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Asymptotically Best Possible Lebesgue-Type Inequalities for the Fourier Sums on Sets of Generalized Poisson Integrals

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Abstract. In this paper we establish Lebesgue-type inequalities for 2π-periodic functions f, which are defined by generalized Poisson integrals of the functions φ from L_p, $1 \le p < \infty$. In these inequalities uniform norms of deviations of Fourier sums $\|f - S_{n-1}\|_C$ are expressed via best approximations $E_n(\varphi)_{L_p}$ of functions φ by trigonometric polynomials in the metric of space L_p. We show that obtained estimates are asymptotically best possible.

1. Introduction

Let L_p, $1 \le p < \infty$, be the space of 2π–periodic functions f summable to the power p on $[0, 2\pi)$, in which the norm is given by the formula $\|f\|_p = \Big(\int\limits_0^{2\pi} |f(t)|^p dt\Big)^{\frac{1}{p}}$; L_∞ be the space of measurable and essentially bounded 2π–periodic functions f with the norm $\|f\|_\infty = \operatorname*{ess\,sup}\limits_t |f(t)|$; C be the space of continuous 2π–periodic functions f, in which the norm is specified by the equality $\|f\|_C = \max\limits_t |f(t)|$.

Denote by $C^{\alpha,r}_\beta L_p$, $\alpha > 0$, $r > 0$, $\beta \in \mathbb{R}$, $1 \le p \le \infty$, the set of all 2π–periodic functions, representable for all $x \in \mathbb{R}$ as convolutions of the form (see, e.g., [20, Ch.3, 7-8])

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t)\varphi(t)dt,\ a_0 \in \mathbb{R},\ \ \varphi \perp 1, \tag{1}$$

where $\varphi \in L_p$ and $P_{\alpha,r,\beta}(t)$ are the following generated kernels

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$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\Big(kt - \frac{\beta\pi}{2}\Big), \quad \alpha, r > 0, \ \beta \in \mathbb{R}. \tag{2}$$

The kernels $P_{\alpha,r,\beta}$ of the form (2) are called generalized Poisson kernels. For $r = 1$ and $\beta = 0$ the kernels $P_{\alpha,r,\beta}$ are usual Poisson kernels of harmonic functions.

If the functions f and φ are related by the equality (1), then the function f in this equality is called generalized Poisson integral of the function φ and is denoted by $\mathcal{J}^{\alpha,r}_{\beta}(\varphi)(f(\cdot) = \mathcal{J}^{\alpha,r}_{\beta}(\varphi,\cdot))$. The function φ in equality (1) is called generalized derivative of the function f and is denoted by $f^{\alpha,r}_{\beta}$ $(\varphi(\cdot) = f^{\alpha,r}_{\beta}(\cdot))$.

The set of functions f from $C^{\alpha,r}_{\beta}L_p$, $1 \le p \le \infty$, such that $f^{\alpha,r}_{\beta} \in B_p$, where

$$B_p = \Big\{\varphi : \ \|\varphi\|_p \le 1\Big\},$$

we will denote by $C^{\alpha,r}_{\beta,p}$.

The sets of generalized Poisson integrals $C^{\alpha,r}_{\beta}L_p$ are closely related with the well–known Gevrey classes (see, e.g. [21]).

Let τ_{2n-1} be the space of all trigonometric polynomials of degree at most $n-1$ and let $E_n(f)_{L_p}$ be the best approximation of the function $f \in L_p$ in the metric of space L_p, $1 \le p \le \infty$, by the trigonometric polynomials t_{n-1} of degree $n-1$, i.e.,

$$E_n(f)_{L_p} = \inf_{t_{n-1}\in\tau_{2n-1}} \|f - t_{n-1}\|_p.$$

Analogously, by $E_n(f)_C$ we denote the best uniform approximation of the function f from C by trigonometric polynomials of degree $n-1$, i.e.,

$$E_n(f)_C = \inf_{t_{n-1}\in\tau_{2n-1}} \|f - t_{n-1}\|_C.$$

Let $\rho_n(f;x)$ be the following quantity

$$\rho_n(f;x) := f(x) - S_{n-1}(f;x), \tag{3}$$

where $S_{n-1}(f;\cdot)$ are the partial Fourier sums of degree $n-1$ of a function f.

One can estimate the norms $\|\rho_n(f;\cdot)\|_C$ via $E_n(f)_C$ by Lebesgue inequalities

$$\|\rho_n(f;\cdot)\|_C \le (1 + L_{n-1})E_n(f)_C, \ n \in \mathbb{N}, \tag{4}$$

where quantites L_{n-1} are Lebesgue constants of the Fourier sums of the form

$$L_{n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_{n-1}(t)|dt = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \frac{|\sin(2n-1)t|}{\sin t}dt,$$

$$D_{n-1}(t) := \frac{1}{2} + \sum_{k=1}^{\infty} \cos kt = \frac{\sin(n-\frac{1}{2})t}{2\sin\frac{t}{2}}.$$

Fejer [3] established the asymptotic equality for Lebesgue constants L_n

$$L_n = \frac{4}{\pi^2}\ln n + O(1), \ \ n \to \infty.$$

More exact estimates for the differences $L_n - \frac{4}{\pi^2}\ln(n+a)$, $a > 0$, as $n \in \mathbb{N}$ were found in works [1], [2], [4], [8], [18] and [28].

