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HUR-Approximation of an ELTA Functional Equation

A. R. Sharifi[a], H. Azadi Kenary[b], B. Yousefi[a], R. Soltani[a]

[a]*Department of Mathematics, College of Sciences, Payam Nour University Tehran, Iran*
[b]*Department of Mathematics,Yasouj University, Yasouj 75914-353, Iran*

Abstract. The main goal of this paper is study of the Hyers-Ulam-Rassias stability (briefly HUR-approximation) of the following Euler-Lagrange type additive(briefly ELTA) functional equation

$$\sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\le i\le n, i\neq j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i f(x_i) = nf\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right)$$

where $r_1, \cdots, r_n \in \mathbb{R}$, $\sum_{i=k}^{n} r_k \neq 0$, and $r_i, r_j \neq 0$ for some $1 \le i < j \le n$, in fuzzy normed spaces.
The concept of HUR-approximation originated from Th. M. Rassias stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?.
If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [36] in 1940.
We are given a group G and a metric group G' with metric $d(.,.)$. Given $\varepsilon > 0$, dose there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$, for all $x, y \in G$, , then a homomorphism $h : G \to G$; exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G$?.
In the next year Hyres [18], gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [26] proved a generalization of Hyres' theorem for additive mappings .
The result of Th. M. Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations. In 1994, a generalization of Rassias' theorem was obtained by Găvruta [14] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.
The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1]-[34]).

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Email addresses: mathsharifi@yahoo.com (A. R. Sharifi), azadi@yu.ac.ir (H. Azadi Kenary), B.yousefi@pnu.ac.ir (B. Yousefi), rsoltani@pnu.ac.ir (R. Soltani)

Definition 1.1. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*
$(N1)$ $N(x,t) = 0$ *for* $t \leq 0$;
$(N2)$ $x = 0$ *if and only if* $N(x,t) = 1$ *for all* $t > 0$;
$(N3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;
$(N4)$ $N(x+y, c+t) \geq min\{N(x,s), N(y,t)\}$;
$(N5)$ $N(x,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 1$;
$(N6)$ *for* $x \neq 0$, $N(x,.)$ *is continuous on* $\mathbb{R}$.

Example 1.2. *Let $(X, \|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then*

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta\|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.3. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - \lim_{t\to\infty} x_n = x$.*

Definition 1.4. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.
We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [6]).

Definition 1.5. *Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:*
(1) $d(x,y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$;
(2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
(3) $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x, y, z \in X$.

Theorem 1.6. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n_0 \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In this paper, we prove the HUR-approximation of the following functional equation

$$\sum_{j=1}^{n} f\left(\frac{1}{2} \sum_{1\leq i\leq n, i\neq j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i f(x_i) = n f\left(\frac{1}{2} \sum_{i=1}^{n} r_i x_i\right) \tag{1}$$

where $r_1, \cdots, r_n \in \mathbb{R}$, $\sum_{k=1}^{n} r_k \neq 0$, and $r_i, r_j \neq 0$ for some $1 \leq i < j \leq n$, in fuzzy normed spaces.

2. Fuzzy stability of functional equation (1): a fixed point method

In this section, using the fixed point alternative approach we prove the HUR-approximation of functional equation (1) in fuzzy Banach spaces. Throughout this section, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Lemma 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $r_1, \ldots, r_n$ be real numbers with $\sum_{i=1}^n r_i \neq 0$ and $r_i \neq 0, r_j \neq 0$ for some $1 \leq i < j \leq n$. Assume that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (1) for all $x_1, \ldots, x_n \in \mathcal{X}$. Then the mapping f is additive. Moreover, $f(r_k x) = r_k f(x)$ for all $x \in \mathcal{X}$ and all $1 \leq k \leq n$.*

Proof. Since $\sum_{k=1}^n r_k \neq 0$, putting $x_1 = \cdots = x_n = 0$ in (1), we get $f(0) = 0$. Without loss of generality, we may assume that $r_1, r_2 \neq 0$. Letting $x_3 = \cdots = x_n = 0$ in (1), we get

$$f\left(\frac{-r_1x_1 + r_2x_2}{2}\right) + f\left(\frac{r_1x_1 - r_2x_2}{2}\right) + r_1f(x_1) + r_2f(x_2) = 2f\left(\frac{r_1x_1 + r_2x_2}{2}\right) \tag{2}$$

for all $x_1, x_2 \in \mathcal{X}$. Letting $x_2 = 0$ in (2), we have

$$r_1f(x_1) = f\left(\frac{r_1x_1}{2}\right) - f\left(\frac{-r_1x_1}{2}\right) \tag{3}$$

for all $x_1 \in \mathcal{X}$. Similarly, by putting $x_1 = 0$ in (2), we have

$$r_2f(x_2) = f\left(\frac{r_2x_2}{2}\right) - f\left(\frac{-r_2x_2}{2}\right) \tag{4}$$

for all $x_2 \in \mathcal{X}$. It follows from (2), (3) and (4) that

$$f\left(\frac{-r_1x_1 + r_2x_2}{2}\right) + f\left(\frac{r_1x_1 - r_2x_2}{2}\right) + f\left(\frac{r_1x_1}{2}\right) + f\left(\frac{r_2x_2}{2}\right) - f\left(\frac{-r_1x_1}{2}\right) - f\left(\frac{-r_2x_2}{2}\right) = 2f\left(\frac{r_1x_1 + r_2x_2}{2}\right) \tag{5}$$

for all $x_1, x_2 \in \mathcal{X}$. Replacing x_1 and x_2 by $\frac{2x}{r_1}$ and $\frac{2x}{r_1}$, respectively, in (5), we get

$$f(-x + y) + f(x - y) + f(x) + f(y) - f(-x) - f(-y) = 2f(x + y) \tag{6}$$

for all $x, y \in \mathcal{X}$. Putting $y = -x$ in (6), we get that $f(-2x) + f(2x) = 0$ for all $x \in \mathcal{X}$. So the mapping f is an odd mapping. Therefore, it follows from (6) that the mapping f is additive. Moreover, let $x \in \mathcal{X}$ and $1 \leq k \leq n$. Setting $x_k = x$ and $x_l = 0$ for all $1 \leq l \leq n$ with $l \neq k$ in (1) and using the oddness of f, we get that $f(r_kx) = r_kf(x)$. □

Lemma 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $r_1, \ldots, r_n$ be real numbers with $\sum_{i=1}^n r_i \neq 0$ and $r_i \neq 0, r_j \neq 0$ for some $1 \leq i < j \leq n$. Assume that a mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfies the functional equation (1) for all $x_1, \ldots, x_n \in \mathcal{X}$. Then the mapping L is additive. Moreover, $f(r_kx) = r_kf(x)$ for all $x \in \mathcal{X}$ and all $1 \leq k \leq n$.*

