Filomat 34:13 (2020), 4453–4472
https://doi.org/10.2298/FIL2013453B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and Nonexistence Results for a System of Integral Boundary Value Problems With Parametric Dependence

Rim Bourguiba[a], Faten Toumi[b], Om Kalthoum Wanassi[a]

[a]*University of Monastir, Faculty of Sciences of Monastir, LR18ES17, 5019 Monastir, Tunisia.*
[b]*King Faisal University, College of Business Administration, P.O. 380 Ahsaa 31982, Saoudi Arabia.*

Abstract. In this paper, we are concerned with a class of system of nonlinear singular fractional differential equations with integral boundary conditions. More precisely, we establish sufficient conditions for existence, multiplicity and nonexistence of positive solutions. The results are derived in terms of different values of the parameters. Our approach relies on the Krasnoselskii's fixed point theorem. Some examples are given to illustrate our main results.

1. Introduction

In this paper, we consider the following class of system of boundary value problems

$$\begin{cases} D^{\alpha}u(t)+\mu_1 a(t)f(t,u(t),v(t))=0, \ \text{ in } (0,1),\ n-1<\alpha \le n,\\ D^{\beta}v(t)+\mu_2 b(t)g(t,u(t),v(t))=0, \ \text{ in } (0,1),\ m-1<\beta \le m,\\ u(0)=u'(0)=...=u^{(n-2)}(0)=0,\ u(1)=\lambda_1\int_0^1 u(s)ds,\\ v(0)=v'(0)=...=v^{(m-2)}(0)=0,\ v(1)=\lambda_2\int_0^1 v(s)ds, \end{cases} \tag{1}$$

where D^{δ} is the standard Reimann-Liouville fractional derivative of order $\delta>0$, $n,m\in\mathbb{N}$, $n,m\geq 3$, $0<\lambda_1<\alpha$, $0<\lambda_2<\beta$, μ_1 and μ_2 are two positive parameters. The functions a,b are continuous nonnegative on $(0,1)$ and they are allowed to be singular at $t=0$ and/or $t=1$. The nonlinearities f,g are in $C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$.
Many previous works have studied boundary value problems with integral boundary conditions in the scalar case, we cite the works [5–7, 22]. Namely, Cabada and Hamdi in [5], gave existence results for the following boundary value problem

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0 \qquad \text{ in } (0,1),\\ u(0)=u'(0)=0, u(1)=\lambda\int_0^1 u(s)ds, \end{cases} \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 34A08; Secondary 34B27, 34B18, 35G60.
Keywords. Fractional differential system, positive solution, integral boundary conditions, Green's functions, dependence on a parameter.
Received: 13 January 2020; Revised: 01 February 2020; Accepted: 13 Febraury 2020
Communicated by Maria Alessandra Ragusa
Email addresses: Rym.Bourguiba@fsm.rnu.tn (Rim Bourguiba), ftowmi@kfu.edu.sa (Faten Toumi), omkalthoum.wannassi@fsm.rnu.tn (Om Kalthoum Wanassi)

where $2 < \alpha \leq 3,\ 0 < \lambda < \alpha$ and f is continuous function.
More recently, in [6], the authors considered the following problem

$$\begin{cases} D^{\alpha}u(t) + \mu g(t)h(u(t)) = 0 \qquad \text{in } [0,1], \\ u(0) = u'(0) = 0, u(1) = \lambda \int_0^1 u(s)ds, \end{cases} \tag{3}$$

depending on the real parameter $\mu > 0$, where $2 < \alpha \leq 3$, h and g are nonnegative continuous functions. Under the condition that $g \in L^1([0,1])$ and $\int_{\frac{1}{2}}^1 g(t)dt > 0$, they derived various existence and multiplicity results of positive solutions depending on the parameter $\mu > 0$.

On the other hand, as a generalization of boundary value problems of differential equation, many authors treated these kind of problems, we cite [1, 9, 11, 14–19, 21]. It is worth to remark that the great importance of such problems came from their applications. In fact, these problems present efficient models for description of different systems and process in engineering, science, economy, chemical, thermo-elasticity, population dynamical and so forth, we refer the reader to [20, 22, 26, 31, 32] and the references therein.

In this work, we shall give sufficient conditions for existence, multiplicity and nonexistence of positive solutions for system (1). The results derived depending on the positive parameters μ_1 and μ_2. We remark here that we are essentially inspired by the works [6, 21], we will see that severel of Cabada and Hamdi's proofs in [6] carry over to some proof's here, quite nicely.

The rest of the paper is organized as follows. In the next section we recall some tools and we present properties of the Green's function. Moreover we state preliminary lemmas. Section 3 and Section 4 are devoted to establish existence of one or more positive solutions for (1), respectively. However, Section 5 concerns nonexistence results. The final Section of the paper contains examples to illustrate our results.

2. Preliminaries

In this section, we recall some results and we prove key lemmas that will be used to prove our main results. We begin with the following background on the fractional calculus. We refer the reader to [26, 31] for more details.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a measurable function $f : (0, +\infty) \to \mathbb{R}$ is defined as*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,\ t > 0,$$

where Γ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a measurable function $f : (0, +\infty) \to \mathbb{R}$ is defined as*

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds = (\frac{d}{dt})^n I^{n-\alpha} f(t),$$

provided that the right-hand side is pointwise defined on $\mathbb{R}^+$.
Here $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3. *Let $\alpha > 0$. Let $u \in C(0,1) \cap L^1(0,1)$. Then*

i) $D^{\alpha}I^{\alpha}u = u$.

ii) *For $\delta > \alpha - 1$, $D^{\alpha}t^{\delta} = \dfrac{\Gamma(\delta+1)}{\Gamma(\delta-\alpha+1)} t^{\delta-\alpha}$. Moreover, we have* $\quad D^{\alpha}t^{\alpha-i} = 0,\ i = 1, 2, .., n.$

iii) $D^{\alpha}u(t) = 0$ *if and only if* $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, i = 1, 2, .., n.

iv) Assume that $D^{\alpha}u \in C(0,1) \cap L^1(0,1)$, *then we have*

$I^{\alpha}D^{\alpha}u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, .., n$.

Now, we give the explicit expression of the Green's function for the linear fractional differential equation associated to the problem (1). More precisely, we assert the following result.

Lemma 2.4. *Let* $n \geq 3$, $n-1 < \alpha \leq n$ *and* $\lambda \in (0, \alpha)$. *Let* $y \in C[0,1]$. *Then the unique solution of the fractional differential equation*

$$D^{\alpha}u(t) + y(t) = 0 \text{ in } (0,1), \tag{4}$$

subject to the boundary conditions

$$u(0) = ... = u^{(n-2)}(0) = 0, u(1) = \lambda \int_0^1 u(s)ds, \tag{5}$$

is given by

$$u(t) = \int_0^1 G_{\alpha,\lambda}(t,s)y(s)ds,$$

where *for all t,s* $\in [0,1]$,

$$G_{\alpha,\lambda}(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)((t-s)^+)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}. \tag{6}$$

$G_{\alpha,\lambda}(t,s)$ is called the Green's function of the boundary value problem (4)-(5). Here, for $x \in \mathbb{R}$, $x^+ = \max(x, 0)$.

Proof. From Lemma 2.3, we have

$$u(t) = -I^{\alpha}y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}.$$

Consequently the solution of (4) is

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}.$$

Since $u(0) = 0$, we find $c_n = 0$. So

$$u(t) = -I^{\alpha}y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_{n-1} t^{\alpha-n+1}. \tag{7}$$

Differentiating (7), we obtain

$$u'(t) = -\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}y(s)ds + c_1(\alpha-1)t^{\alpha-2} + ... + c_{n-1}(\alpha-n+1)t^{\alpha-n},$$

By $u'(0) = 0$, we obtain $c_{n-1} = 0$. Similarly, we obtain $c_2 = c_3 = ... = c_{n-2} = 0$. Therefore

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1 t^{\alpha-1}. \tag{8}$$

Now, the condition $u(1)=\lambda\int_0^1 u(s)ds$ implies that

$$c_1=\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+\lambda\int_0^1 u(s)ds.$$

Let $A=\int_0^1 u(t)dt$. Then by (8), we obtain

$$A=-\int_0^1\frac{(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}y(s)ds+\frac{1}{\alpha}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+\frac{\lambda}{\alpha}A.$$

So, we have

$$A=-\frac{1}{\alpha-\lambda}\int_0^1\frac{(1-s)^{\alpha}}{\Gamma(\alpha)}y(s)ds+\frac{1}{\alpha-\lambda}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds.$$