In particular, it follows from [20] (see also [8, p.97]) that

$$\left|L_{n-1} - \frac{4}{\pi^2}\ln n\right| < 1,271, \quad n \in \mathbb{N}.$$

Then, the inequality (4) can be written in the form

$$\|\rho_n(f;\cdot)\|_C \le \left(\frac{4}{\pi^2}\ln n + R_n\right)E_n(f)_C, \tag{5}$$

where $|R_n| < 2,271$.

On the whole space C the inequality(5) is asymptotically exact. At the same there exist subsets of functions from C and for elements of these subsets the inequality (5) is not exact even by order (see, e.g., [24, p. 435]).

In the paper [10] the following estimate was established

$$\|\rho_n(f;\cdot)\|_C \le \sum_{v=n}^{2n-1} \frac{E_v(f)_C}{v-n+1}, \ f \in C, \ \ n \to \infty,$$

(here K is some absolute constant) and it was proved that this constant is exact by the order on the classes $C(\varepsilon) := \{f \in C : \ E_v(f)_C \le \varepsilon_v, \ v \in \mathbb{N}\}$, where $\{\varepsilon_v\}_{v=0}^{\infty}$ is a sequence of nonnegative numbers, such that $\varepsilon_v \downarrow 0$ as $v \to \infty$.

In [6], [7], [14], [22] and [24] the analogs of the Lebesque inequalities for functions $f \in C^{\alpha,r}_{\beta}L_p$ have been found in the case $r \in (0,1)$ and $p = \infty$, and also in the case $r \ge 1$ and $1 \le p \le \infty$, where the estimates for the deviations $\|f(\cdot) - S_{n-1}(f;\cdot)\|_C$ are expressed in terms of the best approximations $E_n(f^{\alpha,r}_{\beta})_{L_p}$. Namely, in [24] it was proved that for arbitrary $f \in C^{\alpha,r}_{\beta}$, $r \in (0,1)$, $\beta \in \mathbb{R}$, the following inequality holds

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \le \left(\frac{4}{\pi^2}\ln n^{1-r} + O(1)\right)e^{-\alpha n^r}E_n(f^{\alpha,r}_{\beta})_C, \tag{6}$$

where $O(1)$ is a quantity uniformly bounded with respect to n, β and $f \in C^{\alpha,r}_{\beta}C$. It was also shown that for any function $f \in C^{\alpha,r}_{\beta}C$ and for every $n \in \mathbb{N}$ one can find a function $\mathcal{F}(\cdot) = \mathcal{F}(f;n;\cdot)$ in the set $C^{\alpha,r}_{\beta}C$, such that $E_n(\mathcal{F}^{\alpha,r}_{\beta})_C = E_n(f^{\alpha,r}_{\beta})_C$ and for this function the relation (6) becomes an equality.

Least upper bounds of the quantity $\|\rho_n(f;\cdot)\|_C$ over the classes $C^{\alpha,r}_{\beta,p}$, we denote by $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$, i.e.,

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C = \sup_{f\in C^{\alpha,r}_{\beta,p}} \|\rho_n(f;\cdot)\|_C, \quad r > 0, \ \alpha > 0, \ 1 \le p \le \infty. \tag{7}$$

Asymptotic behaviour of the quantities $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$ of the form (7) was studied in [9], [11], [15]–[17], [19], [20], [23], [25], [27].

The present paper is a continuation of [6], [7], [14], [22] and [24] and is devoted to obtaining of asymptotically best possible analogs of Lebesgue-type inequalities on the sets $C^{\alpha,r}_{\beta}L_p$, $r \in (0,1)$ and $p \in [1,\infty)$. This case has not been considered yet.

It should be also noticed that asymptotically best possible Lebesgue inequalities on classes of generalized Poisson integrals $C^{\alpha,r}_{\beta}L_p$ for $r \in (0,1)$, $p = \infty$ and $r \ge 1$, $1 \le p \le \infty$ were also established for approximations by Lagrange trigonometric interpolation polynomials with uniform distribution of interpolation nodes (see, e.g., [12], [13], [26]).

2. Main results

Let us formulate the results of the paper.

By $F(a,b;c;d)$ we denote Gauss hypergeometric function

$$F(a,b;c;z)=1+\sum_{k=1}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{8}$$

$(x)_k := x(x+1)(x+2)...(x+k-1)$.

