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On Relative Essential Spectra of a 3×3 Operator Matrix Involving Relative Generalized Weak Demicompactness

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Abstract. In this paper, we investigate the relative essential spectra of a 3×3 block matrix operator with unbounded entries and with domain consisting of vectors satisfying certain relations between their components. Our results are formulated in term of relative generalized weak demicompactness and measure of non-strict-singularity.

1. Introduction

During the last years, e.g. the papers [2, 27] were devoted to the study of the Wolf essential spectrum of operators represented by a 2×2 block matrix acting on a product of Banach spaces. An account of the research and a wide panorama of methods to investigate the spectrum of the unbounded block operator matrices are presented by C. Tretter in [28, 29, 30].

In the theory of unbounded block operator matrices, the Frobenius-Schur factorization is a basic tool to study the spectrum and various spectral properties. This was first recognized by R. Nagel in [20, 21] and, independently and under slightly different assumptions, later in [2]. In [13], A. Jeribi, N. Moalla and I. Walha extended the results developed by F. V. Atkinson et al in [2] for a 3×3 block matrix operator. In [5], inspired by the ideas of the paper of [3], A. Ben amar, A. Jeribi and B. Krichen extended the previous results to a 3×3 block operator matrices

$$\mathcal{L}_0 = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & L \end{pmatrix}, \tag{1.1}$$

with domain consisting of vectors satisfying certain relations of the form $\Gamma_X x = \Gamma_Y y = \Gamma_Z z$ between the components of its elements.

Definition 1.1. An operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be demicompact if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $x_n - Tx_n \to x \in X$, there exists a convergent subsequence of $(x_n)_n$.

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In [1, 22], a fundamental role is played by demicompact linear operators to establish some results in Fredholm theory. In 2014, B. Krichen [15] gave a generalization of this notion by introducing the class of relative demicompact linear operators. Recently, W. Chaker, A. Jeribi and B. Krichen [6, 7] continued this study to investigate the essential spectra of densely defined linear operators. In 2018, B. Krichen and D. O'Regan developed in [16] some Fredholm and perturbation results involving a new concept, called weakly relative demicompactness for nonlinear operators. In [17], the same authors studied the relationship between the class of weakly demicompact linear operators and an axiomatic measures of weak noncompactness of linear operator. In 2019, I. Ferjani, A. Jeribi and B. Krichen [8], introduced the notion of generalized weak demicompactness as a generalization of the class of demicompact, they gave their relationship with Fredholm and upper semi-Fredholm operators. A characterization by means of upper semi- Browder spectrum, was also given. Moreover, they ensured the generalized weak demicompactness of the closure of a closable block matrix operator. In [9], I. Ferjani, A. Jeribi and B. Krichen continued the analysis started in [8] and extended it to more general classes by introducing the concept of relative generalized weak demicompactness (see Definition 2.6.).

In the present paper, we extend the results of [5] and we focus on the investigation of the closability and the description of the M-essential spectra where,

$$M = \begin{pmatrix} M_1 & M_4 & M_5 \\ M_6 & M_2 & M_7 \\ M_8 & M_9 & M_3 \end{pmatrix}. \tag{1.2}$$

We determine the M-essential spectra of the closure of a 3×3 block operator matrices (1.1) without knowing the M-essential spectra of the operator A but only that of one of its restrictions involving the concept of relative generalized weak demicompactness. Furthermore, we give some results on this last concept by means of measure of non-strict-singularity.

This paper is organized as follows: In the next section, we recall some definitions and preliminary results. Furthermore, we describe the closure of the operator in (1.1) under certain assumptions on its entries. In section 3, we determine the M-essential spectra of this closure involving the concept of relative generalized weak demicompactness. In section 4, a characterization by means of measure of non-strict-singularity is given.

2. Preliminary results

In this section, we will give some notations, definitions and preliminary results that are necessary in the sequel.

Let X and Y be two Banach spaces and let T be an operator acting from X into Y. We denote by $\mathcal{D}(T) \subset X$ its domain and $\mathcal{R}(T) \subset Y$ its range. We denote by $\mathcal{C}(X, Y)$ (resp. $\mathcal{L}(X, Y)$) the set of all closed, densely defined linear operators (resp. the Banach algebra of all bounded linear operators) from X into Y. The subset of $\mathcal{L}(X, Y)$ of all compact operators is denoted by $\mathcal{K}(X, Y)$. For $T \in \mathcal{C}(X, Y)$, $\mathcal{N}(T)$ denotes the Kernel of T. The nullity, $\alpha(T)$ is defined as the dimension of $\mathcal{N}(T)$ and the deficiency, $\beta(T)$ of T is defined as the codimension of $\mathcal{R}(T)$ in Y. We denote by $asc(T)$ the ascent of T, i.e. the smallest non-negative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$. An operator $T \in \mathcal{L}(X)$ is said to be weakly compact, if $T(M)$ is relatively weakly compact for every bounded subset $M \subseteq X$. The family of weakly compact operators on X, is denoted by $\mathcal{W}(X)$.

The set of upper semi-Fredholm operators from X into Y is defined by

$$\Phi_+(X, Y) := \{T \in \mathcal{C}(X, Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$

the set of lower semi-Fredholm operators from X into Y is defined by

$$\Phi_-(X, Y) := \{T \in \mathcal{C}(X, Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$

the set of Fredholm operators from X into Y is defined by

$$\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y).$$

If $T \in \Phi(X,Y)$, the number $i(T) := \alpha(T) - \beta(T)$ is called the index of T.
If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, and $\Phi_-(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$, respectively. If $T \in \mathcal{C}(X)$, we denote by $\rho(T)$ the resolvent set of T and by $\sigma(T)$ the spectrum of T. For $x \in \mathcal{D}(T)$, the graph norm $\|.\|_T$ of x is defined by $\|x\|_T = \|x\| + \|Tx\|$.
Let $T \in \mathcal{C}(X)$. We recall the following essential spectra :

$$\sigma_{e_1}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_+(X)\} := \mathbb{C}\backslash\Phi_{+T},$$

$$\sigma_{e_4}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi(X)\} := \mathbb{C}\backslash\Phi_{T},$$

$$\sigma_{e_5}(T) := \mathbb{C}\backslash\rho_5(T),$$

where $\rho_5(T) := \{\lambda \in \mathbb{C}$ such that $\lambda \in \Phi_T$ and $i(\lambda - T) = 0\}$.

Now, we will recall some well known properties of the Fredholm sets.

Definition 2.1. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$.

(*i*) The operator F is called a Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$.
(*ii*) F is called an upper (resp. lower) semi-Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ (resp. $U + F \in \Phi_-(X,Y)$) whenever $U \in \Phi_+(X,Y)$ (resp. $U \in \Phi_-(X,Y)$).

We denote by $\mathcal{F}(X,Y)$ the set of Fredholm perturbation and by $\mathcal{F}_+(X,Y)$ (resp. $\mathcal{F}_-(X,Y)$) the set of upper (resp. lower) semi-Fredholm perturbations.

Remark 2.1. Let $\Phi^b(X,Y)$ denote the set $\Phi(X,Y) \cap \mathcal{L}(X,Y)$. If in Definition 2.1 we replace $\Phi(X,Y)$ by $\Phi^b(X,Y)$, we obtain the sets $\mathcal{F}^b(X,Y)$, $\mathcal{F}_+^b(X,Y)$ and $\mathcal{F}_-^b(X,Y)$.

Definition 2.2. A Banach space X is said to have the Dunford-Pettis property (in short DP property) if every bounded weakly compact operator T from X into another Banach space Y transforms weakly compact sets on X into norm-compact sets on Y.

Remark 2.2. If X is Banach space with DP property, then

$$\mathcal{W}(X) \subset \mathcal{F}(X).$$

Definition 2.3. Let X and Y be two Banach spaces. An operator $S \in \mathcal{L}(X)$ is said to be strictly singular if the restriction of S to any infinite-dimensional subspace of X is not an homeomorphism.

Let $\mathcal{S}(X,Y)$ denote the set of strictly singular operators from X to Y.
The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [14] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators we refer to [11, 14]. Note that $\mathcal{S}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$. In general, strictly singular operators are not compact and if $X = Y$, $\mathcal{S}(X) = \mathcal{S}(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$.
Let us recall the definition of Hausdorff measure of noncompactness (see [24]).

Definition 2.4. For a bounded subset Ω of X we consider

$$q(\Omega) = \inf\{r > 0, \Omega \text{ can be covred by finite set of open ball of radius } r\}.$$

The Hausdorff measure of noncompactness of $A \in \mathcal{L}(X,Y)$ is defined by

$$q(A) = q[A(B_X)],$$

where B_X denotes the closed unit ball in X, that is, the set of all $x \in X$ satisfying $\|x\| \leq 1$.
It was proved in [18] that

$$q(A) \leq \|A\|,$$

$$q(A) = 0 \text{ if and only if } A \in \mathcal{K}(X,Y),$$
$$q(A+K) = q(A), \text{ for all } K \in \mathcal{K}(X,Y).$$

Definition 2.5. For $A \in \mathcal{L}(X,Y)$, set

$$g_M = \inf_{N \subset M} q(A_{|N}) \text{ and } g(A) = \sup_{M \subset X} g_M(A), \tag{2.1}$$

where M, N represent infinite dimensional subspaces of X, and $A_{|N}$ denotes the restriction of A to the subspace N.The semi-norm g is a measure of non-strict singularity, it was introduced by Schechter in [26].