Replacing A in (8), we obtain the following expression of the function u

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+t^{\alpha-1}\int_0^1\frac{(1-s)^{\alpha-1}(\alpha+\lambda(s-1))}{(\alpha-\lambda)\Gamma(\alpha)}y(s)ds.$$

Finally, we deduce

$$\begin{aligned}u(t)&=\int_0^t\frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha+\lambda(s-1))-(\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}y(s)ds\\&\quad+\int_t^1\frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha+\lambda(s-1))}{(\alpha-\lambda)\Gamma(\alpha)}y(s)ds\\&=\int_0^1 G_{\alpha,\lambda}(t,s)y(s)ds.\end{aligned}$$

This ends the proof. □

In order to give some estimations on the function $G_{\alpha,\lambda}$, we introduce the function G_0 on $[0,1]\times[0,1]$ given by

$$G_0(t,s)=\frac{t^{\alpha-1}(1-s)^{\alpha-1}-((t-s)^+)^{\alpha-1}}{\Gamma(\alpha)}. \tag{9}$$

Lemma 2.5. *Let $n\in\mathbb{N}$, $n\geq 3$, $n-1<\alpha\leq n$. Define the function $H(t,s)$ on $[0,1]\times[0,1]$ by*

$$H(t,s)=\frac{1}{\Gamma(\alpha)}t^{\alpha-2}(1-s)^{\alpha-2}\min(t,s)(1-\max(t,s)). \tag{10}$$

Then the function G_0 has the following property

$$\frac{1}{\alpha-1}H(t,s)\leq G_0(t,s)\leq H(t,s),\ t,s\in[0,1]. \tag{11}$$

To show Lemma 2.5, we recall the following standard results.

Lemma 2.6. *i) Let $\eta,\delta\in(0,+\infty)$, $a,u\in[0,1]$. Then*

$$\min(1,\frac{\delta}{\eta})(1-au^{\delta})\leq(1-au^{\eta})\leq\max(1,\frac{\delta}{\eta})(1-au^{\delta}).$$

ii) For $t,s\in[0,1]$, we have

$$ts\leq\min(t,s)\leq s \text{ and } (1-s)(1-t)\leq 1-\max(t,s)\leq(1-s). \tag{12}$$

Now we are able to prove Lemma 2.5.

Proof. First, remark that for each $t,s \in (0,1)$,

$$\Gamma(\alpha)G_0(t,s) = t^{\alpha-1}(1-s)^{\alpha-1}[1-(\frac{(t-s)^+}{t(1-s)})^{\alpha-1}].$$

Thus by applying Lemma 2.6 (i) for $\eta = \alpha - 1$, $\delta = 1$, $a = 1$ and

$u = (\frac{(t-s)^+}{t(1-s)})^{\alpha-1}$, we obtain

$$\frac{1}{(\alpha-1)}(1-(\frac{(t-s)^+}{t(1-s)})) \leq [1-(\frac{(t-s)^+}{t(1-s)})^{\alpha-1}] \leq (1-(\frac{(t-s)^+}{t(1-s)})).$$

Then

$$\frac{1}{(\alpha-1)}\frac{t(1-s)-(t-s)^+}{t(1-s)} \leq [1-(\frac{(t-s)^+}{t(1-s)})^{\alpha-1}] \leq \frac{t(1-s)-(t-s)^+}{t(1-s)}.$$

By using the fact that $t(1-s)-(t-s)^+ = \min(t,s)(1-\max(t,s))$, we conclude that

$$\tfrac{1}{(\alpha-1)}H(t,s) \leq G_0(t,s) \leq H(t,s).$$

□

The following properties of the Green's function play an important role in this paper.

Proposition 2.7. *Let $n \in \mathbb{N}$, $n \geq 3$, $n-1 < \alpha \leq n$, and $\lambda \in [0,\alpha)$. Then the function $G_{\alpha,\lambda}$ defined by (6) satisfies the following properties*

i) $G_{\alpha,\lambda}$ *is nonnegative continuous function on* $[0,1]\times[0,1]$ *and* $G_{\alpha,\lambda}(t,s) > 0$, *for all* $t,s \in (0,1)$.

ii) *For all* $t \in [0,1]$, $s \in [0,1]$, *we have*

$$p(t)K_\alpha(s) \leq G_{\alpha,\lambda}(t,s) \leq q(t)K_\alpha(s), \tag{13}$$

where

$$K_\alpha(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \tag{14}$$

and

$$p(t) = \left(\frac{(1-t)}{\alpha-1} + \frac{\lambda}{\alpha-\lambda}\right)t^{\alpha-1}, q(t) = \frac{((\alpha-\lambda)t+\lambda)}{(\alpha-\lambda)}\, t^{\alpha-2}.$$

iii) *Let* $\theta \in (0,\frac{1}{2})$, $t,s \in [0,1]$, *then we have*

$$\min_{t\in[\theta,1-\theta]} G_{\alpha,\lambda}(t,s) \geq \gamma_{\alpha,\lambda}K_\alpha(s), \tag{15}$$

where

$$\gamma_{\alpha,\lambda} = (\frac{\theta}{\alpha-1} + \frac{\lambda}{\alpha-\lambda})\theta^{\alpha-1}. \tag{16}$$

iv) For each $t, s \in [0,1]$, we have

$$G_{\alpha,\lambda}(t,s) \leq \eta_{\alpha,\lambda} K_\alpha(s), \tag{17}$$

where

$$\eta_{\alpha,\lambda} = \frac{\alpha}{(\alpha-\lambda)}. \tag{18}$$

Proof. i) It is obvious to see that $G_{\alpha,\lambda}(t,s)$ is continuous on $[0,1]\times[0,1]$

and $G_{\alpha,\lambda}(t,s) > 0$.

ii) Let $t, s \in [0,1]$, using Lemma 2.6 (ii), we conclude that the functin $H(t,s)$ defined by (10) satisfies

$$(1-t)t^{\alpha-1}s(1-s)^{\alpha-1} \leq \Gamma(\alpha)H(t,s) \leq t^{\alpha-2}s(1-s)^{\alpha-1}. \tag{19}$$

Combining (11) and (19), we get

$$\tfrac{1}{(\alpha-1)}(1-t)t^{\alpha-1}s(1-s)^{\alpha-1} \leq \Gamma(\alpha)G_0(t,s) \leq t^{\alpha-2}s(1-s)^{\alpha-1}. \tag{20}$$

On the other hand, the function $G_{\alpha,\lambda}$ defined by (6) satisfies

$$(\alpha-\lambda)\Gamma(\alpha)G_{\alpha,\lambda}(t,s) = (\alpha-\lambda)\Gamma(\alpha)G_0(t,s) + \lambda s(t(1-s))^{\alpha-1}.$$

Thus, using (20), we obtain

$$\frac{(\alpha-\lambda)(1-t)+\lambda(\alpha-1)}{(\alpha-1)}t^{\alpha-1}s(1-s)^{\alpha-1} \leq (\alpha-\lambda)\Gamma(\alpha)G_{\alpha,\lambda}(t,s) \leq ((\alpha-\lambda)t+\lambda)t^{\alpha-2}s(1-s)^{\alpha-1}.$$

So, we deduce that

$$p(t)K_\alpha(s) \leq G_{\alpha,\lambda}(t,s) \leq q(t)K_\alpha(s).$$

where

$$K_\alpha(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)},$$

and

$$p(t) = \left(\frac{(1-t)}{\alpha-1} + \frac{\lambda}{\alpha-\lambda}\right)t^{\alpha-1},\ q(t) = \frac{((\alpha-\lambda)t+\lambda)}{(\alpha-\lambda)}\,t^{\alpha-2}.$$

(iii) and (iv) are consequences of the inequalities (13). □

Remark 2.8. *Note that this estimation on Green's function $G_{\alpha,\lambda}(t,s)$* obtained *in the previous proposition improve those obtained in [5] and [6] in the case where $2 < \alpha \leq 3$.*

The proofs of ours results are based upon the following Krasnoselskii's fixed point theorem.