For arbitrary $\alpha>0$, $r\in(0,1)$ and $1\le p<\infty$ we denote by $n_0=n_0(\alpha,r,p)$ the smallest integer n such that

$$\frac{1}{\alpha r}\frac{1}{n^r}+\frac{\alpha r p}{n^{1-r}}\le\begin{cases}\frac{1}{14}, & p=1,\\ \frac{1}{(3\pi)^3}\cdot\frac{p-1}{p}, & 1<p<\infty.\end{cases} \tag{9}$$

The following theorem takes place.

Theorem 2.1. *Let $0<r<1$, $\alpha>0$, $\beta\in\mathbb{R}$ and $n\in\mathbb{N}$. Then in the case $1<p<\infty$ for any function $f\in C^{\alpha,r}_{\beta}L_p$ and $n\ge n_0(\alpha,r,p)$, the following inequality holds*

$$\|f(\cdot)-S_{n-1}(f;\cdot)\|_C\le e^{-\alpha n^r}n^{\frac{1-r}{p}}\Big(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}}F^{\frac{1}{p'}}\Big(\frac{1}{2},\frac{3-p'}{2};\frac{3}{2};1\Big)$$
$$+\gamma_{n,p}\Big(\Big(1+\frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}}+\frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Big)\Big)E_n(f^{\alpha,r}_{\beta})_{L_p},\quad \frac{1}{p}+\frac{1}{p'}=1, \tag{10}$$

where $F(a,b;c;d)$ is Gauss hypergeometric function.

Moreover, for any function $f\in C^{\alpha,r}_{\beta}L_p$ one can find a function $\mathcal{F}(x)=\mathcal{F}(f;n;x)$, such that $E_n(\mathcal{F}^{\alpha,r}_{\beta})_{L_p}=E_n(f^{\alpha,r}_{\beta})_{L_p}$ and for $n\ge n_0(\alpha,r,p)$ the following equality holds

$$\|\mathcal{F}(\cdot)-S_{n-1}(F;\cdot)\|_C=e^{-\alpha n^r}n^{\frac{1-r}{p}}\Big(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}}F^{\frac{1}{p'}}\Big(\frac{1}{2},\frac{3-p'}{2};\frac{3}{2};1\Big)$$
$$+\gamma_{n,p}\Big(\Big(1+\frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}}+\frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Big)\Big)E_n(f^{\alpha,r}_{\beta})_{L_p},\quad \frac{1}{p}+\frac{1}{p'}=1. \tag{11}$$

In (10) *and* (11) *the quantity $\gamma_{n,p}=\gamma_{n,p}(\alpha,r,\beta)$ is such that $|\gamma_{n,p}|\le(14\pi)^2$.*

Proof. The first part of Theorem 2.1 was proved by the authors in the work [14]. That is why here we will prove only the equality (11).

Denote

$$P^{(n)}_{\alpha,r,\beta}(t):=\sum_{k=n}^{\infty}e^{-\alpha k^r}\cos\Big(kt-\frac{\beta\pi}{2}\Big),\ 0<r<1,\ \alpha>0,\ \beta\in\mathbb{R}. \tag{12}$$

The function $P^{(n)}_{\alpha,r,\beta}(t)$ is orthogonal to any trigonometric polynomial t_{n-1} of degree not greater than $n-1$. Hence, for $f\in C^{\alpha,r}_{\beta}L_p$, $1\le p\le\infty$ and for any polynomial $t_{n-1}\in\tau_{2n-1}$ at every point $x\in\mathbb{R}$ the following equality holds

$$\rho_n(f;x)=f(x)-S_{n-1}(f;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}\delta_n(t)P^{(n)}_{\alpha,r,\beta}(x-t)dt, \tag{13}$$

where

$$\delta_n(x)=\delta_n(\alpha,r,\beta;x):=f^{\alpha,r}_{\beta}(x)-t_{n-1}(x). \tag{14}$$

To prove the second part of Theorem 2.1, according to the equality (13), for arbitrary $\varphi \in L_p$ we should find the function $\Phi(\cdot) = \Phi(\varphi, n; \cdot) \in L_p$, such that $E_n(\Phi)_{L_p} = E_n(\varphi)_{L_p}$ and for all $n \geq n_0(\alpha, r, p)$ the following equality holds

$$\frac{1}{\pi}\left|\int_{-\pi}^{\pi}(\Phi(t) - t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(0-t)dt\right| = e^{-\alpha n^r} n^{\frac{1-r}{p}}\left(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}}F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big)\right.$$
$$\left.+\gamma_{n,p}\Big(\Big(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Big)\right) E_n(\varphi)_{L_p}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{15}$$

where t^*_{n-1} is the polynomial of the best approximation of the degree $n-1$ of the function Φ in the space L_p, $|\gamma_{n,p}| \leq (14\pi)^2$.