We recall the following result established in [23].

Proposition 2.1. For $A \in \mathcal{L}(X,Y)$,

(*i*) $A \in \mathcal{S}(X,Y)$ if, and only if $g(A) = 0$.
(*ii*) $A \in \mathcal{S}(X,Y)$ if, and only if $g(A+T) = g(T)$ for all $T \in \mathcal{L}(X,Y)$.
(*iii*) If Z is a Banach space and $B \in \mathcal{L}(Y,Z)$, then $g(BA) \leq g(B)g(A)$.

Now, we recall the following results founding in [9]:

Definition 2.6. Let X be a Banach space and let $A, S \in \mathcal{C}(X)$ with $\mathcal{D}(A) \subset \mathcal{D}(S)$. A is called a generalized weakly S-demicompact operator if there exists a finite subset E of $\mathbb{C}$ containing 0 such that:

(*i*) For all $\lambda \in \mathbb{C}\backslash E$, $\frac{1}{\lambda}A$ is weakly S-demicompact operator,
(*ii*) for all $\lambda \in \mathbb{C}\backslash E$, $\lambda S - A$ has a finite ascent, and
(*iii*) all $\lambda \in \sigma_S(A)\backslash E$, are eigenvalues of finite multiplicity and have no accumulation points except possibly points of E.

The set E is called a generalized set of A.

Remark 2.3. It should be noted that if, E is a generalized set of A and G is a finite subset of $\mathbb{C}$ containing E, then G is also a generalized set of A.

Theorem 2.1. Let X be a Banach space, $T \in \mathcal{C}(X)$ and $S \in \mathcal{L}(X)$ such that $0 \in \rho(S)$, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$ and $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$. Then, T is a generalized weakly S-demicompact if, and only if, there exists a finite subset $E \subset \mathbb{C}$ containing 0 such that $\lambda S - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$.

Theorem 2.2. Let X be a Banach space, $T \in \mathcal{C}(X)$ and $S \in \mathcal{L}(X)$ such that $0 \in \rho(S)$, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$ and $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$. If μT is a generalized weakly S-demicompact operator for each $\mu \in [0,1]$ with a generalized subset E, then $\lambda S - T \in \Phi(X)$ and $i(\lambda S - T) = i(\lambda S)$, for all $\lambda \in \mathbb{C}\backslash E$.

In this work we are concerned with the M-essential spectra of operators defined by a 3×3 block matrix operators (1.1), where the entries of the matrix are in general unbounded operators. The operator (1.1) is defined on $(\mathcal{D}(A) \cap \mathcal{D} \cap \mathcal{D}(G)) \times (\mathcal{D}(B) \cap \mathcal{D}(E) \cap \mathcal{D}(H)) \times (\mathcal{D}(C) \cap \mathcal{D}(F) \cap \mathcal{D}(L))$.

Let X, Y, Z and W be Banach spaces. We consider the block matrix operator (1.1) in the space $X\times Y\times Z$, that is the linear operator A acts in X, E in Y and L in Z, B from Y to X. We assume that operators Γ_X, Γ_Y, Γ_Z are given, acting from X, Y, Z, respectively, into W. In what follows, we will consider the following assumptions.

($H1$) The operator A is densely defined and closable. Then $\mathcal{D}(A)$, equipped with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ can be completed to a Banach space X_A which coincides with $\mathcal{D}(\overline{A})$, the domain of the closure of A in X.

($H2$) $\mathcal{D}(A) \subset \mathcal{D}(\Gamma_X) \subset X_A$ and $\Gamma_X : X_A \longrightarrow W$ is a bounded mapping. Denote by $\overline{\Gamma}_X$ the extension by continuity which is a bounded operator from X_A into W.

($H3$) $\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)$ is dense in X and the M_1-resolvent set of the restriction $A_1 := A|_{\mathcal{D}(A)\cap\mathcal{N}(\Gamma_X)}$ is not empty: $\rho_{M_1}(A_1) \neq \emptyset$.

($H4$) The operator B is densely defined and for some (and hence for all) $\mu \in \rho_{M_1}(A_1)$ the operator $(A_1 - \mu M_1)^{-1}(B - \mu M_4)$ is bounded.

($H5$) $\mathcal{D}(A) \subset \mathcal{D}(D) \subset X_A$, $\mathcal{D}(A) \subset \mathcal{D}(G) \subset X_A$ and D and G are a closable operators from X_A into Y and X_A into Z, respectively.

Taking into account the assumption ($H5$) and apply the closed graph theorem, it follows that for $\lambda \in \rho_{M_1}(A_1)$ $F_1(\lambda) := (D - \lambda M_6)(A_1 - \lambda M_1)^{-1}$ and $F_2(\lambda) := (G - \lambda M_8)(A_1 - \lambda M_1)^{-1}$ are bounded operators from X into Y and X into Z, respectively.

From Lemma 3.1 in [31], $\mathcal{D}(A)$ was decomposed as follows:

$$\mathcal{D}(A) = \mathcal{D}(A_1) \oplus \mathcal{N}(A - \mu M_1)$$

for every $\mu \in \rho_{M_1}(A_1)$ and the restriction of Γ_X to $\mathcal{N}(A - \mu M_1)$ is injective. Denote the inverse of $\Gamma_X|_{\mathcal{N}(A-\mu M_1)}$ by $K_\mu := (\Gamma_X|_{\mathcal{N}(A-\mu M_1)})^{-1}$.

Remark 2.4. K_μ is closable if, and only if, K_λ is closable, in which case we have $\overline{K}_\mu - \overline{K}_\lambda = (\mu - \lambda)(A_1 - \mu M_1)^{-1} M_1 \overline{K}_\lambda$. ($H6$) For some $\mu \in \rho_{M_1}(A_1)$, K_μ is a bounded operator from $\Gamma_X(\mathcal{D}(A))$ into X, its extension by continuity to $\overline{\Gamma_X(\mathcal{D}(A))}$ is denoted by $\overline{K}_\mu$.

In the following, denote $S(\mu) := E + (D - \mu M_6)[K_\mu \Gamma_Y - (A_1 - \mu M_1)^{-1}(B - \mu M_4)]$. The operator $S(\mu)$ is defined on the domain:

$$Y_1 = \{y \in \mathcal{D}(B) \cap \mathcal{D}(E) : \ \Gamma_Y y \in \Gamma_X(\mathcal{D}(A))\}.$$

For $\mu \in \rho_{M_1}(A_1)$, denote the restriction of $S(\mu)$ to the set $Y_1 \cap \mathcal{N}(\Gamma_Y)$ by $S_1(\mu)$.

($H7$) For some $\mu \in \rho_{M_1}(A_1)$, the operator $S_1(\mu)$ is closed.

Remark 2.5. For every λ, $\mu \in \rho_{M_1}(A_1)$ we have

$$S_1(\mu) - S_1(\lambda) = (\mu - \lambda)[M_6 - F_1(\mu)M_1](A_1 - \lambda M_1)^{-1}(B - \lambda M_4) + (\mu - \lambda)F_1(\mu)M_4. \tag{2.2}$$

Remark 2.6. According to assumptions ($H4$) and ($H5$), we have the operator $F_1(\mu)M_1(A_1 - \lambda M_1)^{-1}(B - \lambda M_4)$ is bounded on its domain, which implies that if $S_1(\mu)$ is closed for some $\mu \in \rho_{M_1}(A_1)$ then it is closed for all such μ.

For $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$, the set Y_1 can be decomposed as follows:

$$Y_1 = \mathcal{D}(S_1(\mu)) \oplus \mathcal{N}(S(\mu) - \mu M_2).$$

As in [3], the inverse of $\Gamma_Y|_{\mathcal{N}(S(\mu)-\mu M_2)}$ is denoted by $J_\mu := (\Gamma_Y|_{\mathcal{N}(S(\mu)-\mu M_2)})^{-1}$,

$$J_\mu : \Gamma_Y(Y_1) \longrightarrow \mathcal{N}(S(\mu) - \mu M_2) \subset Y_1.$$

Remark 2.7. J_μ is closable if, and only if, J_λ is closable. Moreover, $\overline{J}_\mu = (S_1(\mu) - \mu M_2)^{-1}(S_1(\lambda) - \lambda M_2)\overline{J}_\lambda$.

Assume that for some $\mu \in \rho_{M_1}(A_1)$, J_μ is bounded from $\Gamma_Y(Y_1)$ into Y and its extension by continuity to

$\overline{\Gamma_Y(Y_1)}$ is denoted by $\overline{J}_\mu$.