Lemma 2.9. *([26]) Let P be the cone of a real Banach space E and Ω_1, Ω_2 two bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow P$ is completely continuous operator such that either*
(i) $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$, or
(ii) $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$.
holds. Then the operator T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Now, let $E = C([0,1]) \times C([0,1])$, endowed with the norm
$\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \sup_{t\in[0,1]} |u(t)|$. Then E is a Banach space.
Let $\theta \in [0, \frac{1}{2})$, and set $J_\theta = [\theta, 1-\theta]$. In the sequel we need the following notations

$$\sigma_\alpha^\theta = \int_\theta^{1-\theta} a(t)K_\alpha(t)dt \ \text{ and } \ \sigma_\beta^\theta = \int_\theta^{1-\theta} b(t)K_\beta(t)dt,$$

where K_α, K_β are defined by (14).

$$G_\alpha = G_{\alpha,\lambda_1} \text{and } \ G_\beta = G_{\beta,\lambda_2},$$

where G_{α,λ_1}, G_{β,λ_2} are given by (6). And

$$\gamma_\alpha = \gamma_{\alpha,\lambda_1},\ \gamma_\beta = \gamma_{\beta,\lambda_2}. \tag{21}$$

$$\eta_\alpha = \eta_{\alpha,\lambda_1},\ \eta_\beta = \eta_{\beta,\lambda_2}. \tag{22}$$

$$\gamma_\alpha^* = \gamma_\alpha \eta_\alpha^{-1}, \gamma_\beta^* = \gamma_\beta \eta_\beta^{-1}, \gamma = \min(\gamma_\alpha^*, \gamma_\beta^*), \tag{23}$$

where $\gamma_{\alpha,\lambda_1}$, γ_{β,λ_2}, η_{α,λ_1} and η_{β,λ_2} are defined by (16) and (18).
Now, we define the cone Ω in E by

$$\Omega = \{(u,v) \in E:\ u(t) \geq 0,\ v(t) \geq 0 \text{ on } [0,1],\ \min_{t\in J_\theta} u(t) \geq \gamma_\alpha^* \|u\|, \min_{t\in J_\theta} v(t) \geq \gamma_\beta^* \|v\|\},$$

and for $r > 0$, let

$$\Omega_r = \{(u,v) \in \Omega : \|(u,v)\| < r\}.$$

Next, we define the operator $T : E \longrightarrow E$ as follows

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \quad t \in [0,1] \tag{24}$$

where

$$T_1(u,v)(t) = \mu_1 \int_0^1 G_\alpha(t,s)a(s)f(s,u(s),v(s))ds,$$

and

$$T_2(u,v)(t) = \mu_2 \int_0^1 G_\beta(t,s)b(s)g(s,u(s),v(s))ds.$$

In the remainder of the paper, we adopt the following hypotheses:

(H_1) $a, b \in C((0,1), [0+\infty))$, $a(t), b(t) \neq 0$ on any subinterval of $(0,1)$ and $0 < \sigma_\alpha^0, \sigma_\beta^0 < \infty$.

(H_2) $f, g \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$.

(H_3) There exist $t_1, t_2 \in (0,1)$ such that $f(t_1,u,v) > 0$ and $g(t_2,u,v) > 0$ for each $u, v \in (0+\infty)$.

We note that our study on the problem (1) remains to the seek of fixed point of the operator T and this is due to the following lemma.

Lemma 2.10. *Suppose that (H_1)-(H_3) hold. Then* $(u, v) \in C([0, 1]) \times C([0, 1])$ *is a solution of the boundary value problem (1) if and only if* $(u, v) \in C([0, 1]) \times C([0, 1])$ *is a solution of the integral equations*

$$u(t) = \mu_1 \int_0^1 G_\alpha(t, s)a(s)f(s, u(s), v(s))ds. \tag{25}$$

$$v(t) = \mu_2 \int_0^1 G_\beta(t, s)b(s)g(s, u(s), v(s))ds. \tag{26}$$

That is (u, v) *is a fixed point of the operator* T *defined by (24).*

Proof. The proof is immediate from Lemma 2.4, so we omit it. □

We call $G(t, s) = (G_\alpha(t, s), G_\beta(t, s))$ the Green's function of the problem (1).
Now, we state some lemmas which will be used in the proofs of our main results.

Lemma 2.11. *Suppose that conditions (H_1) and (H_2) hold. Then* $T : \Omega \longrightarrow \Omega$ *is completely continuous.*

Proof. Since G_α, G_β, f and g are nonnegatives continuous functions and using (H_1) we conclude that $T : \Omega \longrightarrow \Omega$ is continuous. Let $(u, v) \in \Omega$, then by Proposition 2.7, (15) and (17), we obtain for all $t \in J_\theta$,

$$\begin{aligned}
T_1(u, v)(t) &\geq \mu_1\gamma_\alpha \int_0^1 K_\alpha(s)a(s)f(s, u(s), v(s))ds \\
&\geq \mu_1\frac{\gamma_\alpha}{\eta_\alpha} \int_0^1 G_\alpha(\tau, s)a(s)f(s, u(s), v(s)) \\
&\geq \gamma^*_\alpha \max_{\tau\in[0,1]}\{\mu_1 \int_0^1 G_\alpha(\tau, s)a(s)f(s, u(s), v(s))ds\} \\
&= \gamma^*_\alpha\|T_1(u, v)\|.
\end{aligned}$$

Similarly, it follows that for all $t \in J_\theta$

$$T_2(u, v)(t) \geq \gamma^*_\beta\|T_2(u, v)\|.$$

Then $T(\Omega) \subset \Omega$. Now, let S be a bounded set of Ω, then there exists a positive constant $M > 0$ such that $\|(u, v)\| \leq M$, for all $(u, v) \in S$. Define now

$$M_1 := \max_{t\in[0,1],u+v\leq M} f(t, u, v) \text{ and } M_2 := \max_{t\in[0,1],u+v\leq M} g(t, u, v).$$

From hypothesis (H_1), Proposition 2.7 and (17), we have for all $t \in [0, 1]$, $u, v \in S$,

$$\begin{aligned}
T_1(u, v)(t) &\leq \mu_1\eta_\alpha \int_0^1 K_\alpha(s)a(s)f(s, u(s), v(s))ds \\
&\leq \mu_1\eta_\alpha M_1\sigma^0_\alpha.
\end{aligned}$$

Similarly

$$\begin{aligned}
T_2(u, v)(t) &\leq \mu_2\eta_\beta \int_0^1 K_\beta(s)b(s)g(s, u(s), v(s))ds \\
&\leq \mu_2\eta_\beta M_2\sigma^0_\beta.
\end{aligned}$$

So, we have

$$\|T(u,v)\| \le \mu_1\eta_\alpha M_1\sigma_\alpha^0 + \mu_2\eta_\beta M_2\sigma_\beta^0.$$

Hence T(S) is uniformly bounded.
Now let us prove that T(S) is equicontinuous on $[0,1]$. Using Proposition 2.7, we obtain that $G_\alpha$is uniformly continuous on $[0,1]\times[0,1]$. Thus for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for any $t_1, t_2 \in [0,1]$ satisfying $|t_1 - t_2| < \delta$ and for each $s \in [0,1]$,

$$|G_\alpha(t_2,s) - G_\alpha(t_1,s)| \le \frac{\varepsilon}{2\mu_1\eta_\alpha M_1\sigma_\alpha^0}.$$

Then, for $u, v \in S$, we have

$$|T_1(u,v)(t_2) - T_1(u,v)(t_1)| < \frac{\varepsilon}{2}\ . \tag{27}$$

Similarly

$$|T_2(u,v)(t_2) - T_2(u,v)(t_1)| < \frac{\varepsilon}{2}. \tag{28}$$

Therefore, by (27) and (28) we obtain

$$\| T(u,v)(t_2) - T(u,v)(t_1) \|_1 < \varepsilon.$$

where $\|.\|_1$ is the norm *on* $\mathbb{R}^2$ defined by $\|(u,v)\|_1 = |u| + |v|$. Thus $T(S)$ is equicontinuous. Consequently by Ascoli's theorem, we conclude that$T(S)$ is relatively compact in E. Therfore $T : \Omega \longrightarrow \Omega$ is completely continuous. This completes the proof. □

Hereinafter, we introduce the following notations

$$f^\delta = \lim_{u+v\to\delta} \{\max_{t\in[0,1]} \frac{f(t,u,v)}{u+v}\}, \tag{29}$$

and

$$f_\delta = \lim_{u+v\to\delta} \{\min_{t\in J_\theta} \frac{f(t,u,v)}{u+v}\}. \tag{30}$$

where $\delta = 0$ or $+\infty$.