In this case for an arbitrary function $f \in C^{\alpha,r}_{\beta}L_p$, $1 < p < \infty$, there exists a function $\Phi(\cdot) = \Phi(f^{\alpha,r}_{\beta}; \cdot)$, such that $E_n(\Phi)_{L_p} = E_n(f^{\alpha,r}_{\beta})_{L_p}$, and for $n \geq n_0(\alpha, r, p)$ the formula (15) holds, where as function φ we take the function $f^{\alpha,r}_{\beta}$.

Let us assume

$$\mathcal{F}(\cdot) = \mathcal{J}^{\alpha,r}_{\beta}\Big(\Phi(\cdot) - \frac{a_0}{2}\Big),$$

where

$$a_0 = a_0(\Phi) := \frac{1}{\pi}\int_{-\pi}^{\pi}\Phi(t)dt.$$

The function $\mathcal{F}$ is the function, which we have looked for because $\mathcal{F} \in C^{\alpha,r}_{\beta}L_p$ and

$$E_n(\mathcal{F}^{\alpha,r}_{\beta})_{L_p} = E_n(\Phi - \frac{a_0}{2})_{L_p} = E_n(\Phi)_{L_p} = E_n(f^{\alpha,r}_{\beta})_{L_p},$$

so (13), (10) and (15) imply (11).

At last, let us prove (15). Let $\varphi \in L_p$, $1 < p < \infty$. Then as a function $\Phi(t)$ we consider the function

$$\Phi(t) = \|P^{(n)}_{\alpha,r,-\beta}\|^{1-p'}_{p'}|P^{(n)}_{\alpha,r,-\beta}(t)|^{p'-1}\mathrm{sign}(P^{(n)}_{\alpha,r,-\beta}(t))E_n(\varphi)_{L_p} \tag{16}$$

For this function we have that

$$\|\Phi\|_p = \|P^{(n)}_{\alpha,r,-\beta}\|^{1-p'}_{p'}\||P^{(n)}_{\alpha,r,-\beta}|^{p'-1}\|_p E_n(\varphi)_{L_p}$$
$$= \|P^{(n)}_{\alpha,r,-\beta}\|^{1-p'}_{p'}\|P^{(n)}_{\alpha,r,-\beta}\|^{p'-1}_{p'}E_n(\varphi)_{L_p} = E_n(\varphi)_{L_p}.$$

Now we show that the polynomial t^*_{n-1} of best approximation of the degree $n-1$ in the space L_p of the function $\Phi(t)$ equals identically to zero: $t^*_{n-1} \equiv 0$.

For any $t_{n-1} \in \tau_{2n-1}$

$$\int_0^{2\pi} t_{n-1}(t)|\Phi(t)|^{p-1}\mathrm{sign}(\Phi(t))dt = \|P^{(n)}_{\alpha,r,-\beta}\|^{-1}_{p'}(E_n(\varphi)_{L_p})^{p-1}\int_{-\pi}^{\pi} t_{n-1}(t)P^{(n)}_{\alpha,r,-\beta}(t)dt = 0.$$

Then, according to Proposition 1.4.12 of the work [5, p. 29] we can make conclusion that the polynomial $t^*_{n-1} \equiv 0$ is the polynomial of the best approximation of the function $\Phi(t)$ in the space L_p, $1 < p < \infty$.

For the function $\Phi(t)$ of the form (16) we can write

$$\frac{1}{\pi}\int\limits_{-\pi}^{\pi}(\Phi(t)-t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(-t)dt$$

$$=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\Phi(t)P^{(n)}_{\alpha,r,\beta}(-t)dt=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\Phi(t)P^{(n)}_{\alpha,r,-\beta}(t)dt$$

$$=\frac{1}{\pi}\|P^{(n)}_{\alpha,r,-\beta}\|^{1-p'}_{p'}E_n(\varphi)_{L_p}\int\limits_{-\pi}^{\pi}|P^{(n)}_{\alpha,r,-\beta}(t)|^{p'}dt=\frac{1}{\pi}\|P^{(n)}_{\alpha,r,-\beta}\|_{p'}E_n(\varphi)_{L_p}. \tag{17}$$

It follows from the relation (18) of the work [14] that for $n\geq n_0(\alpha,r,p)$, $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, the following equality holds

$$\frac{1}{\pi}\Big\|P^{(n)}_{\alpha,r,\beta}\Big\|_{p'}=e^{-\alpha n^r}n^{\frac{1-r}{p}}\Big(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}}F^{\frac{1}{p'}}\Big(\frac{1}{2},\frac{3-p'}{2};\frac{3}{2};1\Big)$$
$$+\gamma^{(2)}_{n,p}\Big(\Big(1+\frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}}+\frac{p^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Big)\Big), \tag{18}$$

where the quantities $\gamma^{(2)}_{n,p}=\gamma^{(2)}_{n,p}(\alpha,r,\beta)$, satisfy the inequality $|\gamma^{(2)}_{n,p}|\leq(14\pi)^2$.