$(H8)$ $\mathcal{D}(B) \cap \mathcal{D}(E) \subset \mathcal{D}(\Gamma_Y)$, $\mathcal{D}(B) \cap \mathcal{D}(H) \subset \mathcal{D}(\Gamma_Y)$, the set Y_1 is dense in Y and the restriction of Γ_Y to Y_1 is bounded as an operator from Y into W. The extension by continuity of $\Gamma_Y|_{Y_1}$ to Y is denoted by $\overline{\Gamma}_Y^0$.

$(H9)$ L is densely defined and closed with non empty M_3-resolvent set, i.e., $\rho_{M_3}(L) \neq \emptyset$.

$(H10)$ For some $\mu \in \rho_{M_1}(A_1)$, $G_2(\mu) := (A_1 - \mu M_1)^{-1}(C - \mu M_5)$ is bounded operator.

$(H11)$ $\mathcal{D}(C) \cap \mathcal{D}(F) \cap \mathcal{D}(L) \subset \mathcal{D}(\Gamma_Z)$, the set

$$Z_1 := \{z \in \mathcal{D}(C) \cap \mathcal{D}(F) \cap \mathcal{D}(L) : \Gamma_Z z \in \Gamma_Y(Y_1)\}$$

is dense in Z and the restriction of Γ_Z to Z_1 is bounded from Z into W. The extension by continuity of $\Gamma_Z|_{Z_1}$ to Z is denoted by $\overline{\Gamma}_Z^0$.

$(H12)$ For some (and hence for all) $\mu \in \rho_{M_1}(A_1)$, $F - (D - \mu M_6)(A_1 - \mu M_1)^{-1}(C - \mu M_5)$ is closable and its closure $\overline{F - (D - \mu M_6)(A_1 - \mu M_1)^{-1}(C - \mu M_5)}$ is bounded.

Remark 2.8. These assumptions are sufficient conditions. The optimality condition is a question which is a priori still open.

Under these assumptions, we show the closability of the operator in (1.1) and we describe the closure. As in the 2×2 case, we will use the tool of the factorization of the 3×3 matrix with a diagonal matrix of Schur complements in the middle and invertible factors to the right and to the left (see for example [32]).
We consider the Banach space $X \times Y \times Z$ and define the operator $\mathcal{L}_0$ as follows:

$$\mathcal{D}(\mathcal{L}_0) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{array}{l} x \in \mathcal{D}(A) \\ y \in \mathcal{D}(B) \cap \mathcal{D}(E) \\ z \in \mathcal{D}(C) \cap D(F) \cap \mathcal{D}(L) \end{array} , \ \Gamma_X x = \Gamma_Y y = \Gamma_Z z \right\}.$$

As in the case of a 2×2 matrix operator (see [2,29]), we introduce the following operators:

$$\begin{aligned}
&G_1(\mu) := -K_\mu \Gamma_Y + (A_1 - \mu M_1)^{-1}(B - \mu M_4), \\
&G_3(\mu) := -J_\mu \Gamma_Z + (S_1(\mu) - \mu M_2)^{-1}(F - (D - \mu M_6)(A_1 - \mu M_1)^{-1}(C - \mu M_5)), \\
&\Theta(\mu) := H + (G - \mu M_8)[K_\mu \Gamma_Y - (A_1 - \mu M_1)^{-1}(B - \mu M_4)], \\
&F_3(\mu) := \Theta(\mu)(S_1(\mu) - \mu M_2)^{-1}, \\
&S_2(\mu) := L - F_2(\mu)(C - \mu M_5) + \Theta(\mu)(J_\mu \Gamma_Z - (S_1(\mu) - \mu M_2)^{-1}(F - F_1(\mu)(C - \mu M_5))).
\end{aligned}$$

Remark 2.9.

- (i) If $\Theta(\mu)$ is closable for some $\mu \in \rho_{M_1}(A_1)$, then it is closable for all such μ.
- (ii) If for some $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$ the operator $S_2(\mu)$ is closable, then it is closable for all such μ.

The closure of $S_2(\mu)$ is denoted by $\overline{S}_2(\mu)$. Then we have

$$\begin{aligned}
\overline{S}_2(\mu) = \overline{S}_2(\lambda) &+ (\lambda - \mu)[F_2(\mu)M_1 - M_8]\left[F_3(\mu)F_1(\lambda)\overline{G_2(\mu)} - \overline{G_2(\lambda)}\right] \\
&+ (F_3(\mu) - F_3(\lambda))(S_1(\lambda) - \lambda M_2)\overline{G_3(\lambda)} + (\mu - \lambda)F_2(\mu)M_4\overline{G_3(\mu)}. \qquad (2.3)
\end{aligned}$$

Further, we consider the following operators

$$\widetilde{G}_1(\mu) := -\overline{K}_\mu \overline{\Gamma}_Y^0 + \overline{(A_1 - \mu M_1)^{-1}(B - \mu M_4)}.$$

$$\widetilde{G}_2(\mu) := \overline{(A_1 - \lambda M_1)^{-1}(C - \mu M_5)}.$$

$$\widetilde{G}_3(\mu) := -\overline{J}_\mu \overline{\Gamma}_Z^0 + (S_1(\mu) - \mu M_2)^{-1}\overline{(F - (D - \mu M_6)(A_1 - \mu M_1)^{-1}(C - \mu M_5))}.$$

Now, we give the following result.

Theorem 2.3. Under assumptions $(H1)$-$(H12)$, the operator $\mathcal{L}_0$ is closable if and only if $S_2(\mu)$ is closable for some $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$. In this case the closure $\mathcal{L}$ of $\mathcal{L}_0$ is given by

$$\mathcal{L} = \mu M + \mathcal{G}_l(\mu)\begin{pmatrix} A_1 - \mu M_1 & 0 & 0 \\ 0 & S_1(\mu) - \mu M_2 & 0 \\ 0 & 0 & \overline{S}_2(\mu) - \mu M_3 \end{pmatrix}\mathcal{G}_r(\mu),$$

where

$$\mathcal{G}_l(\mu) := \begin{pmatrix} I & 0 & 0 \\ F_1(\mu) & I & 0 \\ F_2(\mu) & F_3(\mu) & I \end{pmatrix} \text{ and } \mathcal{G}_r(\mu) := \begin{pmatrix} I & \widetilde{G}_1(\mu) & \widetilde{G}_2(\mu) \\ 0 & I & \widetilde{G}_3(\mu) \\ 0 & 0 & I \end{pmatrix}.$$

Now, rewrite the Frobenius-Schur factorization:

$$\alpha\mathcal{L} = \alpha\mu M + \mathcal{G}_l(\mu)\begin{pmatrix} \alpha\mu M_1 - \alpha A_1 & 0 & 0 \\ 0 & \alpha\mu M_2 - \alpha S_1(\mu) & 0 \\ 0 & 0 & \alpha\mu M_3 - \alpha\overline{S}_2(\mu) \end{pmatrix}\mathcal{G}_r(\mu).$$

Let $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\lambda M - \alpha\mathcal{L} &= \mathcal{G}_l(\mu)\begin{pmatrix} \lambda M_1 - \alpha A_1 & 0 & 0 \\ 0 & \lambda M_2 - \alpha S_1(\mu) & 0 \\ 0 & 0 & \lambda M_3 - \alpha\overline{S}_2(\mu) \end{pmatrix}\mathcal{G}_r(\mu) - (\lambda - \alpha\mu)\mathcal{R}(\mu) \\ &:= \mathcal{G}_l(\mu)V_\alpha(\lambda)\mathcal{G}_r(\mu) - (\lambda - \alpha\mu)\mathcal{R}(\mu). \qquad (2.4)\end{aligned}$$

Where

$$\mathcal{R}(\mu) := \begin{pmatrix} 0 & M_1\widetilde{G}_1(\mu) - M_4 & M_1\widetilde{G}_2(\mu) - M_5 \\ F_1(\mu)M_1 - M_6 & F_1(\mu)M_1\widetilde{G}_1(\mu) & U(\mu) \\ F_2(\mu)M_1 - M_8 & W(\mu) & T(\mu) \end{pmatrix},$$

with

$$U(\mu) = F_1(\mu)M_1\widetilde{G}_2(\mu) + M_2\widetilde{G}_3(\mu) - M_7,$$

$$W(\mu) = F_2(\mu)M_1\widetilde{G}_1(\mu) + F_3(\mu)M_2 - M_9,$$

$$T(\mu) = F_2(\mu)M_1\widetilde{G}_2(\mu) + F_3(\mu)M_2\widetilde{G}_3(\mu).$$

3. Generalized weak M-demicompactness for 3×3 matrix operators

Having obtained the closure $\mathcal{L}$ of the operator $\mathcal{L}_0$, in this section we will determine the generalized weak demicompactness of this operator and its M-essential spectra.