Lemma 2.12. *If conditions (H_1)-(H_3) hold. Then, for every* $R > 0$, there *exist* $\mu_1^*(R) > 0$ *and* $\mu_2^*(R) > 0$ *such that for each* $0 < \mu_1 \le \mu_1^*(R)$ *and* $0 < \mu_2 \le \mu_2^*(R)$ *we have*

$$\|T(u,v)\| \le \|(u,v)\|, \text{ for all } (u,v) \in \partial\Omega_R.$$

Proof. Fix $R > 0$ and let $(u,v) \in \Omega$ with $\|(u,v)\| = R$. Let

$$M_1 = \max_{\substack{t\in[0,1] \\ (u,v)\in[0,R]\times[0,R]}} f(t,u,v) \ \text{ and } \ M_2 = \max_{\substack{t\in[0,1] \\ (u,v)\in[0,R]\times[0,R]}} g(t,u,v).$$

Notice that from (H_3) we get that $M_1, M_2 > 0$ for all $R > 0$ and (H_1) implies that $\sigma_\alpha^0, \sigma_\beta^0 > 0$. Thus define $\mu_1^*(R) = \frac{R}{2\eta_\alpha M_1\sigma_\alpha^0}$ and $\mu_2^*(R) = \frac{R}{2\eta_\beta M_2\sigma_\beta^0}$.
Let $0 < \mu_1 \le \mu_1^*(R)$ and $0 < \mu_2 \le \mu_2^*(R)$. Then, for all $t \in [0,1]$, we have

$$\begin{aligned} T_1(u,v)(t) &\le \mu_1\eta_\alpha \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds \\ &\le \mu_1\eta_\alpha M_1\sigma_\alpha^0 \\ &\le \mu_1\eta_\alpha M_1\sigma_\alpha^0 \le \frac{R}{2} = \frac{1}{2}\|(u,v)\|. \end{aligned}$$

So, we obtain

$$\|T_1(u,v)\| \leq \frac{1}{2}\|(u,v)\|. \tag{31}$$

Analogously, we prove

$$\|T_2(u,v)\| \leq \frac{1}{2}\|(u,v)\|. \tag{32}$$

Hence, from (31) and (32), we conlude that $\|T(u,v)\| \leq \|(u,v)\|$, for all $(u,v) \in \partial\Omega_R$. □

Lemma 2.13. *Assume that conditions (H_1)–(H_3) are fullfilled .*
If $f_0 = \infty$ or $g_0 = \infty$, then there exists $r_0(\mu_1, \mu_2) > 0$ such that for every
$0 < r < r_0(\mu_1, \mu_2)$ we have

$$\|T(u,v)\| \geq \|(u,v)\|, \text{ for all } (u,v) \in \partial\Omega_r.$$

Proof. Let $\mu_1, \mu_2 > 0$. Define $A = \max\{\frac{1}{2\mu_1\gamma\gamma_\alpha\sigma_\alpha^\theta}, \frac{1}{2\mu_2\gamma\gamma_\beta\sigma_\beta^\theta}\} > 0$. In addition assume that $f_0 = \infty$ or $g_0 = \infty$, then there exists $r_0 = r_0(\mu_1, \mu_2) > 0$ such that

$$f(t,u,v) \geq A(u+v), \quad t \in J_\theta, \ \ 0 < u+v \leq r_0. \tag{33}$$

or

$$g(t,u,v) \geq A(u+v), \quad t \in J_\theta, \ \ 0 < u+v \leq r_0. \tag{34}$$

First, if $f_0 = \infty$, then (33) holds. Fix $0 < r < r_0$ and $(u,v) \in \partial\Omega_r$. Then for all $t \in J_\theta$, we have

$$\begin{aligned} T_1(u,v)(t) &\geq \mu_1\gamma_\alpha A \int_\theta^{1-\theta} K_\alpha(s)a(s)(u(s)+v(s))ds \\ &\geq \mu_1\gamma_\alpha A \int_\theta^{1-\theta} K_\alpha(s)a(s)(\gamma_\alpha^*\|u\| \ \| + \gamma_\beta^*\|v\|)ds \\ &\geq \mu_1\gamma\gamma_\alpha\sigma_\alpha^\theta A\|(u,v)\|. \end{aligned}$$

So, we obtain

$$\|T_1(u,v)\| \geq \frac{\|(u,v)\|}{2}.$$

Thus

$$\|T(u,v)\| \geq \|(u,v)\|.$$

Now, we suppose that $g_0 = \infty$, then (34) holds. By the same manner we prove that

$$\|T(u,v)\| \geq \|(u,v)\|,$$

which ends the proof. □

Remark 2.14. *Note that condition f_0 and g_0 need only to be satisfied on subinterval J_θ of* $[0,1]$ rather *the entire interval as is often required with this type of assumption.*

Lemma 2.15. *Suppose that conditions* $(H_1)-(H_3)$ *hold. If* $f^0 = 0$ *and* $g^0 = 0$ *then there exists* $r_0(\mu_1, \mu_2) > 0$ *such that* $0 < r < r_0(\mu_1, \mu_2)$ *we have*

$$\|T(u,v)\| \le \|(u,v)\|, \text{ for all } (u,v) \in \partial\Omega_r.$$

Proof. Since $f^0 = 0$ and $g^0 = 0$, then for $\varepsilon = \min\{\frac{1}{2\mu_1\eta_\alpha\sigma_\alpha^0}, \frac{1}{2\mu_2\eta_\beta\sigma_\beta^0}\} > 0$ there exists $r_0(\mu_1, \mu_2) > 0$ such that $f(t,u,v) \le \varepsilon(u+v)$, and $g(t,u,v) \le \varepsilon(u+v)$, $\forall t \in [0,1]$, $0 < u+v \le r_0(\mu_1,\mu_2)$. Fix $0 < r < r_0(\mu_1,\mu_2)$ and let $(u,v) \in \partial\Omega_r$. Then for all $t \in [0,1]$, we have

$$\begin{aligned} T_1(u,v)(t) &\le \mu_1\eta_\alpha \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds \\ &\le \mu_1\eta_\alpha\varepsilon \int_0^1 K_\alpha(s)a(s)(u(s)+v(s))ds \\ &\le \mu_1\eta_\alpha\sigma_\alpha^0\varepsilon(\|u\|+\|v\|) \end{aligned}$$

Thus,

$$\|T_1(u,v)\| \le \frac{1}{2}\|(u,v)\|.$$

By the same manner, we obtain

$$\|T_2(u,v)\| \le \frac{1}{2}\|(u,v)\|.$$

Then it follows that

$$\|T(u,v)\| \le \|(u,v)\|.$$

Which ends the proof. □

Lemma 2.16. *Assume that* (H_1)-(H_3) *are satisfied. If we have* $f_\infty = \infty$ *or* $g_\infty = \infty$, *then there exists* $R_0(\mu_1,\mu_2) > 0$ *such that for every* $R \ge R_0(\mu_1,\mu_2)$ *we have*

$$\|T(u,v)\| \ge \|(u,v)\|, \text{ for all } (u,v) \in \partial\Omega_R.$$

Proof. Let

$$M = \max\{\frac{1}{\mu_1\gamma\gamma_\alpha\sigma_\alpha^\theta}, \frac{1}{\mu_2\gamma\gamma_\beta\sigma_\beta^\theta}\} > 0.$$

Assume that $f_\infty = \infty$ or $g_\infty = \infty$, then there exists $R_1 = R_1(\mu_1,\mu_2) > 0$ such that $f(t,u,v) \ge M(u+v)$ or $g(t,u,v) \ge M(u+v)$, $\forall t \in J_\theta$, $u+v \geqslant R_1$.
Now, define $R_0(\mu_1,\mu_2) = \frac{1}{\gamma}R_1$ and let $R \ge R_0(\mu_1,\mu_2)$. First, if$f_\infty = \infty$, then for any $(u,v) \in \partial\Omega_R$, we get $f(t,u(t),v(t)) \ge M(u(t)+v(t))$, $\forall t \in J_\theta$. It follows that, for $(u,v) \in \partial\Omega_R$, $t \in J_\theta$,

$$\begin{aligned} T_1(u,v)(t) &\ge \mu_1\gamma_\alpha M \int_\theta^{1-\theta} a(s)K_\alpha(s)(u(s)+v(s))ds \\ &\ge \mu_1\gamma_\alpha M \int_\theta^{1-\theta} a(s)K_\alpha(s)(\gamma_\alpha^*\|u\|+\gamma_\beta^*\|v\|)ds \\ &\ge \mu_1\gamma\gamma_\alpha\sigma_\alpha^\theta M\|(u,v)\| \\ &\ge \|(u,v)\|. \end{aligned}$$