Thus, from (18) and (17) we arrive at the equality (11). Theorem 2.1 is proved. □

Theorem 2.2. *Let $0<r<1$, $\alpha>0$, $\beta\in\mathbb{R}$, $n\in\mathbb{N}$. Then, for any $f\in C^{\alpha,r}_{\beta}L_1$ and $n\geq n_0(\alpha,r,1)$ the following inequality holds:*

$$\|f(\cdot)-S_{n-1}(f;\cdot)\|_C\leq e^{-\alpha n^r}n^{1-r}\Big(\frac{1}{\pi\alpha r}+\gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big)\Big)E_n(f^{\alpha,r}_{\beta})_{L_1}. \tag{19}$$

Moreover, for any function $f\in C^{\alpha,r}_{\beta}L_1$ one can find a function $\mathcal{F}(x)=\mathcal{F}(f;n,x)$ in the set $C^{\alpha,r}_{\beta}L_1$, such that $E_n(\mathcal{F}^{\alpha,r}_{\beta})_{L_1}=E_n(f^{\alpha,r}_{\beta})_{L_1}$ and for $n\geq n_0(\alpha,r,1)$ the following equality holds

$$\|\mathcal{F}(\cdot)-S_{n-1}(F;\cdot)\|_C=e^{-\alpha n^r}n^{1-r}\Big(\frac{1}{\pi\alpha r}+\gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big)\Big)E_n(f^{\alpha,r}_{\beta})_{L_1}. \tag{20}$$

In (19) and (20) the quantity $\gamma_{n,1}=\gamma_{n,1}(\alpha,r,\beta)$ is such that $|\gamma_{n,1}|\leq(14\pi)^2$.

Proof. The first part of Theorem 2.2 was proved in [14].

So let us prove the second part of Theorem 2.2. For this we need for any function $\varphi\in L_1$ to find the function $\Phi(\cdot)=\Phi(\varphi,\cdot)\in L_1$, such that $E_n(\Phi)_{L_1}=E_n(\varphi)_{L_1}$ and for all $n\geq n_0(\alpha,r,1)$ the following equality holds

$$\frac{1}{\pi}\left|\int\limits_{-\pi}^{\pi}\Big(\Phi(t)-t^*_{n-1}(t)\Big)P^{(n)}_{\alpha,r,\beta}(0-t)dt\right|=e^{-\alpha n^r}n^{1-r}\Big(\frac{1}{\pi\alpha r}+\gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big)\Big)E_n(\varphi)_{L_1}, \tag{21}$$

where t^*_{n-1} is the polynomial of the best approximation of degree $n-1$ of the function Φ in the space L_1 and $|\gamma_{n,1}|\leq(14\pi)^2$.

In this case for any function $f\in C^{\alpha,r}_{\beta}L_1$ there exists a function $\Phi(\cdot)=\Phi(f^{\alpha,r}_{\beta};\cdot)$, such that $E_n(\Phi)_{L_1}=E_n(f^{\alpha,r}_{\beta})$, and for $n\geq n_0(\alpha,r,1)$ the formula (21) holds, where as function φ we will take the function $f^{\alpha,r}_{\beta}$.

Let us consider the function

$$\mathcal{F}(\cdot) = \mathcal{J}_{\beta}^{\alpha,r}(\Phi(\cdot) - \frac{a_0}{2}),$$

where

$$a_0 = a_0(\Phi) := \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \Phi(t)dt.$$

The function F is the function, which we look for, because $F \in C_{\beta}^{\alpha,r}L_1$ and

$$E_n(\mathcal{F}_{\beta}^{\alpha,r})_{L_1} = E_n(\Phi - \frac{a_0}{2})_{L_1} = E_n(\Phi)_{L_1} = E_n(f_{\beta}^{\alpha,r})_{L_1},$$

and on the basis (13), (19) and (21) the formula (20) holds.

Let us prove (21). Let t^* be the point from the interval $T = \Big[\frac{\pi(1-\beta)}{2n},\ 2\pi + \frac{\pi(1-\beta)}{2n}\Big)$, where the function $|P_{\alpha,r,-\beta}^{(n)}|$ attains its largest value, i.e.,

$$|P_{\alpha,r,-\beta}^{(n)}(t^*)| = \|P_{\alpha,r,-\beta}^{(n)}\|_C = \|P_{\alpha,r,\beta}^{(n)}\|_C.$$

Let put $\Delta_k^n := \Big[\frac{(k-1)\pi}{n} + \frac{\pi(1-\beta)}{2n}, \frac{k\pi}{n} + \frac{\pi(1-\beta)}{2n}\Big)$, $k = 1, ..., 2n$. By k^* we denote the number, such that $t^* \in \Delta_{k^*}^n$. Taking into accoun, that function $P_{\alpha,r,-\beta}^{(n)}$ is absolutely continuous, so for arbitrary $\varepsilon > 0$ there exists a segment $\ell^* = [\xi^*, \xi^* + \delta] \subset \Delta_{k^*}^n$, such that for arbitrary $t \in \ell^*$ the following inequality holds $|P_{\alpha,r,-\beta}^{(n)}(t)| > \|P_{\alpha,r,\beta}^{(n)}\|_C - \varepsilon$. It is clear that mes $\ell^* = |\ell^*| = \delta < \frac{\pi}{n}$.