Theorem 2.3. *Let $\varphi : X^n \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1, x_2, \cdots, x_n) \leq 2L\varphi\left(\frac{x_1}{2}, \frac{x_2}{2} \cdots, \frac{x_n}{2}\right) \tag{7}$$

for all $x_1, \cdots, x_n \in X$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying

$$N\left(\sum_{j=1}^n f\left(\frac{1}{2}\sum_{1\leq i\leq n, i\neq j} r_ix_i - \frac{1}{2}r_jx_j\right) + \sum_{i=1}^n r_if(x_i) - nf\left(\frac{1}{2}\sum_{i=1}^n r_ix_i\right), t\right) \geq \frac{t}{t + \varphi(x_1, \ldots, x_n)} \tag{8}$$

for all $x_1, \dots, x_n \in X$ *and all* $t > 0$*. Then there exists a unique ELTA mapping* $L : X \to Y$ *such that*

$$N(f(x) - L(x), t) \geq \frac{(4-4L)t}{(4-4L)t + 9\Psi(x)} \tag{9}$$

for all $x \in X$ *and all* $t > 0$*, where*

$$\begin{aligned} \Psi(x) \quad := \quad & \varphi_{ij}\left(\frac{2x}{r_i}, \frac{2x}{r_j}\right) + \varphi_{ij}\left(\frac{2x}{r_i}, 0\right) + \varphi_{ij}\left(0, \frac{2x}{r_j}\right) + \varphi_{ij}\left(\frac{x}{r_i}, -\frac{x}{r_j}\right) \\ + \quad & \varphi_{ij}\left(\frac{x}{r_i}, 0\right) + \varphi_{ij}\left(0, -\frac{x}{r_j}\right) \end{aligned}$$

for all $x \in X$.

Proof. Putting $x_k = 0$ in (8) for all $1 \leq k \leq n$ with $k \neq i, j$, we have

$$\begin{aligned} & N\left(f\left(\frac{-r_i x_i + r_j x_j}{2}\right) + f\left(\frac{r_i x_i - r_j x_j}{2}\right) - 2f\left(\frac{r_i x_i + r_j x_j}{2}\right) + r_i f(x_i) + r_j f(x_j), t\right) \\ & \geq \frac{t}{t + \varphi_{ij}(x_i, x_j)} \end{aligned} \tag{10}$$

for all $x_i, x_j \in X$ and all $t > 0$. For convenience, set $\varphi_{ij}(x, y) := \varphi\left(0, \dots, \underbrace{x}_{ith}, \dots, \underbrace{y}_{jth}, \dots, 0, \dots, 0\right)$ for all $x, y \in X$ and all $1 \leq i < j \leq n$. Letting $x_i = 0$ in (10), we get

$$N\left(f\left(-\frac{r_j x_j}{2}\right) - f\left(\frac{r_j x_j}{2}\right) + r_j f(x_j), t\right) \geq \frac{t}{t + \varphi_{ij}(0, x_j)} \tag{11}$$

for all $x_j \in X$ and all $t > 0$. Similarly, letting $x_j = 0$ in (10), we get

$$N\left(f\left(-\frac{r_i x_i}{2}\right) - f\left(\frac{r_i x_i}{2}\right) + r_i f(x_i), t\right) \geq \frac{t}{t + \varphi_{ij}(x_i, 0)} \tag{12}$$

for all $x_i \in X$. It follows from (10), (11) and (12) that

$$\begin{aligned} & N\Big(f\left(\frac{-r_i x_i + r_j x_j}{2}\right) + f\left(\frac{r_i x_i - r_j x_j}{2}\right) - 2f\left(\frac{r_i x_i + r_j x_j}{2}\right) + f\left(\frac{r_i x_i}{2}\right) \\ & + f\left(\frac{r_j x_j}{2}\right) - f\left(-\frac{r_i x_i}{2}\right) - f\left(-\frac{r_j x_j}{2}\right), 3t\Big) \\ & \geq min\left\{\frac{t}{t + \varphi_{ij}(x_i, x_j)}, \frac{t}{t + \varphi_{ij}(x_i, 0)}, \frac{t}{t + \varphi_{ij}(0, x_j)}\right\} \\ & \geq \frac{t}{t + \varphi_{ij}(x_i, x_j) + \varphi_{ij}(x_i, 0) + \varphi_{ij}(0, x_j)} \end{aligned} \tag{13}$$

for all $x_i, x_j \in X$ and all $t > 0$. Replacing x_i and x_j by $\frac{2x}{r_i}$ and $\frac{2y}{r_j}$ in (13), we get that

$$\begin{aligned} & N(f(-x+y) + f(x-y) - 2f(x+y) + f(x) + f(y) - f(-x) - f(-y), 3t) \\ & \qquad \geq \frac{t}{t + \varphi_{ij}\left(\frac{2x}{r_i}, \frac{2y}{r_j}\right) + \varphi_{ij}\left(\frac{2x}{r_i}, 0\right) + \varphi_{ij}\left(0, \frac{2y}{r_j}\right)} \end{aligned} \tag{14}$$

for all $x, y \in X$ and all $t > 0$. Putting $x = y$ in (14), we get

$$N(2f(x) - 2f(-x) - 2f(2x), 3t) \geq \frac{t}{t + \varphi_{ij}\left(\frac{2x}{r_i}, \frac{2x}{r_j}\right) + \varphi_{ij}\left(\frac{2x}{r_i}, 0\right) + \varphi_{ij}\left(0, \frac{2x}{r_j}\right)} \tag{15}$$

for all $x \in X$ and all $t > 0$. Replacing x and y by $\frac{x}{2}$ and $-\frac{x}{2}$ in (14), respectively

$$N(f(x) + f(-x), 3t) \geq \frac{t}{t + \varphi_{ij}\left(\frac{x}{r_i}, -\frac{x}{r_j}\right) + \varphi_{ij}\left(\frac{x}{r_i}, 0\right) + \varphi_{ij}\left(0, -\frac{x}{r_j}\right)} \tag{16}$$

for all $x \in X$ and all $t > 0$. It follows from (15) and (16) that

$$\begin{aligned} & N(2f(2x) - 4f(x), 9t) \\ & = N(2f(x) - 2f(-x) - 2f(2x) + 2(f(x) + f(-x)), 9t) \\ & \geq min\{N(2f(x) - 2f(-x) - 2f(2x), 3t), N(f(x) + f(-x), 3t)\} \\ & \geq min\Bigg\{\frac{t}{t + \varphi_{ij}\left(\frac{2x}{r_i}, \frac{2x}{r_j}\right) + \varphi_{ij}\left(\frac{2x}{r_i}, 0\right) + \varphi_{ij}\left(0, \frac{2x}{r_j}\right)} \\ & , \frac{t}{t + \varphi_{ij}\left(\frac{x}{r_i}, -\frac{x}{r_j}\right) + \varphi_{ij}\left(\frac{x}{r_i}, 0\right) + \varphi_{ij}\left(0, -\frac{x}{r_j}\right)}\Bigg\} \end{aligned} \tag{17}$$

for all $x \in X$ and all $t > 0$. So

$$N\left(\frac{f(2x)}{2} - f(x), \frac{9t}{4}\right) \geq \frac{t}{t + \Psi(x)} \tag{18}$$

for all $x \in X$ and all $t > 0$. Consider the set $S := \{g : X \to Y\ ;\ g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f, g) = \inf\left\{\mu \in \mathbb{R}^+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \Psi(x)}, \forall x \in X,\ t > 0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [21]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{g(2x)}{2}$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then