Then

$$\|T_1(u,v)\| \geq \|(u,v)\|.$$

Therfore

$$\|T(u,v)\| \geq \|(u,v)\|, \forall\ (u,v) \in \partial\Omega_R. \tag{35}$$

Now, suppose that $g_\infty = \infty$, then for any $(u,v) \in \partial\Omega_R$ we have for every $t \in J_\theta$ $g(t,u(t),v(t)) \geq M(u(t)+v(t))$, By the same manner, we obtain for $t \in J_\theta$

$$\begin{aligned} T_2(u,v)(t) &\geq \mu_2\gamma_\beta M \int_\theta^{1-\theta} b(s)K_\beta(s)g(s,u(s),v(s))ds \\ &\geq \mu_2\gamma\gamma_\beta\sigma_\beta^\theta M\|(u,v)\| \geq \|(u,v)\|. \end{aligned}$$

So

$$\|T(u,v)\| \geq \|(u,v)\|, \forall\ (u,v) \in \partial\Omega_R. \tag{36}$$

Hence, from (35) or (36), we obtain

$$\|T(u,v)\| \geq \|(u,v)\|, \forall (u,v) \in \partial\Omega_R.$$

□

Lemma 2.17. *Suppose that conditions (H_1)-(H_3) hold. Assume that $f^\infty = 0$ and $g^\infty = 0$, then there exists $R_0(\mu_1,\mu_2) > 0$ such that for every $R \geq R_0(\mu_1,\mu_2)$ we have*

$$\|T(u,v)\| \leq \|(u,v)\|, \text{ for all } (u,v) \in \partial\Omega_R.$$

Proof. Since $f^\infty = g^\infty = 0$, then for $\varepsilon = \min(\frac{1}{2\mu_1\eta_\alpha\sigma_\alpha^0}, \frac{1}{2\mu_2\eta_\beta\sigma_\beta^0}) > 0$, there exists R_1=$R_1(\mu_1,\mu_2) > 0$ such that

$$f(t,u,v) \leq \varepsilon(u+v) \text{ and } g(t,u,v) \leq \varepsilon(u+v) \text{ for each } (u+v) \geq R_1.$$

Let

$$M_1 = \max_{\substack{t\in[0,1] \\ (u,v)\in[0,R_1]\times[0,R_1]}} f(t,u,v) \text{ and } M_2 = \max_{\substack{t\in[0,1] \\ (u,v)\in[0,R_1]\times[0,R_1]}} g(t,u,v)$$

Let $R_0(\mu_1,\mu_2) > \max\{2R_1, \mu_1\eta_\alpha\sigma_\alpha^0 M_1(\frac{1}{2} - \mu_1\sigma_\alpha^0\eta_\alpha\varepsilon)^{-1}, \mu_2\eta_\beta\sigma_\beta^0 M_2(\frac{1}{2} - \mu_2\sigma_\beta^0\eta_\beta\varepsilon)^{-1}\}$. Fix $R \geq R_0(\mu_1,\mu_2)$ and let $(u,v) \in \partial\Omega_R$. Let $t \in [0,1]$, then we have

$$\begin{aligned} T_1(u,v)(t) &\leq \mu_1\eta_\alpha \int_0^1 K_\alpha(s)a(s)f(s,u(s),v(s))ds \\ &\leq \mu_1\eta_\alpha M_1\sigma_\alpha^0 + \mu_1\eta_\alpha\varepsilon \int_0^1 K_\alpha(s)a(s)(u(s)+v(s))ds \\ &\leq \mu_1\eta_\alpha M_1\sigma_\alpha^0 + \mu_1\eta_\alpha\sigma_\alpha^0\varepsilon(\|u\|+\|v\|) \\ &\leq R_0(\frac{1}{2} - \mu_1\sigma_\alpha^0\eta_\alpha\varepsilon) + \mu_1\eta_\alpha\sigma_\alpha^0\varepsilon R_0 = \frac{1}{2}\|(u,v)\|. \end{aligned}$$

So, we have

$$\|T_1(u,v)\| \leq \frac{1}{2}\|(u,v)\|.$$

Similary, we prove

$$\|T_2(u,v)\| \le \frac{1}{2}\|(u,v)\|.$$

Thus, we obtain

$$\|T(u,v)\| \le \|(u,v)\|, \forall (u,v) \in \partial\Omega_R.$$

This completes the proof. □

Lemma 2.18. *Suppose that conditions* (H_1)-(H_2) *hold. If* f_∞, $g_\infty \in (0,\infty)$, *then there exist* $\mu_1^*, \mu_2^* > 0$ *and* $R_0 > 0$ *such that for each* $R > R_0$ *and for each* $\mu_1 > \mu_1^*$, $\mu_2 > \mu_2^*$ *we have*

$$\|T(u,v)\| \ge \|(u,v)\|, \forall (u,v) \in \partial\Omega_R.$$

Proof. Suppose that f_∞, $g_\infty \in (0,\infty)$. Then for $\epsilon = \min(\frac{f_\infty}{2}, \frac{g_\infty}{2})$ there exists $R_0 > 0$ such that for each $t \in J_\theta$, $u + v \ge R_0$, we have
$f(t,u,v) \ge (f_\infty - \epsilon)(u+v)$ and $g(t,u,v) \ge (g_\infty - \epsilon)(u+v)$.
Define

$$\mu_1^* = \frac{1}{2\gamma\gamma_\alpha\sigma_\alpha^\theta(f_\infty - \epsilon)} > 0 \quad \text{and} \quad \mu_2^* = \frac{1}{2\gamma\gamma_\beta\sigma_\beta^\theta(g_\infty - \epsilon)} > 0.$$

Let $\mu_1 > \mu_1^*$ and $\mu_2 > \mu_2^*$. Fix $R > \frac{R_0}{\gamma}$ and let $(u,v) \in \partial\Omega_R$, $t \in J_\theta$, then we have

$$\begin{aligned} T_1(u,v)(t) &\ge \mu_1\gamma_\alpha \int_\theta^{1-\theta} K_\alpha(s)a(s)(f_\infty - \epsilon)(u(s)+v(s))ds \\ &\ge \mu_1\gamma\gamma_\alpha\sigma_\alpha^\theta(f_\infty - \epsilon)\|(u,v)\| \\ &\ge \frac{\|(u,v)\|}{2}. \end{aligned}$$

Similary, we get

$$T_2(u,v)(t) \ge \frac{\|(u,v)\|}{2}.$$

Thus

$$\|T(u,v)\| \ge \|(u,v)\|, \ \forall (u,v) \in \partial\Omega_R.$$

Which ends the proof. □

By the same manner we prove the following lemma.

Lemma 2.19. *Suppose that conditions* (H_1)-(H_2) *hold. Assume that* f_0, $g_0 \in (0,\infty)$. *Then there exist* $\mu_1^*, \mu_2^* > 0$ *and* $r_0 > 0$ *such that for each* $0 < r < r_0$ and for each $\mu_1 > \mu_1^*$, $\mu_2 > \mu_2^*$, *we have*

$$\|T(u,v)\| \ge \|(u,v)\|, \ \forall (u,v) \in \partial\Omega_r.$$

3. Existence of positive solution

This section is devoted to give existence of postitive solution for the nonlinear boundary value system (1). By a positive solution of problem (1), we mean a pair of functions $(u, v) \in C([0, 1])\times C([0, 1])$ satisfying (1) with $u(t), v(t) \geq 0$ for all $t \in [0, 1]$ and $u(t), v(t) > 0$ for all $t \in (0, 1)$.

Theorem 3.1. *Suppose that conditions* (H_1)–(H_3) *hold. In addition, suppose that* $\{f_0 = \infty,\ f^\infty = g^\infty = 0\}$ *or* $\{g_0 = \infty,\ f^\infty = g^\infty = 0\}$, *then for every* $\mu_1, \mu_2 > 0$, *the system (1) has at least one positive solution.*

Proof. First, suppose that $\{f_0 = \infty, f^\infty = g^\infty = 0\}$.
Choose $R > \max(\ r_0(\mu_1, \mu_2), R_0(\mu_1, \mu_2))$, where $r_0(\mu_1, \mu_2)$ and $R_0(\mu_1, \mu_2)$ are given by Lemmas 2.13, 2.17 respectively. So we obtain

$$\|T(u, v)\| \geq \|(u, v)\|, (u, v) \in \partial\Omega_r, \tag{37}$$

and

$$\|T(u, v)\| \leq \|(u, v)\|, (u, v) \in \partial\Omega_R. \tag{38}$$

Thus Lemma 2.9 implies that the operator T has a fixed point in $\overline{\Omega_R}\backslash\Omega_r$. So by Lemma 2.10, the system (1) has at least one nonnegative solution $(u, v) \in \Omega$.
We suppose now that $\{g_0 = \infty,\ f^\infty = g^\infty = 0\}$, then by the same manner, we deduce that the problem (1) has at least one nonnegative solution
$(u, v) \in \Omega$.

