha_n\|z_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|Rz_n - y_n\|^2 \\ &\leq (1-\alpha_n)(\|x_n - x^*\|^2 - \gamma(1-\gamma\lambda)\|TAx_n - Ax_n\|^2) \\ &\quad + \alpha_n(\|x_n - x^*\|^2 - \delta(1-\delta\lambda)\|SAx_n - Ax_n\|^2) - \alpha_n(1-\alpha_n)\|Rz_n - y_n\|^2 \\ &\leq \|x_n - x^*\|^2 - (1-\alpha_n)\gamma(1-\gamma\lambda)\|TAx_n - Ax_n\|^2 - \alpha_n(1-\alpha_n)\|Rz_n - y_n\|^2 \\ &\quad - \alpha_n\delta(1-\delta\lambda)\|SAx_n - Ax_n\|^2. \end{aligned} \tag{23}$$

We deduce immediately by conditions (i) and (ii) that $\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$, for every $x^* \in \Gamma_2$, and for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ generated by algorithm (3.11) is the Féjer-monotone with respect to Γ_2. So, we obtain $\lim_{n\to\infty}\|x_n - x^*\|$ exists immediately, this implies that $\{x_n\}$ is bounded, the sequence $\{\|x_n - x^*\|\}$ is monotonically decreasing.

From (23), we have

$$\lim_{n\to\infty}\|SAx_n - Ax_n\| = 0 \text{ and } \lim_{n\to\infty}\|TAx_n - Ax_n\| = 0.$$

On the lines similar to Theorem 3.1, we get $p \in \Gamma$. Since there is no more than one weak-cluster point, the weak convergence of the whole sequence $\{x_n\}$ follows by applying Lemma 2.3 with $\mathbb{W} = \Gamma_2$. This completes the proof. □

We suggest an algorithm in the following Corollary to solve (3).

Corollary 3.5. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_2 \to H_2$ and T: $H_1 \to H_1$ are two nonexpansive mappings. Assume that $\Gamma_0 \neq \emptyset$. The following conditions hold for the sequences $\{\alpha_n\}\subset(0,1)$ and constant γ:*

(1) $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$;

(2) $0 < \gamma < \frac{1}{\lambda}$, *with λ being the spectral radius of the operator AA^*.*

Then a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} y_n = x_n + \gamma A^*(S-I)Ax_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 0, \end{cases}$$

converges weakly to a split common fixed point $p \in \Gamma_0$.

Finally, we propose the following split common fixed point problem, that is, finding a point x^* with the property

$$x^* \in H_1 \text{ and } Ax^* \in Fix(T) \cap Fix(S). \tag{24}$$

We use Γ_3 to denote the set of the solutions of (24).

We introduce the following algorithm, which is take $R = I$ in the algorithm (20) in Theorem 3.4.

Corollary 3.6. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let T, $S : H_2 \to H_2$ are two nonexpansive mappings. Assume that $\Gamma_3 \neq \emptyset$. Given $\{\alpha_n\}$ be sequence in (0,1) and constants γ, δ satisfying $0 < \gamma, \delta < \frac{1}{\lambda}$, with λ being the spectral radius of the operator AA^*.*

Then a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} y_n = x_n + \gamma A^*(T-I)Ax_n, \\ z_n = x_n + \delta A^*(S-I)Ax_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n z_n, \quad n \geq 0, \end{cases}$$

converges weakly to a split common fixed point $p \in \Gamma_3$.

Proof. Take $x^* \in \Gamma_3$, i.e., $Ax^* \in Fix(T)\cap Fix(S)$, using the same argument in Theorem 3.4, we have the bounded of $\{x_n\}$, $\omega_w(x_n) \neq \emptyset$, such that $w - \lim_{i\to\infty} x_{n_i} = p$. By setting $\{y_n\}$, $\{z_n\}$ and combined with the demiclosedness of $I - T$, $I - S$ at 0, all the conditions in this corollary are satisfied, the conclusion of Corollary 3.6 can be obtained from Theorem 3.4 immediately. □

We denote $\Gamma_4 = \{x^* : x^* \in H_1, Ax^* \in Fix(S)\}$.

Corollary 3.7. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_2 \to H_2$ be a nonexpansive mapping. Assume that $\Gamma_4 \neq \emptyset$. Given $\{\alpha_n\}$ be sequence in (0,1) and constants γ, satisfying $0 < \gamma < \frac{1}{\lambda}$, with λ being the spectral radius of the operator AA^*.*

Then a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} y_n = x_n + \gamma A^*(S-I)Ax_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \quad n \geq 0, \end{cases}$$

converges weakly to a split common fixed point $p \in \Gamma_4$.