$$N(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \Psi(x)}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) = N\left(g\left(2x\right) - h\left(2x\right), 2L\epsilon t\right) & \geq \frac{2Lt}{2Lt + \Psi\left(2x\right)} \\ & \geq \frac{2Lt}{2Lt + 2L\Psi\left(x\right)} = \frac{t}{t + \Psi(x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$. It follows from (18) that $d(f, Jf) \leq \frac{9}{4}$. By Theorem 2.1, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, that is,

$$L\left(2x\right) = 2L(x) \tag{19}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that L is a unique mapping satisfying (19) such that there exists $\mu \in (0,\infty)$ satisfying $N(f(x) - L(x), \mu t) \geq \frac{t}{t+\Psi(x)}$ for all $x \in X$ and $t > 0$.
(2) $d(J^l f, L) \to 0$ as $l \to \infty$. This implies the equality

$$N\text{-}\lim_{l\to\infty} \frac{f(2^l x)}{2^l} = L(x)$$

for all $x \in X$.
(3) $d(f,L) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f,L) \leq \frac{9}{4-4L}$. This implies that the inequality (9) holds. On the other hand, it follows from (7) and (8) that

$$\begin{aligned}
&N\left(\sum_{j=1}^{n} L\left(\frac{1}{2}\sum_{1\leq i\leq n, i\neq j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i L(x_i) - nL\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right), t\right) \\
&= \lim_{k\to\infty} N\left(\frac{1}{2^k}\sum_{j=1}^{n} f\left(\sum_{1\leq i\leq n, i\neq j} r_i 2^{k-1} x_i - \frac{r_j 2^k x_j}{2}\right) + \frac{\sum_{i=1}^{n} r_i f(2^k x_i)}{2^k} - \frac{n}{2^k} f\left(\sum_{i=1}^{n} r_i 2^{k-1} x_i\right), t\right) \\
&\geq \lim_{k\to\infty} \frac{2^k t}{2^k t + \varphi(2^k x_1, \cdots, 2^k x_n)} \geq \lim_{k\to\infty} \frac{2^k t}{2^k t + 2^k L^k \varphi(x_1, \cdots, x_n)} \to 1
\end{aligned}$$

for all $x_1, \ldots, x_n \in X$ and all $t > 0$. Therefore, the mapping $L : X \to Y$ satisfies the equation (1). □

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying the following inequality*

$$\begin{aligned}
&N\left(\sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\leq i\leq n, i\neq j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i f(x_i) - nf\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right), t\right) \\
&\qquad\qquad \geq \frac{t}{t + \theta\left(\sum_{k=1}^{n} \|x_k\|^p\right)}
\end{aligned} \tag{20}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Then there exists a unique ELTA mapping $L : X \to Y$ such that

$$N(f(x) - L(x), t) \geq \frac{|r_i|^p |r_j|^p (2^{p+2} - 8)t}{|r_i|^p |r_j|^p (2^{p+2} - 8)t + 9.2^p (|r_i|^p + |r_j|^p)(2 + 2^{p+1})\|x\|^p} \tag{21}$$

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x_1, \cdots, x_n) := \theta\left(\sum_{k=1}^{n} \|x_k\|^p\right)$$

for all $x_1, \cdots, x_n \in X$. Then we have

$$\begin{aligned}
\Psi(x) &:= \varphi_{ij}\left(\frac{2x}{r_i}, \frac{2x}{r_j}\right) + \varphi_{ij}\left(\frac{2x}{r_i}, 0\right) + \varphi_{ij}\left(0, \frac{2x}{r_j}\right) + \varphi_{ij}\left(\frac{x}{r_i}, -\frac{x}{r_j}\right) \\
&+ \varphi_{ij}\left(\frac{x}{r_i}, 0\right) + \varphi_{ij}\left(0, -\frac{x}{r_j}\right) \\
&= \frac{(|r_i|^p + |r_j|^p)(2 + 2^{p+1})\|x\|^p}{|r_i|^p |r_j|^p},
\end{aligned}$$

for all $x \in X$. Now if we choose $L = 2^{1-p}$, we get the desired result. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $p > \frac{1}{2}$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying the following inequality*

$$N\left(\sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\leq i\leq n, i\neq j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i f(x_i) - nf\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right), t\right) \geq \frac{t}{t + \theta\Big(\prod_{k=1}^{n} \|x_k\|^p\Big)} \tag{22}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Then there exists a unique ELTA mapping $L : X \to Y$ such that

$$N(f(x) - L(x), t) \geq \frac{|r_i|^p |r_j|^p (2^{2p+2} - 8)t}{|r_i|^p |r_j|^p (2^{2p+2} - 8)t + 9.2^{2p}(1 + 2^{2p})\|x\|^{2p}} \tag{23}$$

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x_1, \cdots, x_n) := \theta\left(\prod_{k=1}^{n} \|x_k\|^p\right)$$

for all $x_1, \cdots, x_n \in X$. Then we have

$$\Psi(x) := \frac{(1 + 2^{2p})\|x\|^{2p}}{|r_i|^p |r_j|^p},$$

for all $x \in X$. Now if we choose $L = 2^{1-2p}$, we get the desired result. □

Theorem 2.6. *Let $\varphi : X^n \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\varphi\left(\frac{x_1}{2}, \frac{x_2}{2} \cdots, \frac{x_n}{2}\right) \leq \frac{L\varphi(x_1, x_2, \cdots, x_n)}{2}$ for all $x_1, \cdots, x_n \in X$. Let $f : X \to Y$ with $f(0) = 0$ is a mapping satisfying (8) for all $x_1, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique ELTA mapping $L : X \to Y$ such that*

$$N(f(x) - L(x), t) \geq \frac{(4 - 4L)t}{(4 - 4L)t + 9L\Psi(x)} \tag{24}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (14) and (16) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{9t}{2}\right) \geq \frac{t}{t + \Psi\left(\frac{x}{2}\right)} \geq \frac{t}{t + \frac{L}{2}\Psi(x)} \tag{25}$$

for all $x \in X$ and all $t > 0$. So

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{9Lt}{4}\right) \geq \frac{t}{t + \Psi(x)}. \tag{26}$$

Let (S, d) be the generalized metric space defined in the Theorem 2.1. Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then $N(g(x) - h(x), \epsilon t) \geq \frac{t}{t+\Psi(x)}$ for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L\epsilon t}{2}\right) &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \Psi\left(\frac{x}{2}\right)} \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\Psi(x)} = \frac{t}{t + \Psi(x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g,h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g,h)$ for all $g, h \in S$. It follows from (26) that $d(f, Jf) \leq \frac{9L}{4}$. By Theorem 2.1, there exists a mapping $L : X \to Y$ satisfying the following:
(1) L is a fixed point of J, that is,

$$L\left(\frac{x}{2}\right) = \frac{L(x)}{2} \tag{27}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that L is a unique mapping satisfying (27) such that there exists $\mu \in (0,\infty)$ satisfying $N(f(x) - L(x), \mu t) \geq \frac{t}{t+\Psi(x)}$ for all $x \in X$ and $t > 0$.
(2) $d(J^l f, L) \to 0$ as $l \to \infty$. This implies the equality

$$N\text{-}\lim_{l\to\infty} 2^l f\left(\frac{x}{2^l}\right) = L(x)$$

for all $x \in X$.
(3) $d(f, L) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f, L) \leq \frac{9L}{4-4L}$. This implies that the inequality (24) holds. The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.7. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying the inequality (20). Then there exists a unique ELTA mapping $L : X \to Y$ such that*

$$N(f(x) - L(x), t) \geq \frac{|r_i|^p |r_j|^p (8 - 2^{p+2})t}{|r_i|^p |r_j|^p (8 - 2^{p+2})t + 9(|r_i|^p + |r_j|^p)(2^{p+1} + 2^{2p+1})\|x\|^p}$$