Now, we shall prove that (u, v) is positive solution of problem (1) that is $u(t) > 0$ and $v(t) > 0$ for each $t \in (0, 1)$. Assume, on contrary, that there exists $t^* \in (0, 1)$ such that $u(t^*) = 0$ or $v(t^*) = 0$. Suppose that $u(t^*) = 0$. Lemma 2.10 implies that

$$u(t^*) = \mu_1 \int_0^1 G_\alpha(t^*, s)a(s)f(s, u(s), v(s))ds.$$

Since the function G_α, a and f are nonnegative and continuous, we obtain

$$G_\alpha(t^*, s)a(s)f(s, u(s), v(s)) = 0 \qquad a.e.(s).$$

From hypothesis (H_1) and the fact that G_α is positive on $(0, 1) \times (0, 1)$ we deduce that

$$f(s, u(s), v(s)) = 0 \qquad a.e.(s). \tag{39}$$

Further, from hypothesis (H_3) and the continuity of the function f, we claim the existence of subset $I \subset (0, 1)$ with $m(I) > 0$, where m is the Lebesgue measure on $(0, 1)$ such that $f(t, u, v) > 0$ on I, which contradict (39). The proof is complete. □

Theorem 3.2. *Suppose that conditions* (H_1)–(H_3) *hold. In addition suppose that* $\{f^0 = g^0 = 0,\ f_\infty = \infty\}$ *or* $\{f^0 = g^0 = 0, g_\infty = \infty\}$, *then for every* $\mu_1, \mu_2 > 0$, *the system (1) has at least one positive solution.*

Proof. We assume that $\{f^0 = g^0 = 0, f_\infty = \infty\}$, then for $0 < r < r_0(\mu_1, \mu_2)$ and $R > \max\{r_0, R_0(\mu_1, \mu_2)\}$, where $r_0(\mu_1, \mu_2)$ and $R_0(\mu_1, \mu_2)$ are given by Lemmas 2.15 and 2.16 respectively, we have

$$\|T(u, v)\| \leq \|(u, v)\|, \text{for } (u, v) \in \partial\Omega_r,$$

and

$$\|T(u, v)\| \geq \|(u, v)\|, \text{for } (u, v) \in \partial\Omega_R.$$

Thus, Lemma 2.10 implies the existence of a nonnegative solution for problem (1). Now, if $\{f^0 = g^0 = 0, g_\infty = \infty\}$, then by a similar approach, we conclude that problem (1) has at least one nonnegative solution $(u, v) \in \Omega$. The positivity of (u, v) is shown as in proof of the previous Theorem. □

Remark 3.3. *We note here that Theorem (3.1) and Theorem (3.2) are generalisations of Theorem (3.2) stated in the work [5] concerning the scalar boundary value problem (2).*

Theorem 3.4. *If conditions $(H_1)-(H_3)$ hold. Suppose also that one of the following conditions $\{f_0 = \infty$ or $g_0 = \infty\}$ or $\{f_\infty = \infty$ or $g_\infty = \infty\}$ is satisfied. Then there exist $\mu_1^*, \mu_2^* > 0$ such that for each $0 < \mu_1 < \mu_1^*$ and $0 < \mu_2 < \mu_2^*$, the system (1) has at least one positive solution.*

Proof. Choose $R > 0$. From Lemma 2.12, there exist $\mu_1^*, \mu_2^* > 0$ such that for $0 < \mu_1 < \mu_1^*$ and $0 < \mu_2 < \mu_2^*$, we obtain

$$\|T(u,v)\| \le \|(u,v)\|, \text{ for all } (u,v) \in \partial\Omega_R.$$

If the first condition holds, that is we have $\{f_0 = \infty$ or $g_0 = \infty\}$, then by Lemma 2.13, there exits $r_0(\mu_1, \mu_2) > 0$ such that for $0 < r_1 < \min\{\frac{R}{2}, r_0(\mu_1, \mu_2)\}$, we have

$$\|T(u,v)\| \ge \|(u,v)\|, \text{ for all } (u,v) \in \partial\Omega_{r_1}.$$

Now, if we have $\{f_\infty = \infty$ or $g_\infty = \infty\}$, then from Lemma 2.16, there exits $R_0(\mu_1, \mu_2) > 0$ such that for $r_2 > \max\{R_0(\mu_1, \mu_2), 2R\}$ we have

$$\|T(u,v)\| \ge \|(u,v)\|, \text{ for all } (u,v) \in \partial\Omega_{r_2}.$$

Therefore Lemma 2.9 implies that the operator T has a fixed point in $\overline{\Omega_R}\backslash\Omega_{r_1}$ or $\overline{\Omega_{r_2}}\backslash\Omega_R$ whether the first assertion or the second one is satisfied, respectively. Thus by Lemma 2.10, the system (1) admits at least one positive solution for all $0 < \mu_1 < \mu_1^*$ and $0 < \mu_2 < \mu_2^*$. □

Theorem 3.5. *Assume that conditions $(H_1)-(H_2)$ are fullfilled.*
If $\{f_\infty, g_\infty \in (0,\infty), f^0 = g^0 = 0\}$ then there exist $\mu_1^*, \mu_2^* > 0$ such that for each $\mu_1 > \mu_1^*$ and $\mu_2 > \mu_2^*$, the system (1) has at least one positive solution.

Proof. Let $\mu_1 > \mu_1^*$, $\mu_2 > \mu_2^*$, $R > R_0$, where μ_1^*, μ_2^* and R_0 are given by Lemma 2.18. Then we have

$$\|T(u,v)\| \ge \|(u,v)\|, \forall (u,v) \in \partial\Omega_R. \tag{40}$$

On the other hand, let $r < \min\{R_0, r_0((\mu_1, \mu_2)\}$ where $r_0(\mu_1, \mu_2)$ is defined by Lemma 2.15, then we obtain

$$\|T(u,v)\| \le \|(u,v)\|, \text{ for all } (u,v) \in \partial\Omega_r. \tag{41}$$

Therfore, by applying Lemma 2.9 to (40) and (41), the system (1) has a positive solution for $\mu_1 > \mu_1^*$ and $\mu_2 > \mu_2^*$. □

Theorem 3.6. *Assume that conditions $(H_1)-(H_2)$ are fullfilled.*
If $\{f^\infty = g^\infty = 0, f_0, g_0 \in (0,\infty)\}$, then there exist $\mu_1^*, \mu_2^* > 0$ such that for each $\mu_1 > \mu_1^*$ and $\mu_2 > \mu_2^*$, the system (1) has at least one positive solution.

Proof. Using Lemma 2.19 combined with Lemma 2.18, by the same manner as the proof of Theorem 3.5, we prove that problem (1) has at least one positive solution for every $\mu_1 > \mu_1^*$ and $\mu_2 > \mu_2^*$. □

4. Multiplicity results

In this Section, we state two existence results.

Theorem 4.1. *Suppose that conditions $(H_1)-(H_3)$ hold. In addition, suppose that $\{f_0 = \infty$ or $g_0 = \infty\}$ and $\{f_\infty = \infty$ or $g_\infty = \infty\}$, then there exist $\mu_1^*, \mu_2^* > 0$ such that for each $0 < \mu_1 < \mu_1^*$ and $0 < \mu_2 < \mu_2^*$, the problem (1) has two positive solutions.*

Proof. Choose two numbers $R_2 > R_1 > 0$. By Lemma 2.12, there exist $\mu_1^* > 0$ and $\mu_2^* > 0$ such that $0 < \mu_1 < \mu_1^*$, $0 < \mu_2 < \mu_2^*$ we have

$$\|T(u,v)\| \leq \|(u,v)\|, \text{for } (u,v) \in \partial\Omega_{R_i}, i = 1,2. \tag{42}$$

Since $f_0 = \infty$ or $g_0 = \infty$, then by Lemma 2.13, we can choose $0 < r < \min\{\frac{R_1}{2}, r_0(\mu_1,\mu_2)\}$ such that

$$\|T(u,v)\| \geq \|(u,v)\|, \text{ for } (u,v) \in \partial\Omega_r.$$

Moreover, since $\{f_\infty = \infty$ or $g_\infty = \infty\}$, by Lemma 2.16 we can choose $R > \max\{2R_2, R_0(\mu_1,\mu_2)\}$ such that

$$\|T(u,v)\| \geq \|(u,v)\|, \text{for } (u,v) \in \partial\Omega_R.$$

Then, from Lemma 2.9, T has two fixed points (u_1,v_1) and (u_2,v_2) such that $(u_1,v_1) \in \overline{\Omega_{R_1}}\backslash\Omega_r$ and $(u_2,v_2) \in \overline{\Omega_R}\backslash\Omega_{R_2}$. Thus (u_1,v_1) and (u_2,v_2) are the desired distinct positive solutions of system (1) for $0 < \mu_1 < \mu_1^*$ and $0 < \mu_2 < \mu_2^*$ satisfying

$$r \leq \|(u_1,v_1)\| \leq R_1 < R_2 \leq \|(u_2,v_2)\| \leq R.$$

This complets the proof. □

Theorem 4.2. *Suppose that conditions* (H_1)–(H_3) *hold. In addition, suppose that* $\{f^0 = g^0 = 0, f^\infty = g^\infty = 0\}$, *then there exist* $\mu_1^*, \mu_2^* > 0$ *such that for each* $0 < \mu_1 < \mu_1^*$ *and* $0 < \mu_2 < \mu_2^*$, *the problem (1) has two positive solutions.*

Proof. By combining Lemma 2.17 and Lemma 2.15, we follow the result of Theorem (4.2) by the same manner as the proof of Theorem (4.1), so we omit it. □

5. Nonexistence of positive solutions

We give in this Section nonexistence results for the system (1).