In all what follows, we will consider the following invertible matrix operator

$$M = \begin{pmatrix} M_1 & M_4 & M_5 \\ M_6 & M_2 & M_7 \\ M_8 & M_9 & M_3 \end{pmatrix}, \qquad (3.1)$$

such that $0 \in \rho(M_1) \cap \rho(M_2) \cap \rho(M_3)$.

As a first step we start by a result for a particular representation of $\mathcal{L}_0$.

Theorem 3.1. Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Z)$. Let consider the matrix operator $M \in \mathcal{L}(X \times Y \times Z)$ with the representation (3.1) and the 3×3 matrix operator

$$L_C := \begin{pmatrix} A & D & E \\ 0 & B & F \\ 0 & 0 & C \end{pmatrix},$$

where $D \in \mathcal{L}(Y, X)$, $E \in \mathcal{L}(Z, X)$, $F \in \mathcal{L}(Z, Y)$.
Assume that $M_6 \in \mathcal{F}(X, Y)$, $M_8 \in \mathcal{F}(X, Z)$ and $M_9 \in \mathcal{F}(Y, Z)$ and that $\sigma_{M_2}(B), \sigma_{M_3}(C)$ be a finite subsets of $\mathbb{C}$. Then, A is a generalized weakly M_1-demicompact, B is a generalized weakly M_2-demicompact and C is a generalized weakly M_3-demicompact operators if, and only if, L_C is a generalized weakly M-demicompact operator.

Proof. Let $\lambda \in \mathbb{C}$. Clearly, we have

$$\begin{aligned} \lambda M - L_C &= \begin{pmatrix} \lambda M_1 - A & \lambda M_4 - D & \lambda M_5 - E \\ \lambda M_6 & \lambda M_2 - B & \lambda M_7 - F \\ \lambda M_8 & \lambda M_9 & \lambda M_3 - C \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ \lambda M_6 & 0 & 0 \\ \lambda M_8 & \lambda M_9 & 0 \end{pmatrix} + \begin{pmatrix} \lambda M_1 - A & \lambda M_4 - D & \lambda M_5 - E \\ 0 & \lambda M_2 - B & \lambda M_7 - F \\ 0 & 0 & \lambda M_3 - C \end{pmatrix}. \end{aligned} \tag{3.2}$$

Since A is generalized weakly M_1-demicompact, B is generalized weakly M_2-demicompact and C is generalized weakly M_3-demicompact, then there exist three finite subsets E_1, E_2 and E_3 of $\mathbb{C}$ containing 0 such that $\lambda M_1 - A \in \Phi_+(X)$ for all $\lambda \in \mathbb{C} \backslash E_1$ and $\lambda M_2 - B \in \Phi_+(Y)$ for all $\lambda \in \mathbb{C} \backslash E_2$, and $\lambda M_3 - C \in \Phi_+(Z)$ for all $\lambda \in \mathbb{C} \backslash E_3$. From Remark 2.3, it follows that A is generalized weakly M_1-demicompact, B is generalized weakly M_2-demicompact and C is generalized weakly M_3-demicompact with a generalized set $E = E_1 \cup E_2 \cup E_3$. This allows us to get $\lambda M_1 - A \in \Phi_+(X)$, $\lambda M_2 - B \in \Phi_+(Y)$ and $\lambda M_3 - C \in \Phi_+(Z)$ for all $\lambda \in \mathbb{C} \backslash E$. Now, when applying Lemma 6.6.1 in [12] and using the fact that $M_6 \in \mathcal{F}(X, Y)$, $M_8 \in \mathcal{F}(X, Z)$ and $M_9 \in \mathcal{F}(Y, Z)$, we get $\lambda M - L_C \in \Phi_+(X \times Y \times Z)$ for all $\lambda \in \mathbb{C} \backslash E$, and for every $D \in \mathcal{L}(Y, X)$, $E \in \mathcal{L}(Z, X)$, $F \in \mathcal{L}(Z, Y)$. Hence, from Theorem 2.1, L_C is generalized weakly M-demicompact with a generalized set E.
To prove the converse, assume that L_C is a generalized weakly M-demicompact operator then, from Theorem 2.1, there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda M - L_C$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C} \backslash (E \cup \sigma_{M_2}(B) \cup \sigma_{M_3}(C))$.
From Equation (3.2), we have

$$\lambda M - L_C = \mathcal{H} + \begin{pmatrix} \lambda M_1 - A & \lambda M_4 - D & \lambda M_5 - E \\ 0 & \lambda M_2 - B & \lambda M_7 - F \\ 0 & 0 & \lambda M_3 - C \end{pmatrix},$$

where $\mathcal{H} = \begin{pmatrix} 0 & 0 & 0 \\ \lambda M_6 & 0 & 0 \\ \lambda M_8 & \lambda M_9 & 0 \end{pmatrix}$.

Now, we put the following factorization

$$\lambda M - L_C = \mathcal{H} + \mathcal{N}\mathcal{B}\mathcal{C}\mathcal{A},$$

where $\mathcal{N} = \begin{pmatrix} I & (\lambda M_4 - D)(\lambda M_2 - B)^{-1} & (\lambda M_5 - E)(\lambda M_3 - C)^{-1} \\ 0 & I & (\lambda M_7 - F)(\lambda M_3 - C)^{-1} \\ 0 & 0 & I \end{pmatrix}$,

$$\mathcal{A} = \begin{pmatrix} \lambda M_1 - A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \mathcal{B} = \begin{pmatrix} I & 0 & 0 \\ 0 & \lambda M_2 - B & 0 \\ 0 & 0 & I \end{pmatrix}, \text{ and } C = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \lambda M_3 - C \end{pmatrix}.$$

Taking into account that $\lambda M - L_C \in \Phi_+(X \times Y \times Z)$ and using the fact that $\mathcal{H} \in \mathcal{F}(X \times Y \times Z)$, it follows from Theorem 5.32 in [25] and Lemma 6.6.2 in [12], that $\lambda M_1 - A$, $\lambda M_2 - B$ and $\lambda M_3 - C$, are upper semi-Fredholm operators, for all $\lambda \in \mathbb{C}\backslash(E \cup \sigma_{M_2}(B) \cup \sigma_{M_3}(C))$.
Consequently, in view of Theorem 2.1, A is a generalized weakly M_1-demicompact operator, B is a generalized weakly M_2-demicompact operator and C is a generalized weakly M_3-demicompact operator. □

Now, we are in position to state the following result.

Theorem 3.2. Let X, Y and Z be Banach spaces and $M \in \mathcal{L}(X \times Y \times Z)$ with the representation (3.1). Assume that the operator $\mathcal{L}_0$ defined in Equation (1.1) satisfies (H_1)-(H_{12}) and let $\mathcal{L}$ be its closure. Let $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$ and $\lambda \in \mathbb{C}$. If for $t \in [0,1]$, the operators tA_1 is a generalized weakly M_1-demicompact, $tS_1(\mu)$ is a generalized weakly M_2-demicompact, $t\overline{S}_2(\mu)$ is a generalized weakly M_3-demicompact and $\mathcal{R}(\mu) \in \mathcal{F}_+(X \times Y \times Z)$, then $\alpha\mathcal{L}$ is generalized weakly M-demicompact for all $\alpha \in [0,1]$.

Proof. Let $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{C}$. According to Equation (2.4), we have

$$\lambda M - \alpha\mathcal{L} = \mathcal{G}_l(\mu)V_\alpha(\lambda)\mathcal{G}_r(\mu) - (\lambda - \alpha\mu)\mathcal{R}(\mu).$$

Since αA_1 is a generalized weakly M_1-demicompact operator, $\alpha S_1(\mu)$ is a generalized weakly M_2-demicompact operator and $\alpha\overline{S}_2(\mu)$ is a generalized weakly M_3-demicompact operator then, in view of Theorem 2.1, there exist three finite subsets E_1, E_2 and E_3 of $\mathbb{C}$ containing 0 such that $\lambda M_1 - \alpha A_1 \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E_1$, $\lambda M_2 - \alpha S_1(\mu) \in \Phi_+(Y)$ for all $\lambda \in \mathbb{C}\backslash E_2$ and $\lambda M_3 - \alpha\overline{S}_2(\mu) \in \Phi_+(Z)$ for all $\lambda \in \mathbb{C}\backslash E_3$. From Remark 2.3, it follows that also αA_1 is generalized weakly M_1-demicompact, $\alpha S_1(\mu)$ is generalized weakly M_2-demicompact and $\alpha\overline{S}_2(\mu)$ is generalized weakly M_3-demicompact with a generalized set $E = E_1 \cup E_2 \cup E_3$. Again from Theorem 2.1, we get $\lambda M_1 - \alpha A_1 \in \Phi_+(X)$, $\lambda M_2 - \alpha S_1(\mu) \in \Phi_+(Y)$ and $\lambda M_3 - \alpha\overline{S}_2(\mu) \in \Phi_+(Z)$ for all $\lambda \in \mathbb{C}\backslash E$. Now, when applying Lemma 6.6.1 in [12], we obtain $V_\alpha(\lambda) \in \Phi_+(X \times Y \times Z)$ for all $\lambda \in \mathbb{C}\backslash E$. Taking into account the fact that $\mathcal{R}(\mu) \in \mathcal{F}_+(X \times Y \times Z)$ and the boundedness of the operators $\mathcal{G}_l(\mu)$, $\mathcal{G}_r(\mu)$ and theirs inverses, we deduce that $\lambda M - \alpha\mathcal{L} \in \Phi_+(X \times Y \times Z)$ for all $\lambda \in \mathbb{C}\backslash E$. Hence, from Theorem 2.1, $\alpha\mathcal{L}$ is a generalized weakly M-demicompact operator with a generalized set E. □

Remark 3.1.