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x_1, \cdots, x_n) := \theta\left(\sum_{k=1}^{n} \|x_k\|^p\right)$$

for all $x_1, \cdots, x_n \in X$. Then we choose $L = 2^{p-1}$ to get the desired result. □

Corollary 2.8. *Let $\theta \geq 0$ and let p be a real number with $0 < 2p < 1$. Let X be a normed vector space with norm $\|.\|$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying the inequality (22). Then there exists a unique ELTA mapping $L : X \to Y$ such that*

$$N(f(x) - L(x), t) \geq \frac{|r_i|^p |r_j|^p (8 - 2^{2p+2})t}{|r_i|^p |r_j|^p (8 - 2^{2p+2})t + 9.2^{2p}(1 + 2^{2p})\|x\|^{2p}}$$

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x_1, \cdots, x_n) := \theta\left(\prod_{k=1}^{n} \|x_k\|^p\right)$$

for all $x_1, \cdots, x_n \in X$. Then we choose $L = 2^{p-1}$ to get the desired result. □

3. Fuzzy stability of functional equation (1): a direct method

In this section, using direct method, we prove the HUR-stability of functional equation (1) in fuzzy Banach spaces. Throughout this section, we assume that X is a linear space, (Y, N) is a fuzzy Banach space and (Z, N') is a fuzzy normed spaces. Moreover, we assume that $N(x, .)$ is a left continuous function on $\mathbb{R}$.

Theorem 3.1. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N\left(\sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i f(x_i) - n f\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right), t\right) \ge N'(\varphi(x_1, \cdots, x_n), t) \tag{28}$$

for all $x_1, \cdots, x_n \in X$, $t > 0$ and $\varphi : X^n \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < \frac{1}{2}$ such that

$$N'\left(\varphi\left(\frac{x_1}{2}, \cdots, \frac{x_n}{2}\right), t\right) \ge N'\left(\varphi(x_1, \cdots, x_n), \frac{t}{|r|}\right) \tag{29}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Then there exists a unique ELTA mapping $L : X \to Y$ satisfying (1) and the inequality

$$\begin{aligned} &N(f(x) - L(x), t) \\ &\ge min\Big\{ \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{2(1-2|r|)t}{9}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2(1-2|r|)t}{9}\right) \\ &, N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{2(1-2|r|)t}{9}\right)\Big\}, \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{2(1-2|r|)t}{9|r|}\right) \\ &, N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2(1-2|r|)t}{9|r|}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{2(1-2|r|)t}{9|r|}\right)\Big\}\Big\} \end{aligned} \tag{30}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (29) that

$$N'\left(\varphi\left(\frac{x_1}{2^k}, \cdots, \frac{x_n}{2^k}\right), t\right) \ge N'\left(\varphi(x_1, \cdots, x_n), \frac{t}{|r|^k}\right)$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Putting $x_k = 0$ in (28) for all $1 \le k \le n$ with $k \ne i, j$, we have

$$\begin{aligned} &N\left(f\left(\frac{-r_i x_i + r_j x_j}{2}\right) + f\left(\frac{r_i x_i - r_j x_j}{2}\right) - 2f\left(\frac{r_i x_i + r_j x_j}{2}\right) + r_i f(x_i) + r_j f(x_j), t\right) \\ &\ge N'(\varphi_{ij}(x_i, x_j), t) \end{aligned} \tag{31}$$

for all $x_i, x_j \in X$ and all $t > 0$. Letting $x_i = 0$ in (31), we get

$$N\left(f\left(-\frac{r_j x_j}{2}\right) - f\left(\frac{r_j x_j}{2}\right) + r_j f(x_j), t\right) \ge N'(\varphi_{ij}(0, x_j), t) \tag{32}$$

for all $x_j \in X$ and all $t > 0$. Similarly, letting $x_j = 0$ in (31), we get

$$N\left(f\left(-\frac{r_i x_i}{2}\right) - f\left(\frac{r_i x_i}{2}\right) + r_i f(x_i), t\right) \ge N'(\varphi_{ij}(x_i, 0), t) \tag{33}$$

for all $x_i \in X$ and all $t > 0$. It follows from (31), (32) and (33) that

$$\begin{aligned} &N\Big(f\left(\frac{-r_i x_i + r_j x_j}{2}\right) + f\left(\frac{r_i x_i - r_j x_j}{2}\right) - 2f\left(\frac{r_i x_i + r_j x_j}{2}\right) + f\left(\frac{r_i x_i}{2}\right) \\ &+ f\left(\frac{r_j x_j}{2}\right) - f\left(-\frac{r_i x_i}{2}\right) - f\left(-\frac{r_j x_j}{2}\right), 3t\Big) \\ &\ge min\left\{N'(\varphi_{ij}(x_i, x_j), t), N'(\varphi_{ij}(x_i, 0), t), N'(\varphi_{ij}(0, x_j), t)\right\} \end{aligned} \tag{34}$$

for all $x_i, x_j \in X$ and all $t > 0$. Replacing x_i and x_j by $\frac{2x}{r_i}$ and $\frac{2y}{r_j}$ in (34), we get that

$$\begin{aligned} &N(f(-x+y)+f(x-y)-2f(x+y)+f(x)+f(y)-f(-x)-f(-y),3t) \\ &\geq \min\Big\{N'\left(\varphi_{ij}\left(\frac{2x}{r_i},\frac{2y}{r_j}\right),t\right),N'\left(\varphi_{ij}\left(\frac{2x}{r_i},0\right),t\right) \\ &,N'\left(\varphi_{ij}\left(0,\frac{2y}{r_j}\right),t\right)\Big\} \end{aligned} \tag{35}$$

for all $x, y \in X$ and all $t > 0$. Putting $x = y$ in (35), we get

$$\begin{aligned} &N(2f(x)-2f(-x)-2f(2x),3t) \\ &\geq \min\left\{N'\left(\varphi_{ij}\left(\frac{2x}{r_i},\frac{2x}{r_j}\right),t\right),N'\left(\varphi_{ij}\left(\frac{2x}{r_i},0\right),t\right),N'\left(\varphi_{ij}\left(0,\frac{2x}{r_j}\right),t\right)\right\} \end{aligned} \tag{36}$$

for all $x \in X$ and all $t > 0$. Replacing x and y by $\frac{x}{2}$ and $-\frac{x}{2}$ in (35), respectively

$$\begin{aligned} &N(f(x)+f(-x),3t) \\ &\geq \min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i},\frac{-x}{r_j}\right),t\right),N'\left(\varphi_{ij}\left(\frac{x}{r_i},0\right),t\right),N'\left(\varphi_{ij}\left(0,\frac{-x}{r_j}\right),t\right)\right\} \end{aligned} \tag{37}$$