Theorem 5.1. *Suppose that conditions* (H_1)–(H_3) *are fulfilled. In addition assume that* f^∞, f^0, g^∞, $g^0 \in (0,\infty)$, *then there exist* μ_1^*, $\mu_2^* > 0$ *such that problem (1) has no positive solution for all* $0 < \mu_1 < \mu_1^*$ *and* $0 < \mu_2 < \mu_2^*$.

Proof. Suppose the contrary that is the problem (1) admits a positive solution. Since f^∞, $f^0 \in (0,\infty)$, then there exist $\epsilon_1, \epsilon_2, l_1, l_2$ such that
$l_1 < l_2$ and for each $t \in [0,1]$ we have

$$f(t,u,v) \leq \epsilon_1 l_1, \quad 0 \leq u+v \leq l_1.$$

and

$$f(t,u,v) \leq \epsilon_2 l_2, \qquad u+v \geq l_2.$$

Let

$$L_1 = \max\{l_1, l_2, \max_{\substack{t\in[0,1]\\ l_1\leq u+v\leq l_2}} \frac{f(t,u,v)}{u+v}\}.$$

Thus, we obtain

$$f(t,u,v) \leq L_1(u+v), \qquad \forall u,v \geq 0.$$

By the same manner, there exists $L_2 > 0$ such that

$$g(t,u,v) \leq L_2(u+v), \quad \forall u,v \geq 0.$$

Define $\mu_1^* = \frac{1}{2\eta_\alpha \sigma_\alpha^0 L_1}$ and $\mu_2^* = \frac{1}{2\eta_\beta \sigma_\beta^0 L_2}$. Let $\mu_1 \in (0, \mu_1^*)$ and $\mu_2 \in (0, \mu_2^*)$. Then, by using Lemma 2.10, we get

$$(u, v) = (T_1(u, v), T_2(u, v)).$$

So

$$\begin{aligned} u(t) &\leq \mu_1 \eta_\alpha \sigma_\alpha^0 L_1(\|u\| + \|v\|) \\ &< \mu_1^* \eta_\alpha \sigma_\alpha^0 L_1 \|(u, v)\| \\ &= \frac{\|(u, v)\|}{2}. \end{aligned}$$

Similarly, we obtain

$$v(t) < \frac{\|(u, v)\|}{2}.$$

Therfore, we get $\|(u, v)\| < \|(u, v)\|$, which is a contradiction. The proof is complete. □

Theorem 5.2. *If (H_1)–(H_3) hold. If $\{f_0 > 0, f_\infty > 0\}$ or $\{g_0 > 0, g_\infty > 0\}$ then there exists $\mu_1 > 0$ or $\mu_2 > 0$ such that problem (1) has no positive solution for all $\mu_1 > \mu_0$ or $\mu_2 > \mu_0$.*

Proof. Suppose that $\{f_0 > 0, f_\infty > 0\}$, then there exists $L_1 > 0$ such that $f(t, u, v) \geq L_1(u + v)$ for all $u \geq 0, v \geq 0$. Define

$$\mu_0 = \frac{1}{\gamma \gamma_\alpha L_1 \sigma_\alpha^\theta}.$$

Let $\mu_1 > \mu_0$ and suppose that (1) has a positive solution (u, v) then, by Lemma 2.10, (u, v) satisfy (25) and (26). Thus (u, v) is necessary in the cone Ω. From Proposition 2.7 (iii), we have for each $t \in J_\theta$

$$\begin{aligned} u(t) &\geqslant \mu_1 \gamma_\alpha \int_\theta^{1-\theta} K_\alpha(s) a(s) f(s, u(s), v(s)) ds \\ &\geqslant \mu_1 \gamma_\alpha L_1 \int_\theta^{1-\theta} K_\alpha(s) a(s)(u(s) + v(s)) ds \\ &\geqslant \mu_1 \gamma_\alpha L_1 \sigma_\alpha^\theta (\gamma_\alpha^* \|u\| + \gamma_\beta^* \|v\|) \\ &> \mu_0 \gamma_\alpha L_1 \sigma_\alpha^\theta \gamma \|(u, v)\| = \|(u, v)\|. \end{aligned}$$

Thus

$$\|u\| > \|(u, v)\|.$$

Which is a contradiction. Now, suppose that $\{g_0 > 0, g_\infty > 0\}$, then there exists $L_2 > 0$ such that $g(t, u, v) \geq L_2(u + v)$ for all $u \geq 0, v \geq 0$. Define

$$\mu_0 = \frac{1}{\gamma \gamma_\beta L_2 \sigma_\beta^\theta}.$$

Let $\mu_2 > \mu_0$, by the same manner, we obtain

$$\|v\| > \|(u, v)\|.$$

Which is a contradiction.
Consequently, we conclude that there exists $\mu_0 > 0$ such that the system (1) has no positive solution if either $\mu_1 > \mu_0$ or $\mu_2 > \mu_0$. □

6. Examples

In this section, we present some examples in order to illustrate our results. We remark that in the following examples, it is immediate to verify that conditions (H_1), (H_2) and (H_3) hold.

Example 6.1. *We consider the following nonlinear fractional differential equations*

$$\begin{cases} D^{\frac{5}{2}}u(t) + \mu_1 \frac{1}{t(1-t)}(u(t))^{\frac{2}{3}} = 0, \quad in \ (0,1), \\ D^{\frac{5}{2}}v(t) + \mu_2 \frac{1}{t(1-t)}(v(t))^{\frac{2}{3}} = 0, \quad in \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = 2\int_0^1 u(s)ds, \\ v(0) = v'(0) = 0, \quad v(1) = 2\int_0^1 v(s)ds. \end{cases} \tag{43}$$

Let $f(t,u,v) = u^{\frac{2}{3}}$, $g(t,u,v) = v^{\frac{2}{3}}$, $a(t) = b(t) = \frac{1}{t(1-t)}$, $\lambda_1 = \lambda_2 = \lambda = 2$. *By direct calculation, we obtain* $f_0 = \infty$ *and* $f^\infty = 0$. *From Theorem 3.1, we deduce that problem (43) has at least one positive solution for every* $\mu_1, \mu_2 > 0$. *In particular for* $\mu_1 = \mu_2 = \mu$, *the problem (43) admits a positive solution. Consequently, the following boundary value problem*

$$\begin{cases} D^{\frac{5}{2}}u(t) + \mu \frac{1}{t(1-t)}(u(t))^{\frac{2}{3}} = 0 \quad in \ (0,1), \\ u(0) = u'(0) = 0, u(1) = 2\int_0^1 u(s)ds, \end{cases} \tag{44}$$

admits at least one positive solution for each $\mu > 0$.
We remark here that problem (44) can not be treated by Theorem (3.2) in [6] concerning problem (3). This due to the fact that the conditions on $g(t) = \frac{1}{t(1-t)}$ *are not required.*

Example 6.2. *Consider the following boundary value problem*

$$\begin{cases} D^{\frac{5}{2}}u(t) + \mu_1 \frac{1}{t(1-t)}((1+t^2)\exp(\sqrt{u+v})) = 0, \quad in \ (0,1), \\ D^{\frac{7}{3}}v(t) + \mu_2 \frac{1}{t}((2+t)\sqrt{u+v}\cos^2(u+v)) = 0, \quad in \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \frac{3}{2}\int_0^1 u(s)ds, \\ v(0) = v'(0) = 0, \quad v(1) = \frac{1}{2}\int_0^1 v(s)ds. \end{cases} \tag{45}$$

Set $\alpha = \frac{5}{2}, \beta = \frac{7}{3}, a(t) = \frac{1}{t(1-t)}, b(t) = \frac{1}{t}$. *Let* $f(t,u,v) = (1+t^2)\exp(\sqrt{u+v})$ *and* $g(t,u,v) = (2+t)\sqrt{u+v}\cos^2(u+v)$. *So, we get* $f_0 = \infty$ *and* $g_0 = \infty$, *then we can apply the first assertion of Theorem 3.4. Let* $R = 1$. *Then, by Lemma 2.12, we get* $\mu_1^*(1) \approx 9.6952 \times 10^{-2}$, $\mu_2^*(1) \approx 1.4855$. *According the proof of Theorem 3.4, we get* $0 < \|(u,v)\| \leq 1$. *So, problem (45) admits a positive solution for* $0 < \mu_1 < 0.096952$ *and* $0 < \mu_2 < 1.4855$.