4. Strong convergence to split common fixed point of nonexpansive mappings

In this section, we suggest a Halpern-type iterative algorithm to solve the SCFP (3) involved in nonexpansive mappings, and prove its strong convergence.

Theorem 4.1. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_2 \to H_2$ and $T : H_1 \to H_1$ are two nonexpansive mappings. Assume that $\Gamma_0 \neq \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and constant δ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}| < \infty$,

(iii) $0 < \beta_n < b < 1$, $0 < \delta < \frac{1-b}{\lambda}$, with λ being the spectral radius of the operator AA^*.

For fixed $u, x_0 \in H_1$ arbitrarily, define a sequence $\{x_n\}$ by algorithm (6), then $\{x_n\}$ converges strongly to x^*, given by $x^* = P_{\Gamma_0}(u)$.

Proof. First, we observe that $\{x_n\}$ is bounded, if we take an arbitrary fixed point x^* of Γ_0. Then we get $Ax^* \in Fix(S)$ and $x^* \in Fix(T)$, noting that S is nonexpansive mapping, using (8) and condition (iii), we have

$$\begin{aligned}\|y_n - x^*\|^2 &\leq \|Tx_n - x^*\|^2 + 2(1-\beta_n)\delta\langle Tx_n - Tx^*, A^*(S-I)ATx_n\rangle + \delta^2\|A^*(S-I)ATx_n\|^2 \\ &\leq \|x_n - x^*\|^2 + 2(1-\beta_n)\delta\langle ATx_n - ATx^*, (S-I)ATx_n\rangle + \lambda\delta^2\|(S-I)ATx_n\|^2 \\ &= \|x_n - x^*\|^2 + \lambda\delta^2\|SATx_n - ATx_n\|^2 \\ &\quad + (1-\beta_n)\delta(\|SATx_n - ATx^*\|^2 - \|ATx_n - ATx^*\|^2 - \|SATx_n - ATx_n\|^2) \\ &= \|x_n - x^*\|^2 - \delta((1-\beta_n) - \lambda\delta)\|SATx_n - ATx_n\|^2 \\ &\leq \|x_n - x^*\|^2.\end{aligned} \tag{25}$$

From (6) and (25), we obtain that

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|y_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_n - x^*\|\}.\end{aligned}$$

The boundedness of the sequence $\{x_n\}$ yields by the induction. So are $\{y_n\}$ and $\{Tx_n\}$. These boundedness play a prominent role in proving the strong convergence theorem.

From (6), (7) and (25), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|y_n - x^*\|^2 \\ &\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \delta((1-b) - \lambda\delta)\|SATx_n - ATx_n\|^2.\end{aligned}$$

It follows that

$$\delta(1 - b - \lambda\delta)\|SATx_n - ATx_n\|^2 \leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{26}$$

Next, we consider two possible cases.

Case 1. Assume there exists some integer $m > 0$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n > m$, in this case, we know that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. From (26) and condition (i), (iii), we deduce

$$\lim_{n\to\infty} \|SATx_n - ATx_n\| = 0. \tag{27}$$

As a result, by setting $y_n = Tx_n + (1-\beta_n)\delta A^*(S-I)ATx_n$, we obtain that

$$\lim_{n\to\infty} \|y_n - Tx_n\| = 0. \tag{28}$$

Since $\{y_n\}$ is bounded, from (6) and condition (i), we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{29}$$

We next show that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{30}$$

As a matter of fact $\|x_n - Tx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - Tx_n\|$, so, it suffices to show that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0, \tag{31}$$

if (31) holds, then (30) holds from (28) and (29).

Before calculate $x_n - x_{n+1}$, letting $z_n = Tx_n + \delta A^*(S-I)ATx_n$, on the lines similar to (11), we get the boundedness of the sequence $\{z_n\}$.