(*i*) When we take $\alpha = 1$, we get a same result of Corollary 4.1 in [9].
(*ii*) It should be noticed that Theorem 3.2 remains true if we assume that X, Y and Z have the Dunford-Pettis property and $\mathcal{R}(\mu) \in \mathcal{W}(X \times Y \times Z)$.

Through the next theorem, we will give a characterization of the M-essential spectra involving the concept of generalized weak demicompactness. Before that, we prove the following stability lemma.

Lemma 3.1. Let $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$. If the sets $\Phi^b(Y,X)$, $\Phi^b(Z,X)$ and $\Phi^b(Z,Y)$ are not empty, and if $F_1(\mu) \in \mathcal{F}^b(X,Y)$, $F_2(\mu) \in \mathcal{F}^b(X,Z)$ and $F_3(\mu) \in \mathcal{F}^b(Y,Z)$, then $\sigma_{e_5}(S_1(\mu))$ and $\sigma_{e_5}(\overline{S}_2(\mu))$ do not depend on the choice of μ.

Proof. Using Equation (2.2), assumption $(H4)$, [4,Theorem 3.1] and [10,Theorem 3.2 (*ii*)], we infer that $\sigma_{e_5}(S_1(\mu)) = \sigma_{e_5}(S_1(\lambda))$. Hence $\sigma_{e_5}(S_1(\mu))$ does not depend on μ. Clearly, $[F_3(\mu)F_1(\mu) - \overline{G_2(\lambda)}] \in \mathcal{F}^b(Z)$ and $(F_3(\mu) - F_3(\lambda))(S_1(\lambda) - \lambda M_2)[\overline{J}_\lambda\overline{\Gamma}^0_Z - (S_1(\lambda) - \lambda M_2)^{-1}\overline{(F - (D - \lambda M_6)(A_1 - \lambda M_1)^{-1}(C - \lambda M_5)}] \in \mathcal{F}^b(Z)$, so in the same way we can deduce from Equation (2.3) and [4, Theorem 3.1] that $\sigma_{e_5}(\overline{S}_2(\mu)) = \sigma_{e_5}(\overline{S}_2(\lambda))$. □

Now, we give the following result.

Theorem 3.3. Let X, Y and Z be Banach spaces and $M \in \mathcal{L}(X \times Y \times Z)$ with the representation (3.1). Assume that the operator $\mathcal{L}_0$ defined in Equation (1.1) satisfies (H_1)-(H_{12}) with closure $\mathcal{L}$ and let E be a finite subset of $\mathbb{C}$ containing 0. If for some $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$, we have $F_1(\mu) \in \mathcal{F}^b(X, Y)$, $F_2(\mu) \in \mathcal{F}^b(X, Z)$, $F_3(\mu) \in \mathcal{F}^b(Y, Z)$ then,
(i) if the operators A_1 is a generalized weakly M_1-demicompact, $S_1(\mu)$ is a generalized weakly M_2-demicompact and $\overline{S}_2(\mu)$ is a generalized weakly M_3-demicompact with a generalized set E and $\mathcal{R}(\mu) \in \mathcal{F}_+(X \times Y \times Z)$, then

$$\sigma_{e_{1,M}}(\mathcal{L})\backslash E = [\sigma_{e_{1,M_1}}(A_1) \cup \sigma_{e_{1,M_2}}(S_1(\mu)) \cup \sigma_{e_{1,M_3}}(\overline{S}_2(\mu))]\backslash E.$$

(ii) If for $t \in [0, 1]$, the operators tA_1 is a generalized weakly M_1-demicompact, $tS_1(\mu)$ is a generalized weakly M_2-demicompact and $t\overline{S}_2(\mu)$ is a generalized weakly M_3-demicompact with a generalized set E and $\mathcal{R}(\mu) \in \mathcal{F}(X \times Y \times Z)$, then

$$\sigma_{e_{i,M}}(\mathcal{L})\backslash E = [\sigma_{e_{i,M_1}}(A_1) \cup \sigma_{e_{i,M_2}}(S_1(\mu)) \cup \sigma_{e_{i,M_3}}(\overline{S}_2(\mu))]\backslash E, \text{ where } i \in \{4, 5\}.$$

Proof. (i) Since A_1 is generalized weakly M_1-demicompact, $S_1(\mu)$ is generalized weakly M_2-demicompact and $\overline{S}_2(\mu)$ is generalized weakly M_3-demicompact with a generalized set E and $\mathcal{R}(\mu) \in \mathcal{F}_+(X \times Y \times Z)$, it follows from Theorem 3.2 that, the matrix operator $\mathcal{L}$ is generalized weakly M-demicompact with a generalized set E. Hence, from Theorem 2.1, we get $\lambda M - \mathcal{L}$ is an upper semi-Fredholm operator for all $\lambda \in \mathbb{C}\backslash E$.
Let $\lambda \in \mathbb{C}\backslash E$, according to Equation (2.4), we have

$$\lambda M - \mathcal{L} = \mathcal{G}_l(\mu)V(\lambda)\mathcal{G}_r(\mu) - (\lambda - \mu)\mathcal{R}(\mu).$$

Using the fact that $\mathcal{R}(\mu) \in \mathcal{F}_+(X \times Y \times Z)$, we infer that $\lambda M - \mathcal{L} \in \Phi_+(X \times Y \times Z)$ if, and only if, the operator $\mathcal{G}_l(\mu)V(\lambda)\mathcal{G}_r(\mu)$ is such too. Now, since $\mathcal{G}_l(\mu)$ and $\mathcal{G}_r(\mu)$ are invertible and have bounded inverses, hence $\lambda M - \mathcal{L} \in \Phi_+(X \times Y \times Z)$ if, and only if, $V(\lambda) \in \Phi_+(X \times Y \times Z)$ which is equivalent to $\lambda M_1 - A_1 \in \Phi_+(X)$, $\lambda M_2 - S_1(\mu) \in \Phi_+(Y)$ and $\lambda M_3 - \overline{S}_2(\mu) \in \Phi_+(Z)$. Thus, in view of Lemma 3.1, we have

$$\sigma_{e_{1,M}}(\mathcal{L})\backslash E = [\sigma_{e_{1,M_1}}(A_1) \cup \sigma_{e_{1,M_2}}(S_1(\mu)) \cup \sigma_{e_{1,M_3}}(\overline{S}_2(\mu))]\backslash E.$$

(ii) Since tA_1 is generalized weakly M_1-demicompact, $tS_1(\mu)$ is generalized weakly M_2-demicompact and $t\overline{S}_2(\mu)$ is generalized weakly M_3-demicompact for $t \in [0, 1]$ and $\mathcal{R}(\mu) \in \mathcal{F}(X \times Y \times Z)$, it follows from Theorem 3.2 that, the matrix operator $t\mathcal{L}$ is generalized weakly M-demicompact with a generalized set E for $t \in [0, 1]$. Hence, from Theorem 2.2, we have $\lambda M - \mathcal{L} \in \Phi(X \times Y \times Z)$ and $i(\lambda M - \mathcal{L}) = i(\lambda M)$ for all $\lambda \in \mathbb{C}\backslash E$.
Now, a similar reasoning as (i) allows us to conclude that

$$\sigma_{e_{i,M}}(\mathcal{L})\backslash E = [\sigma_{e_{i,M_1}}(A_1) \cup \sigma_{e_{i,M_2}}(S_1(\mu)) \cup \sigma_{e_{i,M_3}}(\overline{S}_2(\mu))]\backslash E, \text{ where } i \in \{4, 5\}.$$

□

Before moving to the next section, we give an example of generalized weakly S-demicompact matrix operator:

Example 3.1. Let l_2 be a Banach space with its norm. We define the following operators on l_2 by

$$A_1x = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots)$$

$$A_2x = (x_2, \frac{1}{2}x_3, \frac{1}{3}x_4, \cdots)$$

$$A_3x = (0, x_1, 0, x_3, 0, x_5, \cdots)$$

$$A_4x = (0, x_1, 0, \frac{1}{3}x_2, 0, \frac{1}{5}x_3, \cdots).$$

The operators A_i are compact, for all $i = 1, \cdots, 4$.
Let U and V be the forward and the backward unilateral shifts defined by

$$V(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots) \qquad \text{and} \qquad U(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, \cdots).$$

Now, let $(\alpha_n)_n$ be a sequence of numbers such that $\alpha_n > 0$ for all $n \in \mathbb{Z}$. For $x \in l_2(\mathbb{Z})$ define the weighted bilateral shift $B \in \mathcal{L}(l_2(\mathbb{Z}))$ by

$$B(\cdots, x_{-1}, x_0, x_1, \cdots) = (\cdots, \alpha_{-2}x_{-2}, \alpha_{-1}x_{-1}, \alpha_0 x_0, \cdots).$$

In terms of the standard basis in $l_2(\mathbb{Z})$ that is $Be_n = \alpha_n e_{n+1}$. The operator B is invertible with inverse C defined by $Ce_n = \frac{1}{\alpha_n}e_{n-1}$.