For arbitrary $\varphi \in L_1$ and $\varepsilon > 0$ we consider the function $\Phi_\varepsilon(t)$, which on the segment T is defined with a help of equalities

$$\Phi_\varepsilon(t) = \begin{cases} E_n(\varphi)_{L_1} \frac{1-\varepsilon(2\pi-\delta)}{\delta} \operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big), & t \in \ell^*, \\ E_n(\varphi)_{L_1} \varepsilon \operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big), & t \in \mathrm{T} \setminus \ell^*. \end{cases}$$

For the function $\Phi_\varepsilon(t)$ for arbitrary small values of $\varepsilon > 0$ ($\varepsilon \in (0, \frac{1}{2\pi})$) the following equality holds

$$\begin{aligned} \|\Phi_\varepsilon\|_1 &= E_n(\varphi)_{L_1} \frac{1-\varepsilon(2\pi-\delta)}{\delta} \int\limits_{\ell^*} \Big|\operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big)\Big| dt \\ &+ E_n(\varphi)_{L_1} \varepsilon \int\limits_{\mathrm{T}\setminus\ell^*} \Big|\operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big)\Big| dt \\ &= E_n(\varphi)_{L_1} \Big(\frac{1-\varepsilon(2\pi-\delta)}{\delta}\delta + \varepsilon(2\pi-\delta)\Big) = E_n(\varphi)_{L_1}. \end{aligned} \tag{22}$$

It should be noticed that

$$\operatorname{sign}\Phi_\varepsilon(t) = \operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big). \tag{23}$$

Since for arbitrary trigonometric polynomial $t_{n-1} \in \tau_{2n-1}$

$$\int\limits_0^{2\pi} t_{n-1}(t) \operatorname{sign}\cos\Big(nt + \frac{\beta\pi}{2}\Big) dt = 0,$$

so, taking into account (23)

$$\int_0^{2\pi} t_{n-1}(t)\mathrm{sign}\Big(\Phi_\varepsilon(t)-0\Big)dt=0,\quad t_{n-1}\in\tau_{2n-1}.$$

According to Proposition 1.4.12 of the work [5, p.29], the polynomial $t^*_{n-1}\equiv 0$ is a polynomial of the best approximation of the function Φ_ε in the metric of the space L_1, i.e., $E_n(\Phi_\varepsilon)_{L_1}=\|\Phi_\varepsilon\|_1$, so (22) yields $E_n(\Phi_\varepsilon)_{L_1}=E_n(\varphi)_{L_1}$.

Moreover, for the function Φ_ε

$$\begin{aligned}&\frac{1}{\pi}\int_{-\pi}^{\pi}(\Phi_\varepsilon(t)-t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(-t)dt=\frac{1}{\pi}\int_{-\pi}^{\pi}\Phi_\varepsilon(t)P^{(n)}_{\alpha,r,-\beta}(t)dt\\&=\frac{1-\varepsilon(2\pi-\delta)}{\pi\delta}E_n(\varphi)_{L_1}\int_{\ell^*}\mathrm{sign}\cos\Big(nt+\frac{\beta\pi}{2}\Big)P^{(n)}_{\alpha,r,-\beta}(t)dt\\&+\frac{\varepsilon}{\pi}E_n(\varphi)_{L_1}\int_{\mathrm{T}\setminus\ell^*}\mathrm{sign}\cos\Big(nt+\frac{\beta\pi}{2}\Big)P^{(n)}_{\alpha,r,-\beta}(t)dt.\end{aligned}\tag{24}$$

Taking into account that $\mathrm{sign}\Phi_\varepsilon(t)=(-1)^k$, $t\in\Delta^{(n)}_k$, $k=1,...,2n$, and also the embedding $\ell^*\subset\Delta^{(n)}_{k^*}$, we get