for all $x \in X$ and all $t > 0$. It follows from (36) and (37) that

$$\begin{aligned} &N\left(4f(x)-2f(2x),9t\right) \\ &\geq min\Big\{\min\left\{N'\left(\varphi_{ij}\left(\frac{2x}{r_i},\frac{2x}{r_j}\right),t\right),N'\left(\varphi_{ij}\left(\frac{2x}{r_i},0\right),t\right),N'\left(\varphi_{ij}\left(0,\frac{2x}{r_j}\right),t\right)\right\} \\ &,\min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i},\frac{-x}{r_j}\right),t\right),N'\left(\varphi_{ij}\left(\frac{x}{r_i},0\right),t\right),N'\left(\varphi_{ij}\left(0,\frac{-x}{r_j}\right),t\right)\right\}\Big\} \end{aligned} \tag{38}$$

for all $x \in X$ and all $t > 0$. So

$$\begin{aligned} &N\left(2f\left(\frac{x}{2}\right)-f(x),\frac{9t}{2}\right) \\ &\geq min\Big\{\min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i},\frac{x}{r_j}\right),t\right),N'\left(\varphi_{ij}\left(\frac{x}{r_i},0\right),t\right),N'\left(\varphi_{ij}\left(0,\frac{x}{r_j}\right),t\right)\right\} \\ &,\min\left\{N'\left(\varphi_{ij}\left(\frac{x}{2r_i},\frac{-x}{2r_j}\right),t\right),N'\left(\varphi_{ij}\left(\frac{x}{2r_i},0\right),t\right),N'\left(\varphi_{ij}\left(0,\frac{-x}{2r_j}\right),t\right)\right\}\Big\} \end{aligned} \tag{39}$$

for all $x \in X$ and all $t > 0$. Replacing x by $\frac{x}{2^k}$ in (39), we have

$$\begin{aligned} &N\left(2^{k+1}f\left(\frac{x}{2^{k+1}}\right)-2^kf\left(\frac{x}{2^k}\right),\frac{9.2^kt}{2}\right) \\ &\geq min\Big\{\min\left\{N'\left(\varphi_{ij}\left(\frac{x}{2^kr_i},\frac{x}{2^kr_j}\right),t\right),N'\left(\varphi_{ij}\left(\frac{x}{2^kr_i},0\right),t\right),N'\left(\varphi_{ij}\left(0,\frac{x}{2^kr_j}\right),t\right)\right\} \\ &,\min\left\{N'\left(\varphi_{ij}\left(\frac{x}{2^{k+1}r_i},\frac{-x}{2^{k+1}r_j}\right),t\right),N'\left(\varphi_{ij}\left(\frac{x}{2^{k+1}r_i},0\right),t\right),N'\left(\varphi_{ij}\left(0,\frac{-x}{2^{k+1}r_j}\right),t\right)\right\}\Big\} \\ &\geq min\Big\{\min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i},\frac{x}{r_j}\right),\frac{t}{|r|^k}\right),N'\left(\varphi_{ij}\left(\frac{x}{r_i},0\right),\frac{t}{|r|^k}\right),N'\left(\varphi_{ij}\left(0,\frac{x}{r_j}\right),\frac{t}{|r|^k}\right)\right\} \\ &,\min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i},\frac{-x}{r_j}\right),\frac{t}{|r|^{k+1}}\right),N'\left(\varphi_{ij}\left(\frac{x}{r_i},0\right),\frac{t}{|r|^{k+1}}\right),N'\left(\varphi_{ij}\left(0,\frac{-x}{r_j}\right),\frac{t}{|r|^{k+1}}\right)\right\}\Big\} \end{aligned} \tag{40}$$

for all $x \in X$, all $t > 0$ and any integer $k \geq 0$. So

$$
\begin{aligned}
&N\left(f(x) - 2^l f\left(\frac{x}{2^l}\right), \sum_{k=0}^{l-1} \frac{9.2^k|r|^k t}{2}\right)\\
&= N\left(\sum_{k=0}^{l-1} 2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right), \sum_{k=0}^{l-1} \frac{9.2^k|r|^k t}{2}\right)\\
&\geq \min \bigcup_{k=0}^{l-1} \left\{N\left(2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right), \frac{9.2^k|r|^k t}{2}\right)\right\}\\
&\geq min\Big\{ \min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), t\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), t\right), N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), t\right)\right\}\\
&, \min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{t}{|r|}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{t}{|r|}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{t}{|r|}\right)\right\}\Big\}
\end{aligned}
\tag{41}
$$

which yields

$$
\begin{aligned}
&N\left(2^{l+p} f\left(\frac{x}{2^{l+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), \sum_{k=0}^{l-1} \frac{9.2^{k+p}|r|^k t}{2}\right)\\
&\geq min\Big\{ \min\left\{N'\left(\varphi_{ij}\left(\frac{x}{2^p r_i}, \frac{x}{2^p r_j}\right), t\right), N'\left(\varphi_{ij}\left(\frac{x}{2^p r_i}, 0\right), t\right), N'\left(\varphi_{ij}\left(0, \frac{x}{2^p r_j}\right), t\right)\right\}\\
&, \min\left\{N'\left(\varphi_{ij}\left(\frac{x}{2^p r_i}, \frac{-x}{2^p r_j}\right), \frac{t}{|r|}\right), N'\left(\varphi_{ij}\left(\frac{x}{2^p r_i}, 0\right), \frac{t}{|r|}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{2^p r_j}\right), \frac{t}{|r|}\right)\right\}\Big\}\\
&\geq min\Big\{ \min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{t}{|r|^p}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{t}{|r|^p}\right), N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{t}{|r|^p}\right)\right\}\\
&, \min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{t}{|r|^{p+1}}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{t}{|r|^{p+1}}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{t}{|r|^{p+1}}\right)\right\}\Big\}
\end{aligned}
$$

for all $x \in X$, $t > 0$ and any integers $l > 0$, $p \geq 0$. So

$$
\begin{aligned}
&N\left(2^{l+p} f\left(\frac{x}{2^{l+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), \sum_{k=0}^{l-1} \frac{9.2^{k+p}|r|^{k+p} t}{2}\right)\\
&\geq min\Big\{ \min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), t\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), t\right), N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), t\right)\right\}\\
&, \min\left\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{t}{|r|}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{t}{|r|}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{t}{|r|}\right)\right\}\Big\}
\end{aligned}
$$

for all $x \in X$, $t > 0$ and any integers $l > 0$, $p \geq 0$. Hence one obtains

$$
\begin{aligned}
&N\left(2^{l+p} f\left(\frac{x}{2^{l+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), t\right)\\
&\geq min\Big\{ \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{t}{\sum_{k=0}^{l-1} \frac{9.2^{k+p}|r|^{k+p}}{2}}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{t}{\sum_{k=0}^{l-1} \frac{9.2^{k+p}|r|^{k+p}}{2}}\right)\\
&, N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{t}{\sum_{k=0}^{l-1} \frac{9.2^{k+p}|r|^{k+p}}{2}}\right)\Big\}, \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{t}{\sum_{k=0}^{l-1} \frac{9.2^{k+p}|r|^{k+p+1}}{2}}\right)\\
&, N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{t}{\sum_{k=0}^{l-1} \frac{9.2^{k+p}|r|^{k+p+1}}{2}}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{t}{\sum_{k=0}^{l-1} \frac{9.2^{k+p}|r|^{k+p+1}}{2}}\right)\Big\}\Big\}
\end{aligned}
\tag{42}
$$