For every integer $n \ge 1$, we have

$$\begin{aligned}
\|z_n - z_{n-1}\|^2 &= \|Tx_n - Tx_{n-1}\|^2 + \delta^2\|A^*(S-I)ATx_n - A^*(S-I)ATx_{n-1}\|^2 \\
&\quad + 2\delta\langle Tx_n - Tx_{n-1}, A^*(S-I)ATx_n - A^*(S-I)ATx_{n-1}\rangle \\
&\le \|x_n - x_{n-1}\|^2 + \lambda\delta^2\|(S-I)ATx_n - (S-I)ATx_{n-1}\|^2 \\
&\quad + 2\delta\langle ATx_n - ATx_{n-1}, (S-I)ATx_n - (S-I)ATx_{n-1}\rangle \\
&= \|x_n - x_{n-1}\|^2 + \lambda\delta^2\|(S-I)ATx_n - (S-I)ATx_{n-1}\|^2 \\
&\quad + \delta(\|SATx_n - SATx_{n-1}\|^2 - \|ATx_n - ATx_{n-1}\|^2 - \|(S-I)ATx_n - (S-I)ATx_{n-1}\|^2) \\
&\le \|x_n - x_{n-1}\|^2 - \delta(1-\lambda\delta)\|(S-I)ATx_n - (S-I)ATx_{n-1}\|^2 \\
&\le \|x_n - x_{n-1}\|^2.
\end{aligned} \tag{32}$$

In order to prove (30), we now calculate $x_{n+1} - x_n$ for every integer $n \ge 1$,

$$\begin{aligned}
x_{n+1} - x_n &= (\alpha_n u + (1-\alpha_n)((1-\beta_n)z_n + \beta_n Tx_n)) - (\alpha_{n-1}u + (1-\alpha_{n-1})((1-\beta_{n-1})z_{n-1} + \beta_{n-1}Tx_{n-1})) \\
&= (\alpha_n - \alpha_{n-1})u + (1-\alpha_n)(1-\beta_n)(z_n - z_{n-1}) + (1-\alpha_n)(1-\beta_n)z_{n-1} - (1-\alpha_{n-1})\beta_{n-1}Tx_{n-1} \\
&\quad + (1-\alpha_n)\beta_n(Tx_n - Tx_{n-1}) + (1-\alpha_n)\beta_n Tx_{n-1} - (1-\alpha_{n-1})(1-\beta_{n-1})z_{n-1} \\
&= (\alpha_n - \alpha_{n-1})u + (1-\alpha_n)(1-\beta_n)(z_n - z_{n-1}) + (1-\alpha_n)(1-\beta_n)Tx_{n-1} \\
&\quad + (1-\alpha_n)(1-\beta_n)(z_{n-1} - Tx_{n-1}) - (1-\alpha_{n-1})(1-\beta_{n-1})(z_{n-1} - Tx_{n-1}) + (1-\alpha_n)\beta_n Tx_{n-1} \\
&\quad - (1-\alpha_{n-1})(1-\beta_{n-1})Tx_{n-1} + (1-\alpha_n)\beta_n(Tx_n - Tx_{n-1}) - (1-\alpha_{n-1})\beta_{n-1}Tx_{n-1} \\
&= (\alpha_n - \alpha_{n-1})u + (1-\alpha_n)Tx_{n-1} - (1-\alpha_{n-1})Tx_{n-1} + (1-\alpha_n)(1-\beta_n)(z_n - z_{n-1}) \\
&\quad + ((1-\alpha_n)(1-\beta_n) - (1-\alpha_{n-1})(1-\beta_{n-1}))(z_{n-1} - Tx_{n-1}) + (1-\alpha_n)\beta_n(Tx_n - Tx_{n-1}) \\
&= (\alpha_n - \alpha_{n-1})(u - Tx_{n-1}) + (1-\alpha_n)(1-\beta_n)(z_n - z_{n-1}) \\
&\quad + ((\beta_{n-1} - \beta_n)(1-\alpha_n) + (\alpha_{n-1} - \alpha_n)(1-\beta_{n-1}))(z_{n-1} - Tx_{n-1}) + (1-\alpha_n)\beta_n(Tx_n - Tx_{n-1}).
\end{aligned}$$

It follows from (32) that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le (1-\alpha_n)\beta_n\|Tx_n - Tx_{n-1}\| + (1-\alpha_n)(1-\beta_n)\|z_n - z_{n-1}\| \\
&\quad + (|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|)\|z_{n-1} - Tx_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u - Tx_{n-1}\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + M(|\beta_n - \beta_{n-1}| + 2|\alpha_n - \alpha_{n-1}|),
\end{aligned} \tag{33}$$

where M is a constant such that $M = \max\{\|z_{n-1} - Tx_{n-1}\|, \|u - Tx_{n-1}\|\}$ for all $n \ge 1$. By assumptions (i) and (ii), we have $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty}(|\beta_n - \beta_{n-1}| + 2|\alpha_n - \alpha_{n-1}|) < \infty$. Hence, Lemma 2.4 is applicable to (33) and we obtain $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$.