Further, we consider the operator $A \in \mathcal{L}(l_2(\mathbb{N}))$ defined by

$$A((x_n)_{n\geq 0}) := (\lambda_n x_n)_{n\geq 0}),$$

where $(\lambda_n)_n$ is a bounded real sequence.
For $\lambda \notin \overline{\sigma_p(A)}$, the operator $\widetilde{A}((x_n)_{n\geq 0}) := (\frac{x_n}{\lambda_n - \lambda})_{n\geq 0}$ is invertible with inverse $(A - \lambda I)$.

Let the operator $T : l_2 \longrightarrow l_2$ be the backward weighted shift defined by

$$Te_0 = 0 \qquad \text{and} \quad Te_{n+1} = \tau_n e_n \quad n \geq 0,$$

where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis of l_2 and the weight sequence $\{\tau_n\}_{n=0}^{\infty}$ is given by

$$\{\frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{16}}, \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{64}}, \frac{1}{2}, \frac{1}{2^4}, \cdots\},$$

then T is quasinilpotent and hence Riesz operator.
Now, we introduce the following matrix operators defined on $X \times X \times X$, where $X = l_2(\mathbb{Z})$.

$$\mathcal{L} = \begin{pmatrix} \widetilde{A}T & A_1 & A_2 \\ A_3 & U & 0 \\ A_4 & 0 & V \end{pmatrix} \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} \widetilde{A} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

Let $\lambda \in \mathbb{C}$, we write

$$\lambda\mathcal{M} - \mathcal{L} = \mathcal{A}_\lambda - \mathcal{B},$$

where $\mathcal{A}_\lambda = \begin{pmatrix} \lambda\widetilde{A} - \widetilde{A}T & 0 & 0 \\ 0 & \lambda B - U & 0 \\ 0 & 0 & \lambda C - V \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} 0 & A_1 & A_2 \\ A_3 & 0 & 0 \\ A_4 & 0 & 0 \end{pmatrix}$.

Since T is a Riesz operator, we infer that $\lambda I - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$ which implies that $\widetilde{A}(\lambda I - T) \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$.
Now, since U and V are Fredholm operators, it follows that U and V are upper semi-Fredholm operators. Thus, we obtain $\lambda B - U \in \Phi_+(X)$ and $\lambda C - V \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}$. Then, $\mathcal{A}_\lambda$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C}\backslash\{0\}$.
Consequently, in view of Theorem 2.1, we infer that U is a generalized weakly B-demicompact, $\widetilde{A}T$ is a generalized weakly $\widetilde{A}$-demicompact and V is a generalized weakly C-demicompact operators with a generalized set $E = \{0\}$.
Consequently, taking into account the fact that $\mathcal{B} \in \mathcal{K}(X \times X \times X)$ and by applying Theorem 3.2, we conclude that $\mathcal{L}$ is a generalized weakly $\mathcal{M}$-demicompact operator with a generalized set $E = \{0\}$.

4. Generalized weak M-demicompactness for block operators matrices by means of measure of non-strict-singularity

We recall the following result which describes the closure of the operator $\mathcal{L}_0$.

Theorem 4.1. Under assumptions $(H1)$-$(H12)$, the operator $\mathcal{L}_0$ is closable if and only if $S_2(\mu)$ is closable for some $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$. In this case the closure $\mathcal{L}$ of $\mathcal{L}_0$ is given by

$$\mathcal{L} = \mu M + \mathcal{G}_l(\mu) \begin{pmatrix} A_1 - \mu M_1 & 0 & 0 \\ 0 & S_1(\mu) - \mu M_2 & 0 \\ 0 & 0 & \overline{S}_2(\mu) - \mu M_3 \end{pmatrix} \mathcal{G}_r(\mu).$$

For $n \in \mathbb{N}$, let

$$\mathcal{I}_n(X) = \{K \in \mathcal{L}(X) \text{ satisfying } g((KB)^n) < 1 \text{ for all } B \in \mathcal{L}(X)\},$$

where $g(.)$ is a measure of non-strict-singularity, given in (2.1). We have the following inclusion

$$\mathcal{S}(X) \subset \mathcal{I}_n(X).$$

Theorem 4.2. [19] Let $A \in \Phi(X)$, then for all $K \in \mathcal{I}_n(X)$, we have $A + K \in \Phi(X)$ and $i(A + K) = i(A)$.

Remark 4.1.

(i) If $K \in \mathcal{I}_n(X)$ and $A \in \mathcal{L}(X)$, then $KA \in \mathcal{I}_n(X)$.

(ii) If $K \in \mathcal{I}_n(X)$ and $S \in \mathcal{S}(X)$, then $K + S \in \mathcal{I}_n(X)$.

Let $g(.)$ be a measure of non-strict-singularity, given in (2.1).

Lemma 4.1. For all bounded operator

$$\mathcal{T} = \begin{pmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{pmatrix},$$

on $X \times Y \times Z$, we consider

$$G(\mathcal{T}) = \max\left(g(T_1) + g(T_2) + g(T_3), g(T_4) + g(T_5) + g(T_6), g(T_7) + g(T_8) + g(T_9)\right).$$

Then G defines a measure of non-strict-singularity on the space $X \times Y \times Z$.

Proof. In the first step, we will check that G is a semi-norm on $X \times Y \times Z$.

(i) Let $\mathcal{T} = \begin{pmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{pmatrix}$ and $\mathcal{S} = \begin{pmatrix} S_1 & S_2 & S_3 \\ S_4 & S_5 & S_6 \\ S_7 & S_8 & S_9 \end{pmatrix} \in \mathcal{L}(X \times Y \times Z)$.

Then, we get

$$\begin{aligned}
G(\mathcal{T}+\mathcal{S}) &= \max\{g(T_1+S_1)+g(T_2+S_2)+g(T_3+S_3);\\
&\quad g(T_4+S_4)+g(T_5+S_5)+g(T_6+S_6);\\
&\quad g(T_7+S_7)+g(T_8+S_8)+g(T_9+S_9)\}\\
&\leq \max\{g(T_1)+g(S_1)+g(T_2)+g(S_2)+g(T_3)+g(S_3);\\
&\quad g(T_4)+g(S_4)+g(T_5)+g(S_5)+g(T_6)+g(S_6);\\
&\quad g(T_7)+g(S_7)+g(T_8)+g(S_8)+g(T_9)+g(S_9)\}\\
&\leq \max\{g(T_1)+g(T_2)+g(T_3)+g(S_1)+g(S_2)+g(S_3);\\
&\quad g(T_4)+g(T_5)+g(T_6)+g(S_4)+g(S_5)+g(S_6);\\
&\quad g(T_7)+g(T_8)+g(T_9)+g(S_7)+g(S_8)+g(S_9)\}\\
&\leq \max\{g(T_1)+g(T_2)+g(T_3); g(T_4)+g(T_5)+g(T_6); g(T_7)+g(T_8)+g(T_9)\}\\
&\quad +\max\{g(S_1)+g(S_2)+g(S_3); g(S_4)+g(S_5)+g(S_6); g(S_7)+g(S_8)+g(S_9)\}.
\end{aligned}$$

Hence, we conclude that

$$G(\mathcal{T}+\mathcal{S}) \leq G(\mathcal{T})+G(\mathcal{S}).$$

(*ii*) Let $\lambda \in \mathbb{C}$ and $\mathcal{T} = \begin{pmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{pmatrix}$

$$\begin{aligned}
G(\lambda\mathcal{T}) &= \max\{g(\lambda T_1)+g(\lambda T_2)+g(\lambda T_3); g(\lambda T_4)+g(\lambda T_5)+g(\lambda T_6);\\
&\quad g(\lambda T_7)+g(\lambda T_8)+g(\lambda T_9)\}\\
&= \max\{|\lambda|[g(T_1)+g(T_2)+g(T_3)]; |\lambda|[g(T_4)+g(T_5)+g(T_6)];\\
&\quad |\lambda|[g(T_7)+g(T_8)+g(T_9)]\}\\
&= |\lambda|\max\{g(T_1)+g(T_2)+g(T_3); g(T_4)+g(T_5)+g(T_6); g(T_7)+g(T_8)+g(T_9)\}\\
&= |\lambda| G(\mathcal{T}).
\end{aligned}$$