Example 6.3. *Consider the following boundary value problem*

$$\begin{cases} D^{\frac{7}{3}}u(t) + \mu_1 \frac{1}{1-t}(1+t^2)\exp(u+v) = 0, \quad in \ (0,1), \\ D^{\frac{5}{2}}v(t) + \mu_2 \frac{1}{t(1-t)}((u+v)^2 + \ln(2+(u+v)) = 0, \quad in \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = 2\int_0^1 u(s)ds, \\ v(0) = v'(0) = 0, \quad v(1) = \int_0^1 v(s)ds. \end{cases} \tag{46}$$

Let $f(t,u,v) = (1+t^2)\exp(u+v)$, $g(t,u,v) = (u+v)^2 + \ln(2+u+v)$, $a(t) = \frac{1}{1-t}$ *and* $b(t) = \frac{1}{t(1-t)}$. *By direct calculation, we obtain* $f_0 = f_\infty = \infty$, $g_0 = g_\infty = \infty$. *From proof of Theorem 4.1 and using the same notations we choose* $R_1 = \frac{1}{2}$ *and* $R_2 = 1$. *A simple calculs yields to* $\mu_1^* = 0.024334$, $\mu_2^* = 0.017904$. *So Theorem 4.1 ensures the existence of two solutions* (u_1, v_1) *and* (u_2, v_2) *of problem (1) for every* $0 < \mu_1 < \mu_1^*$ *and* $0 < \mu_2 < \mu_2^*$, *such that*

$$0 \leq \|(u_1, v_1)\| \leq \frac{1}{2} < 1 \leq \|(u_2, v_2)\|.$$

Example 6.4. *Consider the following boundary value problem*

$$\begin{cases} D^{\frac{7}{3}}u(t) + \mu_1 \frac{1}{t^2}(2t+1)(u+v) = 0, \quad in \ (0,1), \\ D^{\frac{5}{2}}v(t) + \mu_2 \frac{1}{\sqrt{t}}(u+v+\ln(1+u+v)) = 0, \quad in \ (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = 2\int_0^1 u(s)ds, \\ v(0) = v'(0) = 0, \quad v(1) = \int_0^1 v(s)ds, \end{cases} \tag{47}$$

Let $\alpha = \frac{7}{3}$, $\beta = \frac{5}{2}$, $a(t) = \frac{1}{t^2}$, $b(t) = \frac{1}{\sqrt{t}}$. *Let* $f(t,u,v) = (2t+1)(u+v)$ and $g(t,u,v) = (u+v+\ln(1+u+v))$, *we verify that* $f^0 = f^\infty = 3$, $g^\infty = 1$ *and* $g^0 = 2$, *then we can apply Theorem 5.1.*
A simple calculation shows that $\mu_1^* = 0.198\,44$ *and* $\mu_2^* = 1.\,527\,9$. *Then for* $0 < \mu_1 < 0.198\,44$ *and* $0 < \mu_2 < 1.\,527\,9$ *problem (47) has no positive solution.*

References

[1] B. Ahmed, A. Alsaedi, Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fractional differential equations, Fixed point Theory Appl., (2010), 1-17.
[2] R. L. Bagley, P. J. Torvik. A Theoretical Basis for the Application of Fractional Calculus to Viscoelasticity. Journal of Rheology. 27, (1983), 201-210.
[3] R. L. Bagley, P. J. Torvik. On the appearance of the fractional derivative in the behavior of real materials. J. Appl. Mec., 51, (1984), 294-298.
[4] Z. Bai, H. L¨u, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311, (2005), 495-505.
[5] A. Cabada, Z. Hamdi. Nonlinear fractional differential equations with integral boundary value conditions, Applied Mathematics and Computation,228, (2014), 251-257.
[6] A. Cabada, Z. Hamdi. Multiplicity Results for Integral Boundary Value Problems of Fractional Order with Parametric Dependence. Fractional Calculus and Applied Analysis 18.1, (2015), 223-237.
[7] A. Cabada, G. Wang. Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. J. Math. Anal. Appl. 389(1), (2012), 403-411.
[8] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, J. Math. Anal. Appl. 204, (1996), 609-625.
[9] S. M. Ege, F. S. Topal, Existence of Multiple Positive Solutions for Semipositone Fractional Boundary Value Problems, Filomat, 33 (3), (2019), 749–759.
[10] M. El-Shahed, Positive solutions for boundary value problems of nonlinear fractional differential equation, Abs. Appl. Anal. 18, Article ID 10368,(2007).
[11] M. Feng, X.Zhang,W.Ge, New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions, Bound. Value. Probl., (2011), 1–20.
[12] V.D. Gejji, A. Babakhani, Analysis of a system of fractional differential equations, J. Math. Anal. Appl. 293,(2004), 511-522.
[13] L. Gaul, P. Klein, S. Kempfle, Damping description involving fractional operators, Mech. Syst. Signal Process. 5, (1991), 81-88.
[14] J. Henderson, R. Luca. Positive solutions for a system of nonlocal fractional boundary value problems. Fract. Calc. Appl. Anal. 16(4), (2013), 985-1008.
[15] J. Henderson, R. Luca. Positive solutions for a system of fractional differential equations with coupled integral boundary conditions. Appl. Math. Comput. 249, (2014), 182-197.
[16] J. Henderson,R. Luca, A. Tudorache. Positive Solutions for Systems of Coupled Fractional Boundary Value Problems. Open Journal of Applied Sciences, 5(10), (2015), 600-608.
[17] M. Kratou, Ground State Solutions of p-Laplacian Singular Kirchhoff Problem Involving a Riemann-Liouville Fractional Derivative, Filomat, 33 (7), (2019), 2073–2088.
[18] M.A. Ragusa, Elliptic boundary value problem in Vanishing Mean Oscillation hypothesis, Commentationes Mathematicae Universitatis Carolinae, 40 (4), (1999), 651–663.
[19] M.A. Ragusa, Necessary and sufficient condition for a VMO function, Applied Mathematics and Computation 218 (24), (2012), 11952–11958.
[20] W. R. Schneider and W. Wyss, Fractional Diffusion and Wave Equations, Journal of Mathematical Physics, Vol. 30, No. 1, (1989), 134-144.
[21] F. Toumi. Existence and nonexistence of positive solutions for a system of nonlinear singular fractional differential equations. Fractional Differential Calculus. Vol. 5, (2015), 183-198.
[22] G. Wang, B. Ahmad, L. Zhang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order, Nonlinear Anal., 74, (2011), 792-804.
[23] W. Wyss, The fractional diffusion equation, J. Math. Phys. 27, 2782 (1986).
[24] S.Q. Zhang, Existence of positive solution for some class of nonlinear fractional differential equations, J. Math. Anal. Appl. 278, (2003), 136-148.

[25] S.Q. Zhang, The existence of a positive solution for a nonlinear fractional differential equation, J. Math. Anal. Appl. 252, (2000), 804-812.

[26] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics studies, Vol. 204, Elsevier, Amsterdam, (2006).

[27] M. A. Krasnosel'skii, Positive Solutions of Operator Equations, Noordhoff, Groningen, (1964).

[28] I. Podlubny, Fractional Differential Equations. Ser. Mathematics in Science and Engineering, Academic Press, New York (1999).

[29] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation. Fractional Calculus and Applied Analysis. 5, , 367-386, (2002).

[30] I. Podlubny, The Laplace transform method for linear differential equations of the fractional order, Inst. Expe. Phys., Slov. Acad. Sci., UEF-02-94, Kosice, 1994.

[31] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, Switzerland, (1993).

[32] W.R. Schneider. Fractional diffusion. Lecture Notes Phys., 355, 276-286, (1990).