Since the sequence $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x$. Consequently, we derive from (27) and (30) that

$$Tx_{n_i} \rightharpoonup x,\ ATx_{n_i} \rightharpoonup Ax.$$

Applying demiclosedness principle, we deduce

$$x \in Fix(T) \text{ and } Ax \in Fix(S).$$

That is to say, $x \in \Gamma_0$.
Therefore,

$$\limsup_{n\to\infty}\langle u - x^*, x_{n+1} - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{n_i} - x^*\rangle = \langle u - x^*, x - x^*\rangle \leq 0. \tag{34}$$

Using (9), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1-\alpha_n)\|y_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle. \end{aligned} \tag{35}$$

Applying Lemma 2.4 and (34) and (35), we deduce $x_n \to x^*$, i.e., sequence $\{x_n\}$ converges strongly to a split common fixed point x^*, given by $x^* = P_{\Gamma_0}(u)$.

Case 2. Assume there exists an integers n_0, such that

$$\|x_{n_0} - x^*\| \leq \|x_{n_0+1} - x^*\|.$$

Set $u_n = \{\|x_n - x^*\|\}$, then we have

$$u_{n_0} \leq u_{n_0+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \geq n_0$ as follows:

$$\tau(n) = \max\{l \geq 1 : n_0 \leq l \leq n, u_l \leq u_{l+1}\}.$$

It is clear that $\tau(n)$ is non-decresing sequence satisfying

$$\lim_{n\to\infty}\tau(n) = \infty,$$

and

$$u_{\tau(n)} \leq u_{\tau(n)+1},$$

for all $n \geq n_0$.
By a similar argument to that of Case 1, we can obtain that

$$\lim_{n\to\infty}\|SATx_{\tau(n)} - ATx_{\tau(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|x_{\tau(n)} - Tx_{\tau(n)}\| = 0.$$

This implies that

$$\omega(x_{\tau(n)}) \subset \Gamma_0.$$

Thus, we obtain

$$\limsup_{n\to\infty}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle \leq 0. \tag{36}$$

Since $u_{\tau(n)} \leq u_{\tau(n)+1}$, we have from (35) that

$$u^2_{\tau(n)} \leq u^2_{\tau(n)+1} \leq (1-\alpha_{\tau(n)})u^2_{\tau(n)} + 2\alpha_{\tau(n)}\langle u - x^*, x_{\tau(n)+1} - x^*\rangle. \tag{37}$$

It follows that

$$u^2_{\tau(n)} \leq 2\langle u - x^*, x_{\tau(n)+1} - x^*\rangle. \tag{38}$$

Combining (36) and (38), we have $\limsup_{n\to\infty} u_{\tau(n)} \le 0$, and hence

$$\lim_{n\to\infty} u_{\tau(n)} = 0. \tag{39}$$

By (37), we obtain

$$\limsup_{n\to\infty} u^2_{\tau(n)+1} \le \limsup_{n\to\infty} u^2_{\tau(n)}.$$

This together with (39) implies that

$$\lim_{n\to\infty} u_{\tau(n)+1} = 0.$$

Applying Lemma 2.5 to get

$$0 \le u_n \le \max\{u_{\tau(n)}, u_{\tau(n)+1}\}.$$

therefore, $u_n \to 0$. That is, $x_n \to x^*$, i.e., sequence $\{x_n\}$ converges strongly to a split common fixed point x^*, given by $x^* = P_{\Gamma_0}(u)$. This completes the proof. □

If taking $u = 0$ in the algorithm in Theorem 4.1, we get the following corollary immediately.

Corollary 4.2. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S : H_2 \to H_2$ and $T : H_1 \to H_1$ are two nonexpansive mappings. Assume that $\Gamma_0 \ne \emptyset$. Given $\{\alpha_n\}$, $\{\beta_n\} \subset$ (0,1) and constant δ satisfying the following conditions:*
(1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(2) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}| < \infty$,
(3) $0 < \beta_n < b < 1$, $0 < \delta < \frac{1-b}{\lambda}$, with λ being the spectral radius of the operator AA^*.
For fixed $x_0 \in H_1$ arbitrarily, define a sequence $\{x_n\}$ by the following algorithm,

$$\begin{cases} y_n = Tx_n + (1-\beta_n)\delta A^*(S-I)ATx_n, \\ x_{n+1} = (1-\alpha_n)y_n, \quad n \ge 0, \end{cases}$$

then $\{x_n\}$ converges strongly to a x^*, given by $x^* = P_{\Gamma_0}(0)$, which is the minimum norm in Γ_0.

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