So, Combining together (*i*) and (*ii*), we get G is a semi-norm.
Furthermore, we have $G(\mathcal{T}) = 0$ if, and only if,

$$\max\{g(T_1)+g(T_2)+g(T_3); g(T_4)+g(T_5)+g(T_6); g(T_7)+g(T_8)+g(T_9)\} = 0,$$

which equivalent to

$$g(T_1)+g(T_2)+g(T_3) = 0,\ g(T_4)+g(T_5)+g(T_6) = 0 \text{ and } g(T_7)+g(T_8)+g(T_9) = 0.$$

Thus, we obtain $g(T_i) = 0$ for all $i \in \{1, \cdots 9\}$. As g is a measure of non-strict-singularity, then it yields from the fact that $g(T_i) = 0$, that T_i are strictly singular operators on their respective spaces for all $1 \leq i \leq 9$. Hence, we conclude from Proposition 2.1, that $\mathcal{T}$ is strictly singular operator on $X \times Y \times Z$. Consequently, we get $G(\mathcal{T}) = 0$ if, and only if, $\mathcal{T}$ is strictly singular operator.
As a conclusion, we have G is a measure of non-strict-singularity on $X \times Y \times Z$. □

In all that follows we will make the following assumption

$$(\mathcal{A}): \begin{cases} g(L\widetilde{G}_i(\mu)H\widetilde{G}_j(\mu)K) < \frac{1}{36} \quad g(F_i(\mu)HF_j(\mu)K) < \frac{1}{36} \\[2ex] g(L\widetilde{G}_i(\mu)HF_j(\mu)K) < \frac{1}{36} \quad g(F_i(\mu)H\widetilde{G}_j(\mu)K) < \frac{1}{36} \\[2ex] \text{for some } \mu \in \rho_{M_1}(A_1) \text{ and all bounded operators } L,\ H \text{ and } K, \end{cases}$$

where $i, j \in \{1, 2, 3\}$.

Remark 4.2.

(*i*) Note that if $\widetilde{G}_i(\mu)$ and $F_j(\mu)$ are strictly singular operators, then hypothesis $(\mathcal{A})$ is satisfied.

(*ii*) If the hypothesis

$$g(F_i(\mu)H\widetilde{G}_j(\mu)K) < \frac{1}{36}. \tag{4.1}$$

holds for all bounded operators H and K, then $F_i(\mu)\widetilde{G}_j(\mu)$ is strictly singular. Indeed, since Equation (4.1) is valid for all bounded operators H and K, we can consider $K = nI_Z, n \in \mathbb{N}^*$ (resp. I_Y) and $H = I_Y$ (resp. I_X), we obtain

$$g(F_i(\mu)\widetilde{G}_j(\mu)) < \frac{1}{36n}.$$

So,

$$g\left(F_i(\mu)\widetilde{G}_j(\mu)\right) = 0$$

and this implies that $F_i(\mu)\widetilde{G}_j(\mu)$ is strictly singular.

Theorem 4.3. Let the matrix operator $\mathcal{L}_0$ satisfy conditions (H_1)-(H_{12}) and the matrix operator $M \in \mathcal{L}(X \times Y \times Z)$ with the representation (3.1) such that M_i are compact operators, for all $i \in \{4, \cdots, 9\}$. Assume that the hypothesis $(\mathcal{A})$ is satisfied. Let $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$ and E be a finite subset of $\mathbb{C}$ containing 0. If the operators tA_1 is generalized weakly M_1-demicompact, $tS_1(\mu)$ is generalized weakly M_2-demicompact and $t\overline{S}_2(\mu)$ is generalized weakly M_3-demicompact for all $t \in [0, 1]$ with a generalized set E, then $\mathcal{L}$ is a generalized weakly M-demicompact operator.

Proof. Let $\mu \in \rho_{M_1}(A_1) \cap \rho_{M_2}(S_1(\mu))$ be such that hypothesis $(\mathcal{A})$ is satisfied and set λ be a complex number. It follows from Equation (2.4) that

$$\lambda M - \mathcal{L} = \mathcal{G}_l(\mu)V(\lambda)\mathcal{G}_r(\mu) - (\lambda - \mu)\mathcal{R}(\mu),$$

where

$$\mathcal{R}(\mu) := \begin{pmatrix} 0 & M_1\widetilde{G}_1(\mu) - M_4 & M_1\widetilde{G}_2(\mu) - M_5 \\ F_1(\mu)M_1 - M_6 & F_1(\mu)M_1\widetilde{G}_1(\mu) & U(\mu) \\ F_2(\mu)M_1 - M_8 & W(\mu) & T(\mu) \end{pmatrix}.$$

Let

$$\mathcal{K} = \begin{pmatrix} K_1 & K_2 & K_3 \\ K_4 & K_5 & K_6 \\ K_7 & K_8 & K_9 \end{pmatrix},$$

be a bounded operator on $X \times Y \times Z$. Then

$$\left[\begin{pmatrix} 0 & M_1\widetilde{G}_1(\mu) & M_1\widetilde{G}_2(\mu) \\ F_1(\mu)M_1 & 0 & M_2\widetilde{G}_3(\mu) \\ F_2(\mu)M_1 & F_3(\mu)M_2 & 0 \end{pmatrix}\mathcal{K}\right]^2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix},$$

$$\text{where}\begin{cases} a_1 = (M_1\widetilde{G}_1(\mu)K_4 + M_1\widetilde{G}_2(\mu)K_7)^2 + [M_1\widetilde{G}_1(\mu)K_5F_1(\mu)M_1K_1 + M_1\widetilde{G}_1(\mu)K_5M_2\widetilde{G}_3(\mu)K_7 \\ +M_1\widetilde{G}_2(\mu)K_8F_1(\mu)M_1K_1 + M_1\widetilde{G}_2(\mu)K_8M_2\widetilde{G}_3(\mu)K_7] + [M_1\widetilde{G}_1(\mu)K_6F_2(\mu)M_1K_1 \\ +M_1\widetilde{G}_1(\mu)K_6F_3(\mu)M_2K_4 + M_1\widetilde{G}_2(\mu)K_9F_2(\mu)M_1K_1 + M_1\widetilde{G}_2(\mu)K_9F_3(\mu)M_2K_4] \\ a_4 = [F_1(\mu)M_1K_1M_1\widetilde{G}_1(\mu)K_4 + F_1(\mu)M_1K_1M_1\widetilde{G}_2(\mu)K_7 + M_2\widetilde{G}_3(\mu)K_7M_1\widetilde{G}_1(\mu)K_4 \\ +M_2\widetilde{G}_3(\mu)K_7M_1\widetilde{G}_2(\mu)K_7] + [F_1(\mu)M_1K_2F_1(\mu)M_1K_1 + F_1(\mu)M_1K_2M_2\widetilde{G}_3(\mu)K_7 \\ +M_2\widetilde{G}_3(\mu)K_8F_1(\mu)M_1K_1 + M_2\widetilde{G}_3(\mu)K_8M_2\widetilde{G}_3(\mu)K_7] + [F_1(\mu)M_1K_3F_2(\mu)M_1K_1 \\ +F_1(\mu)M_1K_3F_3(\mu)M_2K_4 + M_2\widetilde{G}_3(\mu)K_9F_2(\mu)M_1K_1 + M_2\widetilde{G}_3(\mu)K_9F_3(\mu)M_2K_4] \\ a_7 = [F_2(\mu)M_1K_1M_1\widetilde{G}_1(\mu)K_4 + F_2(\mu)M_1K_1M_1\widetilde{G}_2(\mu)K_7 + F_3(\mu)M_2K_4M_1\widetilde{G}_1(\mu)K_4 \\ +F_3(\mu)M_2K_4M_1\widetilde{G}_2(\mu)K_7] + [F_2(\mu)M_1K_2F_1(\mu)M_1K_1 + F_2(\mu)M_1K_2M_2\widetilde{G}_3(\mu)K_7 \\ +F_3(\mu)M_2K_5F_1(\mu)M_1K_1 + F_3(\mu)M_2K_5M_2\widetilde{G}_3(\mu)K_7] + [F_2(\mu)M_1K_3F_2(\mu)M_1K_1 \\ +F_2(\mu)M_1K_3F_3(\mu)M_2K_4 + F_3(\mu)M_2K_6F_2(\mu)M_1K_1 + F_3(\mu)M_2K_6F_3(\mu)M_2K_4] \end{cases}$$