for all $x \in X$, $t > 0$ and any integers $l > 0$, $p \geq 0$. Since, the series $\sum_{k=0}^{+\infty} 2^k |r|^k$ is convergent series, we see by taking the limit $p \to \infty$ in the last inequality that the sequence $\left\{2^l f\left(\frac{x}{2^l}\right)\right\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N) and so it converges in Y. Therefore a mapping $L : X \to Y$ defined by $L(x) := N - \lim_{n\to\infty} 2^l f\left(\frac{x}{2^l}\right)$ is well defined for all $x \in X$. It means that

$$\lim_{n\to\infty} N\left(L(x) - 2^l f\left(\frac{x}{2^l}\right), t\right) = 1 \tag{43}$$

for all $x \in X$ and all $t > 0$. In addition, it follows from (42) that

$$\begin{aligned} & N\left(2^l f\left(\frac{x}{2^l}\right) - f(x), t\right) \\ & \geq min\Bigg\{ \min\Bigg\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{2t}{\sum_{k=0}^{l-1} 9.2^k|r|^k}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2t}{\sum_{k=0}^{l-1} 9.2^k|r|^k}\right) \\ & , N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{2t}{\sum_{k=0}^{l-1} 9.2^k|r|^k}\right)\Bigg\}, \min\Bigg\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{2t}{\sum_{k=0}^{l-1} 9.2^k|r|^{k+1}}\right) \\ & , N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2t}{\sum_{k=0}^{l-1} 9.2^k|r|^{k+1}}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{2t}{\sum_{k=0}^{l-1} 9.2^k|r|^{k+1}}\right)\Bigg\}\Bigg\} \end{aligned} \tag{44}$$

for all $x \in X$ and all $t > 0$. So

$$\begin{aligned} & N(f(x) - L(x), t) \\ & \geq \min\left\{N\left(f(x) - 2^l f\left(\frac{x}{2^l}\right), (1-\epsilon)t\right), N\left(L(x) - 2^l f\left(\frac{x}{2^l}\right), \epsilon t\right)\right\} \\ & \geq min\Bigg\{ \min\Bigg\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{2\epsilon t}{\sum_{k=0}^{l-1} 9.2^k|r|^k}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2\epsilon t}{\sum_{k=0}^{l-1} 9.2^k|r|^k}\right) \\ & , N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{2\epsilon t}{\sum_{k=0}^{l-1} 9.2^k|r|^k}\right)\Bigg\}, \min\Bigg\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{2\epsilon t}{\sum_{k=0}^{l-1} 9.2^k|r|^{k+1}}\right) \\ & , N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2\epsilon t}{\sum_{k=0}^{l-1} 9.2^k|r|^{k+1}}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{2\epsilon t}{\sum_{k=0}^{l-1} 9.2^k|r|^{k+1}}\right)\Bigg\}\Bigg\} \\ & \geq min\Bigg\{ \min\Bigg\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{2\epsilon(1-2|r|)t}{9}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2\epsilon(1-2|r|)t}{9}\right) \\ & , N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{2\epsilon(1-2|r|)t}{9}\right)\Bigg\}, \min\Bigg\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{2\epsilon(1-2|r|)t}{9|r|}\right) \\ & , N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2\epsilon(1-2|r|)t}{9|r|}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{2\epsilon(1-2|r|)t}{9|r|}\right)\Bigg\}\Bigg\} \end{aligned}$$

for sufficiently large l and for all $x \in X$, $t > 0$ and ϵ with $0 < \epsilon < 1$. Since ϵ is arbitrary and N' is left continuous, we obtain

$$\begin{aligned} & N(f(x) - L(x), t) \\ & \geq min\Bigg\{ \min\Bigg\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{2(1-2|r|)t}{9}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2(1-2|r|)t}{9}\right) \\ & , N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{2(1-2|r|)t}{9}\right)\Bigg\}, \min\Bigg\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{2(1-2|r|)t}{9|r|}\right) \\ & , N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2(1-2|r|)t}{9|r|}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{2(1-2|r|)t}{9|r|}\right)\Bigg\}\Bigg\} \end{aligned}$$

for all $x \in X$ and $t > 0$. It follows from (28) that

$$
\begin{aligned}
&N\left(2^l \sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j}\frac{r_i x_i}{2^l} - \frac{r_j x_j}{2^{l+1}}\right) + 2^l \sum_{i=1}^{n} r_i f\left(\frac{x_i}{2^l}\right) - 2^l n f\left(\sum_{i=1}^{n}\frac{r_i x_i}{2^{l+1}}\right), t\right)\\
&\ge N'\left(\varphi\left(\frac{x_1}{2^l}, \cdots, \frac{x_n}{2^l}\right), \frac{t}{2^l}\right) \ge N'\left(\varphi(x_1, \cdots, x_n), \frac{t}{2^l |r|^l}\right)
\end{aligned}
$$

for all $x_1, \cdots, x_n \in X$, $t > 0$ and all $l \in \mathbb{N}$. Since $\lim_{l\to\infty} N'\left(\varphi(x_1, \cdots, x_n), \frac{t}{2^l|r|^l}\right) = 1$ and so

$$
N\left(2^l \sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j}\frac{r_i x_i}{2^l} - \frac{r_j x_j}{2^{l+1}}\right) + 2^l \sum_{i=1}^{n} r_i f\left(\frac{x_i}{2^l}\right) - 2^l n f\left(\sum_{i=1}^{n}\frac{r_i x_i}{2^{l+1}}\right), t\right) \to 1
$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Therefore, we obtain in view of (43)

$$
\begin{aligned}
&N\left(\sum_{j=1}^{n} L\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i L(x_i) - nL\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right), t\right)\\
&\ge \min\Big\{N\Big(\sum_{j=1}^{n} L\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i L(x_i) - nL\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right)\\
&-2^k \sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j}\frac{r_i x_i}{2^k} - \frac{r_j x_j}{2^{k+1}}\right) - 2^k \sum_{i=1}^{n} r_i f\left(\frac{x_i}{2^k}\right) + 2^k n f\left(\sum_{i=1}^{n}\frac{r_i x_i}{2^{k+1}}\right), \frac{t}{2}\Big)\\
&, N\Big(2^k \sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j}\frac{r_i x_i}{2^k} - \frac{r_j x_j}{2^{k+1}}\right) + 2^k \sum_{i=1}^{n} r_i f\left(\frac{x_i}{2^k}\right) - 2^k n f\left(\sum_{i=1}^{n}\frac{r_i x_i}{2^{k+1}}\right), \frac{t}{2}\Big)\Big\}\\
&\ge N\left(2^k \sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j}\frac{r_i x_i}{2^k} - \frac{r_j x_j}{2^{k+1}}\right) + 2^k \sum_{i=1}^{n} r_i f\left(\frac{x_i}{2^k}\right) - 2^k n f\left(\sum_{i=1}^{n}\frac{r_i x_i}{2^{k+1}}\right), \frac{t}{2}\right)\\
&\ge N'\left(\varphi(x_1, \cdots, x_n), \frac{t}{2^{k+1}|r|^k}\right) \to 1 \quad \text{as } k \to \infty
\end{aligned}
$$

which implies

$$
\sum_{j=1}^{n} L\left(\frac{1}{2}\sum_{1\le i\le n, i\ne j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i L(x_i) - nL\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right) = 0
$$

for all $x_1, \cdots, x_n \in X$. Thus $L : X \to Y$ is a mapping satisfying the equation (1) and the inequality (30). To prove the uniqueness, let there is another mapping $\mathfrak{R} : X \to Y$ which satisfies the inequality (30). Since