$$\begin{aligned}&\left|\frac{1-\varepsilon(2\pi-\delta)}{\pi\delta}E_n(\varphi)_{L_1}\int_{\ell^*}\mathrm{sign}\cos\Big(nt+\frac{\beta\pi}{2}\Big)P^{(n)}_{\alpha,r,-\beta}(t)dt\right|\\&=\left|(-1)^{k^*}\frac{1-\varepsilon(2\pi-\delta)}{\pi\delta}E_n(\varphi)_{L_1}\int_{\ell^*}P^{(n)}_{\alpha,r,-\beta}(t)dt\right|\\&\geq\frac{1-\varepsilon(2\pi-\delta)}{\pi}E_n(\varphi)_{L_1}\Big(\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon\Big)\\&>\frac{1-2\pi\varepsilon}{\pi}E_n(\varphi)_{L_1}\Big(\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon\Big)\\&=\frac{1}{\pi}E_n(\varphi)_{L_1}\Big(\|P^{(n)}_{\alpha,r,\beta}\|_C-2\pi\varepsilon\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon+2\pi\varepsilon^2\Big)\\&>E_n(\varphi)_{L_1}\Big(\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon\Big(2\|P^{(n)}_{\alpha,r,\beta}\|_C+\frac{1}{\pi}\Big)\Big).\end{aligned}\tag{25}$$

Also, it is not hard to see that

$$\left|\frac{\varepsilon}{\pi}E_n(\varphi)_{L_1}\int_{\mathrm{T}\setminus\ell^*}\mathrm{sign}\cos\Big(nt+\frac{\beta\pi}{2}\Big)P^{(n)}_{\alpha,r,-\beta}(t)dt\right|\leq\frac{\varepsilon}{\pi}E_n(\varphi)_{L_1}\|P^{(n)}_{\alpha,r,\beta}\|_C.\tag{26}$$

Formulas (24)–(26) yield the following inequality

$$\begin{aligned}&\left|\int_{-\pi}^{\pi}\frac{1}{\pi}(\Phi_\varepsilon(t)-t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(-t)dt\right|\\&>E_n(\varphi)_{L_1}\Big(\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_C-\varepsilon\Big(\Big(2+\frac{1}{\pi}\Big)\|P^{(n)}_{\alpha,r,\beta}\|_C+\frac{1}{\pi}\Big)\Big).\end{aligned}\tag{27}$$

It should be noticed that asymptotic estimate for the quantity $\|P^{(n)}_{\alpha,r,\beta}\|_\infty$ was obtained in [17]. Let us show that this estimate can be improved, if we decrease the diapason for the remainder.

Formulas (34), (50)–(52) of the work [17], and also Remark 1 from [17] allow us to write that for any $n \in \mathbb{N}$

$$\|P^{(n)}_{\alpha,r,\beta}\|_\infty = \|P_{\alpha,r,n}\|_\infty\Big(1+\delta^{(1)}_n \frac{M_n}{n}\Big), \tag{28}$$

where

$$P_{\alpha,r,n}(t) := \sum_{k=0}^{\infty} e^{-\alpha(k+n)^r} e^{ikt},$$

$$M_n := \sup_{t\in\mathbb{R}} \frac{|P'_{\alpha,r,n}(t)|}{|P_{\alpha,r,n}(t)|},$$

and for $\delta^{(1)}_n = \delta^{(1)}_n(\alpha,r,\beta)$ the following estimate takes place $|\delta^{(1)}_n| \leq 5\sqrt{2}\pi$.

Then, as it follows from the estimates (87) and (99) of the work [17] for $n \geq n_0(\alpha,r,1)$

$$\|P_{\alpha,r,n}\|_\infty = \frac{e^{-\alpha n^r}}{\alpha r} n^{1-r}\left(1+\theta_{\alpha,r,n}\Big(\frac{1-r}{\alpha r n^r}+\frac{\alpha r}{n^{1-r}}\Big)\right), \quad |\theta_{\alpha,r,n}| \leq \frac{14}{13} \tag{29}$$

and

$$M_n \leq \frac{784\pi^2}{117}\Big(\frac{n^{1-r}}{\alpha r}+\alpha r n^r\Big). \tag{30}$$

Combining formulas (28)–(30), we obtain that for $n \geq n_0(\alpha,r,1)$

$$\begin{aligned}\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_\infty &= \frac{e^{-\alpha n^r}}{\alpha r\pi} n^{1-r}\left(1+\theta_{\alpha,r,n}\Big(\frac{1-r}{\alpha r n^r}+\frac{\alpha r}{n^{1-r}}\Big)\right)\Big(1+\delta^{(1)}_n\frac{M_n}{n}\Big)\\ &= e^{-\alpha n^r} n^{1-r}\Big(\frac{1}{\alpha r\pi}+\gamma_{n,1}\Big(\frac{1}{(\alpha r)^2 n^r}+\frac{\alpha r}{n^{1-r}}\Big)\Big),\end{aligned} \tag{31}$$

where

$$|\gamma_{n,1}| \leq \frac{1}{\pi}\left(\frac{14}{13}+\frac{784\pi^2 5\sqrt{2}\pi}{117}+\frac{14\cdot 5\sqrt{2}\pi\cdot 784\pi^2}{13\cdot 117\cdot 14}\right) = \frac{14}{13\pi}\Big(1+\frac{3920\sqrt{2}\pi^3}{117}\Big). \tag{32}$$