$$\left\{\begin{array}{l}
a_2 = [M_1\widetilde{G}_1(\mu)K_4M_1\widetilde{G}_1(\mu)K_1 + M_1\widetilde{G}_1(\mu)K_4M_1\widetilde{G}_2(\mu)K_8 + M_1\widetilde{G}_2(\mu)K_7M_1\widetilde{G}_1(\mu)K_1\\
+M_1\widetilde{G}_2(\mu)K_7M_1\widetilde{G}_2(\mu)K_8] + [M_1\widetilde{G}_1(\mu)K_5F_1(\mu)M_1K_2 + M_1\widetilde{G}_1(\mu)K_5M_2\widetilde{G}_3(\mu)K_8\\
+M_1\widetilde{G}_2(\mu)K_8F_1(\mu)M_1K_2 + M_1\widetilde{G}_2(\mu)K_8M_2\widetilde{G}_3(\mu)K_8] + [M_1\widetilde{G}_1(\mu)K_6F_2(\mu)M_1K_2\\
+M_1\widetilde{G}_1(\mu)K_6F_3(\mu)M_2K_5 + M_1\widetilde{G}_2(\mu)K_9F_2(\mu)M_1K_2 + M_1\widetilde{G}_2(\mu)K_9F_3(\mu)M_2K_5]\\
a_5 = [F_1(\mu)M_1K_1M_1\widetilde{G}_1(\mu)K_1 + F_1(\mu)M_1K_1M_1\widetilde{G}_2(\mu)K_8 + M_2\widetilde{G}_3(\mu)K_7M_1\widetilde{G}_1(\mu)K_1\\
+M_2\widetilde{G}_3(\mu)K_7M_1\widetilde{G}_2(\mu)K_8] + (F_1(\mu)M_1K_2 + M_2\widetilde{G}_3(\mu)K_8)^2 + [F_1(\mu)M_1K_3F_2(\mu)M_1K_2\\
+F_1(\mu)M_1K_3F_3(\mu)M_2K_5 + M_2\widetilde{G}_3(\mu)K_9F_2(\mu)M_1K_2 + M_2\widetilde{G}_3(\mu)K_9F_3(\mu)M_2K_5]\\
a_8 = [F_2(\mu)M_1K_1M_1\widetilde{G}_1(\mu)K_1 + F_2(\mu)M_1K_1M_1\widetilde{G}_2(\mu)K_8 + F_3(\mu)M_2K_4M_1\widetilde{G}_1(\mu)K_1\\
+F_3(\mu)M_2K_4M_1\widetilde{G}_2(\mu)K_8] + [F_2(\mu)M_1K_2F_1(\mu)M_1K_2 + F_2(\mu)M_1K_2M_2\widetilde{G}_3(\mu)K_8\\
+F_3(\mu)M_2K_5F_1(\mu)M_1K_2 + F_3(\mu)M_2K_5M_2\widetilde{G}_3(\mu)K_8] + [F_2(\mu)M_1K_3F_2(\mu)M_1K_2\\
+F_2(\mu)M_1K_3F_3(\mu)M_2K_5 + F_3(\mu)M_2K_6F_2(\mu)M_1K_2 + F_3(\mu)M_2K_6F_3(\mu)M_2K_5]
\end{array}\right.$$

$$\left\{\begin{array}{l}
a_3 = [M_1\widetilde{G}_1(\mu)K_4M_1\widetilde{G}_1(\mu)K_6 + M_1\widetilde{G}_1(\mu)K_4M_1\widetilde{G}_2(\mu)K_9 + M_1\widetilde{G}_2(\mu)K_7M_1\widetilde{G}_1(\mu)K_6\\
+M_1\widetilde{G}_2(\mu)K_7M_1\widetilde{G}_2(\mu)K_9] + [M_1\widetilde{G}_1(\mu)K_5F_1(\mu)M_1K_3 + M_1\widetilde{G}_1(\mu)K_5M_2\widetilde{G}_3(\mu)K_9\\
+M_1\widetilde{G}_2(\mu)K_8F_1(\mu)M_1K_3 + M_1\widetilde{G}_2(\mu)K_8M_2\widetilde{G}_3(\mu)K_9] + [M_1\widetilde{G}_1(\mu)K_6F_2(\mu)M_1K_3\\
+M_1\widetilde{G}_1(\mu)K_6F_3(\mu)M_2K_6 + M_1\widetilde{G}_2(\mu)K_9F_2(\mu)M_1K_3 + M_1\widetilde{G}_2(\mu)K_9F_3(\mu)M_2K_6]\\
a_6 = [F_1(\mu)M_1K_1M_1\widetilde{G}_1(\mu)K_6 + F_1(\mu)M_1K_1M_1\widetilde{G}_2(\mu)K_9 + M_2\widetilde{G}_3(\mu)K_7M_1\widetilde{G}_1(\mu)K_6\\
+M_2\widetilde{G}_3(\mu)K_7M_1\widetilde{G}_2(\mu)K_9] + [F_1(\mu)M_1K_2F_1(\mu)M_1K_3 + F_1(\mu)M_1K_2M_2\widetilde{G}_3(\mu)K_9\\
+M_2\widetilde{G}_3(\mu)K_8F_1(\mu)M_1K_3 + M_2\widetilde{G}_3(\mu)K_8M_2\widetilde{G}_3(\mu)K_9] + [F_1(\mu)M_1K_3F_2(\mu)M_1K_3\\
+F_1(\mu)M_1K_3F_3(\mu)M_2K_6 + M_2\widetilde{G}_3(\mu)K_9F_2(\mu)M_1K_3 + M_2\widetilde{G}_3(\mu)K_9F_3(\mu)M_2K_6]\\
a_9 = [F_2(\mu)M_1K_1M_1\widetilde{G}_1(\mu)K_6 + F_2(\mu)M_1K_1M_1\widetilde{G}_2(\mu)K_9 + F_3(\mu)M_2K_4M_1\widetilde{G}_1(\mu)K_6\\
+F_3(\mu)M_2K_4M_1\widetilde{G}_2(\mu)K_9] + [F_2(\mu)M_1K_2F_1(\mu)M_1K_3 + F_2(\mu)M_1K_2M_2\widetilde{G}_3(\mu)K_9\\
+F_3(\mu)M_2K_5F_1(\mu)M_1K_3 + F_3(\mu)M_2K_5M_2\widetilde{G}_3(\mu)K_9] + (F_2(\mu)M_1K_3 + F_3(\mu)M_2K_6)^2
\end{array}\right.$$

It follows from hypothesis $(\mathcal{A})$ and Lemma 4.1 that

$$G\left((\lambda-\mu)^2\left[\begin{pmatrix} 0 & M_1\widetilde{G}_1(\mu) & M_1\widetilde{G}_2(\mu)\\ F_1(\mu)M_1 & 0 & M_2\widetilde{G}_3(\mu)\\ F_2(\mu)M_1 & F_3(\mu)M_2 & 0\end{pmatrix}\mathcal{K}\right]^2\right) < 1.$$

Which implies that, the operator

$$(\lambda-\mu)\begin{pmatrix} 0 & M_1\widetilde{G}_1(\mu) & M_1\widetilde{G}_2(\mu)\\ F_1(\mu)M_1 & 0 & M_2\widetilde{G}_3(\mu)\\ F_2(\mu)M_1 & F_3(\mu)M_2 & 0\end{pmatrix} \in \mathcal{I}_2(X\times Y\times Z).$$

Then, we can deduce from Remark 4.1 (*ii*) and the facts that $F_i(\mu)\widetilde{G}_j(\mu)$ is strictly singular and M_i are compacts for all $i \in \{4, \cdots, 9\}$, that

$$(\lambda-\mu)\mathcal{R}(\mu) \in \mathcal{I}_2(X\times Y\times Z).$$

Since tA_1 is a generalized weakly M_1-demicompact operator, $tS_1(\mu)$ is a generalized weakly M_2-demicompact operator and $t\overline{S}_2(\mu)$ is a generalized weakly M_3-demicompact operator with a generalized set E, we infer from Theorem 2.2, that $\lambda M_1 - A_1 \in \Phi(X)$, $\lambda M_2 - S_1(\mu) \in \Phi(Y)$ and $\lambda M_3 - \overline{S}_2(\mu) \in \Phi(Z)$ for all $\lambda \in \mathbb{C}\backslash E$. Now, when applying Lemma 6.6.1 in [12], we get $V(\lambda) \in \Phi(X\times Y\times Z)$ for all $\lambda \in \mathbb{C}\backslash E$. Furthermore, we observe that the operators $\mathcal{G}_r(\mu)$ and $\mathcal{G}_l(\mu)$ are bounded and have bounded inverses. Hence, the operator $\mathcal{G}_l(\mu)V(\lambda)\mathcal{G}_r(\mu) \in \Phi(X\times Y\times Z)$ for all $\lambda \in \mathbb{C}\backslash E$. Now, if we use Equation (2.4) and apply Theorem 4.2, we conclude that $\lambda M - \mathcal{L} \in \Phi(X\times Y\times Z)$ for all $\lambda \in \mathbb{C}\backslash E$, which implies that $\lambda M - \mathcal{L} \in \Phi_+(X\times Y\times Z)$ for all $\lambda \in \mathbb{C}\backslash E$. So, by Theorem 2.1, we deduce that $\mathcal{L}$ is generalized weakly M-demicompact. □

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