$\mathfrak{R}(2^k x) = 2^k\mathfrak{R}(x)$ for all $x \in X$, we have

$$
\begin{aligned}
&N(L(x) - \mathfrak{R}(x), t)\\
&= N\left(2^k L\left(\frac{x}{2^k}\right) - 2^k\mathfrak{R}\left(\frac{x}{2^k}\right), t\right)\\
&\geq \min\left\{N\left(2^k L\left(\frac{x}{2^k}\right) - 2^k f\left(\frac{x}{2^k}\right), \frac{t}{2}\right), N\left(2^k f\left(\frac{x}{2^k}\right) - 2^k\mathfrak{R}\left(\frac{x}{2^k}\right), \frac{t}{2}\right)\right\}\\
&\geq min\Big\{ \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{2^k r_i}, \frac{x}{2^{k+1} r_j}\right), \frac{(1-2|r|)t}{9.2^k}\right), N'\left(\varphi_{ij}\left(\frac{x}{2^k r_i}, 0\right), \frac{(1-2|r|)t}{9.2^{k+1}}\right)\\
&, N'\left(\varphi_{ij}\left(0, \frac{x}{2^k r_j}\right), \frac{(1-2|r|)t}{9.2^{k+1}}\right)\Big\}, \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{2^k r_i}, \frac{-x}{r_j}\right), \frac{(1-2|r|)t}{9.2^{k+1}|r|}\right)\\
&, N'\left(\varphi_{ij}\left(\frac{x}{2^k r_i}, 0\right), \frac{(1-2|r|)t}{9.2^{k+1}|r|}\right), N'\Big(\varphi_{ij}\left(0, \frac{-x}{2^k r_j}\right), \frac{(1-2|r|)t}{9.2^{k+1}|r|}\Big)\Big\}\Big\}\\
&\geq min\Big\{ \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{(1-2|r|)t}{9.2^{k+1}|r|^k}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{(1-2|r|)t}{9.2^{k+1}|r|^k}\right)\\
&, N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{(1-2|r|)t}{9.2^{k+1}|r|^k}\right)\Big\}, \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{(1-2|r|)t}{9.2^{k+1}|r|^{k+1}}\right)\\
&, N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{(1-2|r|)t}{9.2^{k+1}|r|^{k+1}}\right), N'\Big(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{(1-2|r|)t}{9.2^{k+1}|r|^{k+1}}\Big)\Big\}\Big\} \to 1 \text{ as } k \to \infty
\end{aligned}
$$

for all $t > 0$. Therefore $L(x) = \mathfrak{R}(x)$ for all $x \in X$. This completes the proof. □

Corollary 3.2. *Let X be a normed spaces and that $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $p > 1$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following inequality*

$$
\begin{aligned}
N\left(\sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\leq i\leq n, i\neq j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i f(x_i) - nf\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right), t\right)&\\
\geq N'\left(\theta\left(\sum_{k=1}^{n} \|x_k\|^p\right), t\right)& \quad (45)
\end{aligned}
$$

for all $x_1, \cdots, x_n \in X$ and $t > 0$. Then there is a unique ELTA mapping $L : X \to Y$ that satisfying (1) and the inequality

$$N(f(x) - L(x), t) \geq N'\left(\theta\|x\|^p, \frac{(2^p - 2)|r_i|^p |r_j|^p t}{9(|r_i|^p + |r_j|^p)2^{p-1}}\right)$$

Proof. Defining $\varphi : X^n \to Z$ by $\varphi(x_1, \cdots, x_n) := \theta\left(\sum_{k=1}^{n} \|x_k\|^p\right)$ and $|r| = 2^{-p}$. Apply Theorem 3.1, we get desired results. □

Corollary 3.3. *Let X be a normed spaces and that $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $2p > 1$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following inequality*

$$
\begin{aligned}
N\left(\sum_{j=1}^{n} f\left(\frac{1}{2}\sum_{1\leq i\leq n, i\neq j} r_i x_i - \frac{1}{2} r_j x_j\right) + \sum_{i=1}^{n} r_i f(x_i) - nf\left(\frac{1}{2}\sum_{i=1}^{n} r_i x_i\right), t\right)&\\
\geq N'\left(\theta\left(\prod_{k=1}^{n} \|x_k\|^p\right), t\right)& \quad (46)
\end{aligned}
$$

for all $x_1, \cdots, x_n \in X$ and $t > 0$. Then there is a unique ELTA mapping $L : X \to Y$ that satisfying (1) and the inequality

$$N(f(x) - L(x), t) \geq N'\left(\theta\|x\|^{2p}, \frac{(2^p - 2)|r_i|^p|r_j|^p t}{9.2^{p-1}}\right)$$

Proof. Defining $\varphi : X^n \to Z$ by $\varphi(x_1, \cdots, x_n) := \theta\left(\prod_{k=1}^n \|x_k\|^p\right)$ and $|r| = 2^{-p}$. Apply Theorem 3.1, we get desired results. □

Theorem 3.4. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (28) and $\varphi : X^n \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < 2$ such that*

$$N'\left(\varphi(x_1, \cdots, x_n), |r|t\right) \geq N'\left(\varphi\left(\frac{x_1}{2}, \cdots, \frac{x_n}{2}\right), t\right) \tag{47}$$

for all $x_1, \cdots, x_n \in X$ and all $t > 0$. Then there exists a unique ELTA mapping $L : X \to Y$ that satisfying (1) and the following inequality

$$\begin{aligned}
&N\left(f(x) - L(x), t\right)\\
&\geq min\Big\{\min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{2(2-|r|)t}{9|r|}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2(2-|r|)t}{9|r|}\right)\\
&, N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{2(2-|r|)t}{9|r|}\right)\Big\}, \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{2(2-|r|)t}{9}\right)\\
&, N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{2(2-|r|)t}{9}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{2(2-|r|)t}{9}\right)\Big\}\Big\},
\end{aligned}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (38) that

$$\begin{aligned}
&N\left(f(x) - \frac{f(2x)}{2}, \frac{9t}{4}\right)\\
&\geq min\Big\{\min\Big\{N'\left(\varphi_{ij}\left(\frac{2x}{r_i}, \frac{2x}{r_j}\right), t\right), N'\left(\varphi_{ij}\left(\frac{2x}{r_i}, 0\right), t\right), N'\left(\varphi_{ij}\left(0, \frac{2x}{r_j}\right), t\right)\Big\}\\
&, \min\Big\{N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), t\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), t\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), t\right)\Big\}\Big\},
\end{aligned} \tag{48}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2^l x$ in (48), we obtain