Let us choose ε small enough that

$$\varepsilon < \frac{\left((14\pi)^2-\frac{14}{13\pi}\left(1+\frac{3920\sqrt{2}\pi^3}{117}\right)\right)e^{-\alpha n^r} n^{1-r}\big(\frac{1}{\alpha r n^r}+\frac{\alpha r}{n^{1-r}}\big)}{(2+\frac{1}{\pi})\|P^{(n)}_{\alpha,r,\beta}\|_\infty+\frac{1}{\pi}} \tag{33}$$

and for this ε we put

$$\Phi(t) = \Phi_\varepsilon(t). \tag{34}$$

The function $\Phi(t)$ is the function, which we have looked for, because $E_n(\Phi)_{L_1} = E_n(\varphi)_{L_1}$ and according to (27), (31)–(33) for $n \geq n_0(\alpha,r,1)$

$$\begin{aligned}&\left|\frac{1}{\pi}(\Phi(t)-t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(-t)dt\right|\\ &> E_n(\varphi)_{L_1}\left(\frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_C-\left((14\pi)^2-\frac{14}{13\pi}\Big(1+\frac{3920\sqrt{2}\pi^3}{117}\Big)\right)e^{-\alpha n^r} n^{1-r}\Big(\frac{1}{\alpha r n^r}+\frac{\alpha r}{n^{1-r}}\Big)\right)\\ &\geq e^{-\alpha n^r} n^{1-r}\left(\frac{1}{\alpha r\pi}-(14\pi)^2\Big(\frac{1}{(\alpha r)^2 n^r}+\frac{\alpha r}{n^{1-r}}\Big)\right)E_n(\varphi)_{L_1}.\end{aligned} \tag{35}$$

On the other hand, according to (13) for $f \in C^{\alpha,r}_{\beta}L_1$ we get

$$\|f(\cdot)-S_{n-1}(f;\cdot)\|_C = \frac{1}{\pi}\int\limits_{-\pi}^{\pi}(f^{\alpha,r}_{\beta}(t)-t^*_{n-1}(t))P^{(n)}_{\alpha,r,\beta}(x-t)dt \le \frac{1}{\pi}\|P^{(n)}_{\alpha,r,\beta}\|_{\infty}E_n(f^{\alpha,r}_{\beta})_{L_1}, \tag{36}$$

where $t^*_{n-1} \in \tau_{2n-1}$ is the polynomial of the best approximation of the function $f^{\alpha,r}_{\beta}$ in the space L_1.

Formulas (35), (36), (31) and (32) imply (21). Theorem 2.2 is proved. □

As it was already mentioned, the inequalities (10) and (19) were proved in [14]. At the same time the problem about asymptotically best possible upper estimates of uniform norms of deviations of partial Fourier sums of the function f from $C^{\alpha,r}_{\beta}L_p$, $1 \le p < \infty$, remains open. Theorems 2.1 and 2.2 give the full answer on this question: the asymptotic equalities (11) and (20) prove that the estimates (10) and (19) are asymptotically best possible for functions from $C^{\alpha,r}_{\beta}L_p$ in the cases $1 < p < \infty$ snd $p = 1$ respectively. At the very end, we notice that inequalities (10) and (19) are asymptotically best possible on such important subsets from $C^{\alpha,r}_{\beta}L_p$ as sets $C^{\alpha,r}_{\beta,p}$, $1 \le p < \infty$.

Indeed, if $f \in C^{\alpha,r}_{\beta,p}$, then $\|f^{\alpha,r}_{\beta}\|_p \le 1$ and $E_n(f^{\alpha,r}_{\beta})_{L_p} \le 1$, $1 \le p < \infty$. Considering the least upper bounds of both sides of inequality (10) over the classes $C^{\alpha,r}_{\beta,p}$, $1 < p < \infty$, we arrive at the inequality

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \le e^{-\alpha n^r} n^{\frac{1-r}{p}}\Big(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}}F^{\frac{1}{p'}}\Big(\frac{1}{2},\frac{3-p'}{2};\frac{3}{2};1\Big)$$
$$+\gamma_{n,p}\Big(\Big(1+\frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}}+\frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Big)\Big),\quad \frac{1}{p}+\frac{1}{p'}=1. \tag{37}$$

Comparing this relation with the estimate of Theorem 4 from [16] (see also [17]), we conclude that inequality (10) on the classes $C^{\alpha,r}_{\beta,p}$, $1 < p < \infty$, is asymptotically best possible.

In the same way, the asymptotic sharpness of the estimate (19) on the classes $C^{\alpha,r}_{\beta,1}$ follows from comparing inequality

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \le e^{-\alpha n^r} n^{1-r}\Big(\frac{1}{\pi\alpha r}+\gamma_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r}+\frac{1}{n^{1-r}}\Big)\Big) \tag{38}$$

and formula (18) from [17].

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