$$\begin{aligned}
&N\left(\frac{f(2^l x)}{2^l} - \frac{f(2^{l+1}x)}{2^{l+1}}, \frac{9t}{2^{l+2}}\right)\\
&\geq min\Big\{\min\Big\{N'\left(\varphi_{ij}\left(\frac{2^{l+1}x}{r_i}, \frac{2^{l+1}x}{r_j}\right), t\right), N'\left(\varphi_{ij}\left(\frac{2^{l+1}x}{r_i}, 0\right), t\right)\\
&, N'\left(\varphi_{ij}\left(0, \frac{2^{l+1}x}{r_j}\right), t\right)\Big\}, \min\Big\{N'\left(\varphi_{ij}\left(\frac{2^l x}{r_i}, \frac{-2^l x}{r_j}\right), t\right), N'\left(\varphi_{ij}\left(\frac{2^l x}{r_i}, 0\right), t\right)\\
&, N'\left(\varphi_{ij}\left(0, \frac{-2^l x}{r_j}\right), t\right)\Big\}\Big\}
\end{aligned} \tag{49}$$

$$\geq min\Big\{\min\Big\{N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},\frac{x}{r_j}\Big),\frac{t}{|r|^{l+1}}\Big),N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},0\Big),\frac{t}{|r|^{l+1}}\Big)$$
$$,N'\Big(\varphi_{ij}\Big(0,\frac{x}{r_j}\Big),\frac{t}{|r|^{l+1}}\Big)\Big\},\min\Big\{N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},\frac{-x}{r_j}\Big),\frac{t}{|r|^{l}}\Big),N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},0\Big),\frac{t}{|r|^{l}}\Big)$$
$$,N'\Big(\varphi_{ij}\Big(0,\frac{-x}{r_j}\Big),\frac{t}{|r|^{l}}\Big)\Big\}\Big\}.$$

for all $x \in X$ and all $t > 0$. So

$$N\Big(\frac{f(2^l x)}{2^l}-\frac{f(2^{l+1}x)}{2^{l+1}},\frac{9|r|^l t}{2^{l+2}}\Big)$$
$$\geq min\Big\{\min\Big\{N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},\frac{x}{r_j}\Big),\frac{t}{|r|}\Big),N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},0\Big),\frac{t}{|r|}\Big) \tag{50}$$
$$,N'\Big(\varphi_{ij}\Big(0,\frac{x}{r_j}\Big),\frac{t}{|r|}\Big)\Big\},\min\Big\{N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},\frac{-x}{r_j}\Big),t\Big),N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},0\Big),t\Big)$$
$$,N'\Big(\varphi_{ij}\Big(0,\frac{-x}{r_j}\Big),t\Big)\Big\}\Big\}.$$

for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 3.1, we obtain that

$$N\Big(f(x)-\frac{f(2^l x)}{2^l},\sum_{k=0}^{l-1}\frac{9|r|^k t}{2^{k+2}}\Big)$$
$$\geq min\Big\{\min\Big\{N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},\frac{x}{r_j}\Big),\frac{t}{|r|}\Big),N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},0\Big),\frac{t}{|r|}\Big) \tag{51}$$
$$,N'\Big(\varphi_{ij}\Big(0,\frac{x}{r_j}\Big),\frac{t}{|r|}\Big)\Big\},\min\Big\{N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},\frac{-x}{r_j}\Big),t\Big),N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},0\Big),t\Big)$$
$$,N'\Big(\varphi_{ij}\Big(0,\frac{-x}{r_j}\Big),t\Big)\Big\}\Big\},$$

for all $x \in X$, all $t > 0$ and any integer $l > 0$. So

$$N\Big(f(x)-\frac{f(2^l x)}{2^l},t\Big)$$
$$\geq min\Big\{\min\Big\{N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},\frac{x}{r_j}\Big),\frac{t}{\sum_{k=0}^{l-1}\frac{9|r|^{k+1}}{2^{k+2}}}\Big),N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},0\Big),\frac{t}{\sum_{k=0}^{l-1}\frac{9|r|^{k+1}}{2^{k+2}}}\Big)$$
$$,N'\Big(\varphi_{ij}\Big(0,\frac{x}{r_j}\Big),\frac{t}{\sum_{k=0}^{l-1}\frac{9|r|^{k+1}}{2^{k+2}}}\Big)\Big\},\min\Big\{N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},\frac{-x}{r_j}\Big),\frac{t}{\sum_{k=0}^{l-1}\frac{9|r|^{k}}{2^{k+2}}}\Big) \tag{52}$$
$$,N'\Big(\varphi_{ij}\Big(\frac{x}{r_i},0\Big),\frac{t}{\sum_{k=0}^{l-1}\frac{9|r|^{k}}{2^{k+2}}}\Big),N'\Big(\varphi_{ij}\Big(0,\frac{-x}{r_j}\Big),\frac{t}{\sum_{k=0}^{l-1}\frac{9|r|^{k}}{2^{k+2}}}\Big)\Big\}\Big\},$$

for all $x \in X$ and all $t > 0$. Thus

$$N(f(x) - L(x), t)$$
$$\geq min\Big\{ \min\Big\{ N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \frac{t}{\sum_{k=0}^{+\infty} \frac{9|r|^{k+1}}{2^{k+2}}}\right), N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{t}{\sum_{k=0}^{+\infty} \frac{9|r|^{k+1}}{2^{k+2}}}\right)$$
$$, N'\left(\varphi_{ij}\left(0, \frac{x}{r_j}\right), \frac{t}{\sum_{k=0}^{+\infty} \frac{9|r|^{k+1}}{2^{k+2}}}\right)\Big\}, \min\Big\{ N'\left(\varphi_{ij}\left(\frac{x}{r_i}, \frac{-x}{r_j}\right), \frac{t}{\sum_{k=0}^{+\infty} \frac{9|r|^{k}}{2^{k+2}}}\right)$$
$$, N'\left(\varphi_{ij}\left(\frac{x}{r_i}, 0\right), \frac{t}{\sum_{k=0}^{+\infty} \frac{9|r|^{k}}{2^{k+2}}}\right), N'\left(\varphi_{ij}\left(0, \frac{-x}{r_j}\right), \frac{t}{\sum_{k=0}^{+\infty} \frac{9|r|^{k}}{2^{k+2}}}\right)\Big\}\Big\},$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.5. *Let X be a normed spaces and that $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $p < 1$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the following inequality (45). Then there is a unique ELTA mapping $L : X \to Y$ that satisfying (1) and the inequality*

$$N(f(x) - L(x), t) \geq N'\left(\theta \|x\|^p, \frac{2(2-2^p)|r_i|^p |r_j|^p t}{9(|r_i|^p + |r_j|^p)2^p}\right)$$

Proof. Defining $\varphi : X^n \to Z$ by $\varphi(x_1, \cdots, x_n) := \theta\left(\sum_{k=1}^{n} \|x_k\|^p\right)$ and $|r| = 2^p$. Apply Theorem 3.1, we get desired results. □

Corollary 3.6. *Let X be a normed spaces and that $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $2p < 1$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies (46). Then there is a unique ELTA mapping $L : X \to Y$ that satisfying (1) and the inequality*

$$N(f(x) - L(x), t) \geq N'\left(\theta \|x\|^{2p}, \frac{(2-2^p)|r_i|^p |r_j|^p t}{9.2^p}\right)$$

Proof. Defining $\varphi : X^n \to Z$ by $\varphi(x_1, \cdots, x_n) := \theta\left(\prod_{k=1}^{n} \|x_k\|^p\right)$ and $|r| = 2^p$. Apply Theorem 3.1, we get desired results. □

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